Kakeya Sets and its Applications

Kin Yau James Wong

Advisor: Ioan Bejenaru

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Abstract

In this thesis, we review the construction of the Kakeya sets, and its applications to fundamental concepts of real and harmonic analysis. In particular, we use the Kakeya sets to deepen our understanding of the conditions needed for the Lebesgue Differentiation Theorem to hold, as well as when multiplier operators can be extended to $L^p \to L^p$ bounded operators.

Contents

1 Introduction

The study of analysis is often saturated with lots of fine detail due to the existence of "pathological objects" that yield counterintuitive results. The construction of these pathological objects are often quite complicated, and **Kakeya sets** are no exception to this rule. We shall see its applications in certain facets of analysis, so the reader should at least be equipped with some working knowledge of measure theory, functional analysis and L^p spaces before we proceed. In particular, we shall often note the following remark:

Remark 1.1. *$C_c^\infty(\mathbb{R}^n) \coloneqq C^\infty(\mathbb{R}^n) \cap C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ with respect to the L^p norm for all $p \in [1,\infty)$, and dense in $C_0(\mathbb{R}^n)$ with respect to the uniform norm where $C_c(\mathbb{R}^n)$ is the space of compactly supported functions on $\mathbb{R}^n$, $C_0(\mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity, and $C^\infty(\mathbb{R}^n)$ is the space of smooth, i.e. infinitely differentiable, functions on $\mathbb{R}^n$.*

The proof of the above remark is omitted because it is quite involved (see [1] Proposition 8.17). However, one should be able to see, at least intuitively, why it is true.

Now, the avid learner of analysis may well be familiar with the Fundamental Theorem of Calculus, a theorem that identifies differentiation as the inverse operator of integration. In the world of measure theory, there is another notion of differentiation in terms of measures. One can show that for a locally integrable function f, taking averages of f over shrinking balls centered at some $\mathbf{x}$ in $\mathbb{R}^n$ will most likely result in convergence to $f(\mathbf{x})$. Formally,

Theorem 1.2. *If $f \in L^1_{loc}(\mathbb{R}^n)$, then*

$$\lim_{r\to 0} \frac{1}{m(B_r(\mathbf{0}))} \int_{B_r(\mathbf{0})} f(\mathbf{x}-\mathbf{y})\ d\mathbf{y} = f(\mathbf{x})$$

for almost every $\mathbf{x} \in \mathbb{R}^n$, where m is the Lebesgue measure and $B_r(\mathbf{0})$ is the open ball of radius r centered at $\mathbf{0}$.

Although this is a neat result, the averages need not be taken over balls in $\mathbb{R}^n$, nor do they need to contain $\mathbf{x}$ itself (but they ought to be close to $\mathbf{x}$). Folland shows that if a collection of subsets $\{E_r\}_{r>0}$ of $\mathbb{R}^n$ have bounded eccentricity [1], the result above will still hold. Such collections are defined as follows:

Definition 1.3 (Nicely Ordered Shrinking Sets)**.** *A collection of **nicely ordered shrinking sets** is a collection of measurable subsets $\{E_r\}_{r>0}$ of $\mathbb{R}^n$ such that*

1. *$E_r \subseteq B_r(\mathbf{0})$ for all $r > 0$, and*
2. *there exists $\alpha > 0$ such that $m(E_r) > \alpha \cdot m(B_r(\mathbf{0}))$ for all $r > 0$.*

The more general theorem involving nicely ordered shrinking sets is known as the **Lebesgue Differentiation Theorem**. It is stated below:

Theorem 1.4 (Lebesgue Differentiation Theorem)**.** *If $f \in L^1_{loc}(\mathbb{R}^n)$ and $\{E_r\}_{r>0}$ is a collection of nicely ordered shrinking sets, then*

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(\mathbf{x} - \mathbf{y})\, d\mathbf{y} = f(\mathbf{x})$$

for almost every $\mathbf{x} \in \mathbb{R}^n$.

The construction of nicely ordered shrinking sets can still be quite restrictive since they necessitate indexing by positive real numbers (hence the name 'nicely ordered'). However, it is entirely possible to come up with collections that are not indexed by the positive reals, but may still satisfy suitable criteria for the Lebesgue Differentiation Theorem to hold (after generalizing the notion of limits). In that regard, we will provide a generalized version of the Lebesgue Differentiation Theorem in Section 1.3.

Nevertheless, the conditions for nicely ordered shrinking sets are certainly sufficient, so one might wonder to what extent are the conditions necessary. In that regard, we shall find a collection of subsets that will cause the Lebesgue Differentiation Theorem to fail, and in the process use Kakeya sets to prove the result.

1.1 Basic Weak L^p Theory and Introduction to Maximal Operators

Some of our analysis will be on operators on functions. In that regard, it is useful to know some basic definitions and concepts related to such operators. We begin by introducing a relaxed version of the L^p norm, aptly named the **weak L^p functional**. The following definitions can be found from [1].

Definition 1.5 (Weak L^p)**.** *Let f be a measurable function on some measure space $(X, \mathcal{M}, \mu)$. Then,*

$$\begin{aligned} \lambda_f : (0, \infty) &\to [0, \infty] \\ \alpha &\mapsto \mu(|f|^{-1}((\alpha, \infty])) \end{aligned}$$

*is called the **distribution function**. If the measure space is $\mathbb{R}^n$, we will always assume $\mu = m$. The distribution function defines the **weak L^p functional** for each $p \in (0, \infty)$ like so:*

$$[f]_{L^{p,w}} \coloneqq \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p}.$$

*The space of all measurable functions $L^{p,w}(\mu)$ with $[f]_{L^{p,w}} < \infty$ is the **weak L^p space**.*

One should note that $[\cdot]_{L^{p,w}}$ is not a norm because it can be easily checked that it does not satisfy the triangle inequality. However, $[\cdot]_{L^{p,w}}$ can still be used to endow $L^{p,w}$ with a topology in a similar fashion to how metrics induce metric topologies, i.e. the topology of $L^{p,w}$ is generated by sets of the form $\{g \in L^{p,w} : [f - g]_{L^{p,w}} < r\}$ for $f \in L^{p,w}$ and $r > 0$.

This makes $L^{p,w}$ a topological vector space. Note also that $[\cdot]_{L^{p,w}} \leq \|\cdot\|_{L^p}$ which means that $L^p \subseteq L^{p,w}$.

Some of the operators on functions that we will work with will be non-linear, but will still share some useful properties with linear operators.

Definition 1.6 (Sublinear Operators and Boundedness Conditions)**.** *Let $T : \mathcal{V} \to L^0(Y, \mathcal{N}, \nu)$ where $\mathcal{V}$ is a vector subspace of $L^0(X, \mathcal{M}, \mu)$. Then, T is* ***sublinear*** *iff*

1. *$|T(f+g)| \leq |Tf| + |Tg|$ for all $f, g \in \mathcal{V}$, and*
2. *$|T(cf)| = c|Tf|$ for all $f, g \in \mathcal{V}$ and $c > 0$.*

Furthermore, T is ***strong type*** *(p, q) with $p, q \in [1, \infty]$ iff*

1. *T is sublinear,*
2. *$L^p(\mu) \subseteq \mathcal{V}$, and*
3. *$T(L^p(\mu)) \subseteq L^q(\nu)$; in particular, there exists $A > 0$ such that $\|Tf\|_{L^q} \leq A\|f\|_{L^p}$ for all $f \in L^p(\mu)$.*

Similarly, T is ***weak type*** *(p, q) with $p \in [1, \infty]$ and $q \in [1, \infty)$ iff*

1. *T is sublinear,*
2. *$L^p(\mu) \subseteq \mathcal{V}$, and*
3. *$T(L^p(\mu)) \subseteq L^{q,w}(\nu)$; in particular, there exists $A > 0$ such that $[Tf]_{L^{q,w}} \leq A\|f\|_{L^p}$ for all $f \in L^p(\mu)$.*

We say that T is weak type (p, ∞) iff T is strong type (p, ∞).

Remark 1.7. *It is often more practical to show that an operator T is weak type (p, q) for some $p \in [1, \infty]$ and $q \in [1, \infty)$ by showing that there exists $A > 0$ such that $\lambda_{Tf}(\alpha) \leq \frac{A}{\alpha^q}\|f\|_{L^p}^q$ for all $\alpha > 0$ and $f \in L^p(\mu)$. One can easily verify that this is equivalent to the third weak type condition in the definition above.*

An important type of operator is known as the **maximal operator**. The following definition is inspired by [3].

Definition 1.8 (Maximal Operator)**.** *Let μ be a σ-finite measure on $\mathbb{R}^n$. Then,*

$$\begin{aligned} \mathcal{A}_\mu : L^0(\mathbb{R}^n) &\to L^0(\mathbb{R}^n) \\ f &\mapsto \int |f(\cdot - \mathbf{y})| \, d\mu(\mathbf{y}) \end{aligned}$$

is the ***absolute expectation of*** *f* ***with respect to*** *μ. Note that $\mathcal{A}_\mu$ is defined on $L^0(\mathbb{R}^n)$ because $\mathcal{A}_\mu f$ is measurable for all measurable f by the Tonelli Theorem.*

Let $\mathcal{C}$ be a collection of positive σ-finite measures on $\mathbb{R}^n$. Suppose $\sup_{\mu\in\mathcal{C}} \mathcal{A}_\mu f \in L^0(\mathbb{R}^n)$ for all $f \in \mathcal{V}$ where $\mathcal{V}$ is some vector subspace of $L^0(\mathbb{R}^n)$. Then,

$$\begin{aligned} \mathcal{M}_{\mathcal{C}} : \mathcal{V} &\to L^0(\mathbb{R}^n) \\ f &\mapsto \sup_{\mu\in\mathcal{C}} \mathcal{A}_\mu f \end{aligned}$$

*is a **maximal operator on $\mathcal{V}$ induced by $\mathcal{C}$**.*

Remark 1.9.

1. $(\mathcal{M}_{\mathcal{C}} f)^{-1}((a,\infty]) = \bigcup_{\mu\in\mathcal{C}} (\mathcal{A}_\mu f)^{-1}((a,\infty])$ *for all* $a \in \mathbb{R}$.
   (a) *If $\mathcal{C}$ is countable, then $(\mathcal{M}_{\mathcal{C}} f)^{-1}((a,\infty])$ is measurable for all measurable f and $a \in \mathbb{R}$ since σ-algebras are closed under countable unions. Hence, $\mathcal{M}_{\mathcal{C}}$ would be a maximal operator on $L^0(\mathbb{R}^n)$.*
   (b) *If $\mathcal{A}_\mu f$ is continuous for all $f \in \mathcal{V}$ and $\mu \in \mathcal{C}$, then $(\mathcal{M}_{\mathcal{C}} f)^{-1}((a,\infty])$ is open thus measurable for all $f \in \mathcal{V}$ since topologies are closed under arbitrary unions. Hence, $\mathcal{M}_{\mathcal{C}}$ would be a maximal operator on $\mathcal{V}$.*
2. *One can easily verify that absolute expectations and maximal operators are sublinear.*
3. ***Warning:** Not every collection of σ-finite measures on $\mathbb{R}^n$ will induce a maximal operator on a non-trivial subspace of $L^0(\mathbb{R}^n)$.*

It is worthwhile to ponder upon which σ-finite measures μ and which measurable functions f result in $\mathcal{A}_\mu f$ being continuous. Since the Lebesgue Differentiation Theorem is concerned with taking averages of functions over subsets of $\mathbb{R}^n$, we will mostly work with finite measures of the following form:

Definition 1.10 (Conditional Lebesgue Measure). *Let E be a measurable subset of $\mathbb{R}^n$ with non-zero finite measure. Then,*

$$m_E := \frac{m(\cdot \cap E)}{m(E)}$$

*is the **Lebesgue measure conditioned on** E.*

The definition of the conditional Lebesgue measure is akin to the definition of conditional probability. Furthermore, one should notice that

$$\int f(\mathbf{x})\ m_E(\mathbf{x}) = \frac{1}{m(E)} \int_E f(\mathbf{x})\ d\mathbf{x}$$

whenever the integral can be evaluated. On that note, it is not hard to see that for certain measurable E, $\mathcal{A}_{m_E} f$ will "smooth out" some measurable f.

Proposition 1.11. *Let U be a measurable subset of $\mathbb{R}^n$ such that*

1. $m(U) > 0$,

2. $m(\partial U) = 0$ *where* $\partial U \coloneqq \overline{U} \setminus U^\circ$*, and*

3. U *is bounded.*

Then, $\mathcal{A}_{m_U} f$ *is continuous for all* $f \in L^1_{loc}(\mathbb{R}^n)$*. Therefore, if* $\mathcal{C} \subseteq \{m_U : m(U) > 0, m(\partial U) = 0, U \text{ is bounded}\}$*, then* $\mathcal{M}_\mathcal{C}$ *is a maximal operator on* $L^1_{loc}(\mathbb{R}^n)$ *by Remark 1.9.*

Proof. Let $f \in L^1_{loc}(\mathbb{R}^n)$, and let $\{\mathbf{x}_k\}_{k=1}^\infty$ be a sequence of points converging to some $\mathbf{x} \in \mathbb{R}^n$. Given that $(\partial U)^C = (\overline{U})^C \cup U^\circ$ is open, we have that for each $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} - \mathbf{y} \notin \partial U$, there exists $N \in \mathbb{N}$ for which

1. $\mathbf{x}_k - \mathbf{y} \in (\overline{U})^C$ if $\mathbf{x} - \mathbf{y} \in (\overline{U})^C$, or
2. $\mathbf{x}_k - \mathbf{y} \in U^\circ$ if $\mathbf{x} - \mathbf{y} \in U^\circ$

thus $\chi_U(\mathbf{x}_k - \mathbf{y}) = \chi_U(\mathbf{x} - \mathbf{y})$ for all $k \geq N$. Since $m(\partial U) = 0$, this means that $\chi_U(\mathbf{x}_k - \cdot) \to \chi_U(\mathbf{x} - \cdot)$ almost everywhere.

Now, given that $\mathbf{x}_k \to \mathbf{x}$, there exists $N \in \mathbb{N}$ such that $\mathbf{x}_k \in B_1(\mathbf{x})$ for all $k \geq N$. By the boundedness of $B_1(\mathbf{x})$ and U, there exists a compact set K such that $\mathbf{y}, \mathbf{x}_k - \mathbf{y} \in K$ for all $k \geq N$ and $\mathbf{y} \in U$. Hence, $|f \cdot \chi_K| \chi_U(\mathbf{x}_k - \cdot) \leq |f \cdot \chi_K| \in L^1(\mathbb{R}^n)$ for all $k \geq N$. This (on top of almost everywhere convergence) will enable the use of the Dominated Convergence Theorem. On that note,

$$\begin{aligned}
(\mathcal{A}_{m_U} f)(\mathbf{x}_k) &= \frac{1}{m(U)} \int |f \cdot \chi_K|(\mathbf{x}_k - \mathbf{y}) \chi_U(\mathbf{y}) \, d\mathbf{y} && \text{(for all } k \geq N) \\
&= \frac{1}{m(U)} \int |f \cdot \chi_K|(\mathbf{y}) \chi_U(\mathbf{x}_k - \mathbf{y}) \, d\mathbf{y} \\
& && \text{(by commutativity of convolutions of } L^1 \text{ functions)} \\
&\to \frac{1}{m(U)} \int |f \cdot \chi_K|(\mathbf{y}) \chi_U(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \\
& && \text{(by the Dominated Convergence Theorem)} \\
&= (\mathcal{A}_{m_U} f)(\mathbf{x}).
\end{aligned}$$

By the arbitrariness of $\{\mathbf{x}_k\}_{k=1}^\infty$ and $f \in L^1_{loc}(\mathbb{R}^n)$, we are done. □

It is also worthwhile to consider conditions needed for maximal operators to satisfy some boundedness conditions as discussed above. Note that if $\mathcal{B} \coloneqq \{m_{B_r(\mathbf{0})} : r > 0\}$, then $\mathcal{M}_\mathcal{B}$ is a maximal operator on $L^1_{loc}(\mathbb{R}^n)$ by Proposition 1.11. $\mathcal{M}_\mathcal{B}$ is known as the **Hardy-Littlewood Maximal Operator**. We shall later prove that $\mathcal{M}_\mathcal{B}$ is weak-type $(1,1)$. On a slightly more general note, if $\mathcal{C}$ is a collection of finite positive measures supported in a fixed compact set such that $\mathcal{M}_\mathcal{C}$ is a maximal operator on some $\mathcal{V} \supseteq L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$ and $\mathcal{M}_\mathcal{C} f$ does not diverge almost everywhere for each $f \in L^p(\mathbb{R}^n)$, then $\mathcal{M}_\mathcal{C}$ will also be weak-type (p,p) [3].

The proofs of $\mathcal{M}_\mathcal{B}$ and $\mathcal{M}_\mathcal{C}$ having weak boundedness will actually be very different. However, they will both incorporate the use of some covering lemmas. In particular, the **Vitali**

Covering Lemma will be used to prove weak boundedness of $\mathcal{M}_{\mathcal{B}}$, and a lemma that is akin to the Borel-Cantelli Lemma (to which we shall call the **Pseudo-Borel-Cantelli Lemma** [3]) will be used to prove weak boundedness of $\mathcal{M}_{\mathcal{C}}$. The Vitali Covering Lemma roughly states that any collection of balls in $\mathbb{R}^n$ admits a subcollection of disjoint balls with comparable measure, and the Pseudo-Borel-Cantelli Lemma roughly states that if the sum of measures of sets diverges, then there is a way to cover $\mathbb{R}^n$ by translating the sets strategically. These lemmas are stated and proved below.

Lemma 1.12 (Vitali Covering Lemma). *Let $\mathcal{B}$ be some collection of balls in $\mathbb{R}^n$. For all $c < m(\bigcup_{B\in\mathcal{B}} B)$, there exists a finite subcollection $\mathcal{B}_0 \subseteq \mathcal{B}$ of disjoint balls such that $c < 3^n m(\bigcup_{B\in\mathcal{B}_0} B) = 3^n \sum_{B\in B_0} m(B)$.*

Proof. Let $c < m(\bigcup_{B\in\mathcal{B}} B)$. By the regularity of the Lebesgue measure, there exists a compact $K \subseteq \bigcup_{B\in\mathcal{B}} B)$ such that $m(K) > c$. Since $\mathcal{B}$ is an open cover of K, there exists a finite subcover $\mathcal{B}' := \{B_1, \ldots, B_k\}$ (of which we may assume to be ordered by non-increasing measure) of K. Let $j_1 := 1$, and let $j_{i+1} > j_i$ be the smallest index such that $B_{j_{i+1}} \cap (B_{j_1} \cup \cdots \cup B_{j_i}) = \varnothing$ if it exists. Since $\mathcal{B}'$ is finite, the inductive definitions of the j_i's must terminate. Let $\mathcal{B}_0$ be the collection of all B_{j_i}'s (note that $\mathcal{B}_0$ is also a mutually disjoint collection of balls). By our construction, we have that for any $B_\ell \in \mathcal{B}_0 \setminus \mathcal{B}'$, there exists $j_i < \ell$ for which $B_{j_i} \cap B_\ell \neq \varnothing$. Note that $m(B_{j_i}) \geq m(B_\ell)$ by the ordering, so if $3B$ denotes the 3-fold scaling of a ball B from its center, then $B_\ell \subseteq 3B_{j_i}$. In that regard, $K \subseteq \bigcup_{B\in\mathcal{B}'} B \subseteq \bigcup_{B\in\mathcal{B}_0} 3B$ which means that

$$\begin{aligned} c < m(K) &\leq m\left(\bigcup_{B\in\mathcal{B}_0} 3B\right) \\ &\leq 3^n m\left(\bigcup_{B\in\mathcal{B}_0} B\right) && \text{(from the scaling of the balls)} \\ &= 3^n \sum_{B\in\mathcal{B}_0} m(B). && \text{(by dsjointedness of the balls)} \end{aligned}$$

By the arbitrariness of c, we are done. $\square$

Lemma 1.13 (Pseudo-Borel-Cantelli Lemma). *Let $K \subseteq \mathbb{R}^n$ be a compact set. Let $\{E_j\}_{j=1}^\infty$ be a collection of measurable subsets of K such that $\sum_{j=1}^\infty m(E_j) = \infty$. Then, there exists $\{\mathbf{x}_j\}_{j=1}^\infty \subseteq \mathbb{R}^n$ such that $m(\mathbb{R}^n \setminus \bigcap_{k=1}^\infty \bigcup_{j=k}^\infty (E_j + \mathbf{x}_j)) = 0$.*

Proof. We begin by extracting disjoint subsequences $\{F_{i,j}\}_{j=1}^\infty$ for $i \in \mathbb{N}$ from $\{E_j\}_{j=1}^\infty$ such that the sum of measures of terms in each subsequence is also infinite. Let $j_0 := 1$. Given that $\sum_{j=1}^\infty m(E_j) = \infty$, there exists $\{j_i\}_{i=1}^\infty \subseteq \mathbb{N}$ with $j_i > j_{i-1}$ for all $i \in \mathbb{N}$ such that $\sum_{j=j_{i-1}}^{j_i - 1} m(E_j) \geq 1$. Let $J_i := \{j_{i-1}, \ldots, j_i - 1\}$ for each $i \in \mathbb{N}$ and let $\{p_i\}_{i=1}^\infty$ be the enumeration of primes. Then, $\bigcup_{k=1}^\infty J_{p_i^k}$ is countable for each $i \in \mathbb{N}$. Let $\pi_i : \mathbb{N} \to \bigcup_{k=1}^\infty J_{p_i^k}$ be an order-preserving bijection for each i. Then, $F_{i,j} := E_{\pi_i(j)}$ for $i, j \in \mathbb{N}$ yields the desired result. Note that $\pi : \mathbb{N}^2 \to \mathbb{N}, \pi : (i,j) \mapsto \pi_i(j)$ is an injective function because $\bigcup_{k=1}^\infty J_{p_{i_0}^k} \cap \bigcup_{k=1}^\infty J_{p_{i_1}^k} = \varnothing$ for all $i_0 \neq i_1$.

Now, $\mathbb{R}^n$ can be partitioned into countably many unit cubes, and the cubes can be enumerated in a way such that each cube appears infinitely often in the sequence (consider the index sequence $(1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,\dots)$). Let $\{Q_i\}_{i=1}^{\infty}$ be such an enumeration. We will inductively show that there exists $\{\mathbf{v}_{i,j}\}_{j=1}^{\infty} \subseteq \mathbb{R}^n$ such that $m(Q_i \setminus \bigcup_{j=1}^{\infty}(F_{i,j} + \mathbf{v}_{i,j})) = 0$ for each $i \in \mathbb{N}$. Let $\mathbf{v}_{i,1} = 0$ for all $i \in \mathbb{N}$. Fix $i \in \mathbb{N}$ and suppose $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,j-1} \in \mathbb{R}^n$ have been determined for some $j \in \mathbb{N}$. Let $G_{i,j} := Q_i \setminus \bigcup_{k=1}^{j-1}(F_{i,k} + \mathbf{v}_{i,k})$ and let $\eta_{i,j} := \chi_{G_{i,j}} * \chi_{-F_{i,j}}$. By the Tonelli Theorem,

$$\int \eta_{i,j}\ dm = m(G_{i,j}) \cdot m(F_{i,j}).$$

Note that regardless of i, j, if $Q_i \cap (K + \mathbf{x}) = \varnothing$ for some $\mathbf{x} \in \mathbb{R}^n$, then $G_{i,j} \cap (F_{i,j} + \mathbf{x}) = \varnothing$ by containment, and so $\chi_{G_{i,j}}(\cdot)\chi_{-F_{i,j}}(\mathbf{x} - \cdot) = \chi_{G_{i,j}}(\cdot)\chi_{F_{i,j}+\mathbf{x}}(\cdot) \equiv 0$ which means that $\eta_{i,j}(\mathbf{x}) = 0$. In that regard, supposing that K is contained in a cube of side length $r \in \mathbb{N}$ by compactness, we have that $\operatorname{supp}(\eta_{i,j})$ is contained in a cube C of side length $r + 1 =: s$ ($+1$ is from the unit cube Q_i). Note that C can thus be partitioned into s^n unit cubes $\{C_k\}_{k=1}^{s^n}$. Hence,

$$m(G_{i,j}) \cdot m(F_{i,j}) = \int \eta_{i,j}(\mathbf{x})\ d\mathbf{x} = \sum_{k=1}^{s^n} \int_{C_k} \eta_{i,j}(\mathbf{x})\ d\mathbf{x}$$

which means that there exists $1 \le k' \le s^n$ for which $\int_{C_{k'}} \eta_{i,j}(\mathbf{x})\ d\mathbf{x} \ge s^{-n} \cdot m(G_{i,j}) \cdot m(F_{i,j})$. Given that $C_{k'}$ is a unit cube, there exists $\mathbf{v} \in C_{k'}$ for which $m(G_{i,j} \cap (F_{i,j} + \mathbf{v})) = \eta_{i,j}(\mathbf{v}) \ge \int_{C_{k'}} \eta_{i,j}(\mathbf{x})\ d\mathbf{x} \ge s^{-n} \cdot m(G_{i,j}) \cdot m(F_{i,j})$. Letting $\mathbf{v}_{i,j} := \mathbf{v}$ completes our inductive definition of $\mathbf{v}_{i,j}$ and $G_{i,j}$ for all $j \in \mathbb{N}$. By the arbitrariness of i, we have indeed defined $\mathbf{v}_{i,j}$ and $G_{i,j}$ for all $i, j \in \mathbb{N}$. Now, let $H_{i,j} := Q_i \cap \bigcup_{k=1}^{j}(F_{i,k} + \mathbf{v}_{i,k})$ for all $i, j \in \mathbb{N}$. Then,

$$\begin{aligned} m(H_{i,j}) &= m(H_{i,j-1}) + m(G_{i,j} \cap (F_{i,j} + \mathbf{v}_{i,j})) \\ &\ge m(H_{i,j-1}) + s^{-n} \cdot m(G_{i,j}) \cdot m(F_{i,j}) \\ &= m(H_{i,j-1}) + s^{-n} \cdot (1 - m(H_{i,j-1})) \cdot m(F_{i,j}) \end{aligned}$$

which means that $m(H_{i,j}) - m(H_{i,j-1}) \ge s^{-n}(1 - m(H_{i,j-1})) \cdot m(F_{i,j})$ for all $i, j \in \mathbb{N}$. Given that $m(H_{i,j}) \le m(Q_i) = 1$ and $H_{i,j-1} \subseteq H_{i,j}$ for all i, j, we have that $\{m(H_{i,j})\}_{j=1}^{\infty}$ is increasing and bounded above 1 which means that $\lim_{j\to\infty} m(H_{i,j}) \le 1$ for all i. Observe that

$$\lim_{j\to\infty} m(H_{i,j}) \ge \sum_{j=2}^{\infty}(m(H_{i,j}) - m(H_{i,j-1})) \ge s^{-n} \cdot \sum_{j=2}^{\infty}(1 - m(H_{i,j-1})) \cdot m(F_{i,j}).$$

If $\lim_{j\to\infty} m(H_{i,j}) < 1$, then there exists $\varepsilon > 0$ such that $1 - m(H_{i,j}) > \varepsilon$ for all j. However, this would mean that the sum on the right would diverge to ∞ thus $\lim_{j\to\infty} m(H_{i,j}) = \infty$ which is a contradiction. Therefore, $m(Q_i \cap \bigcup_{j=1}^{\infty}(F_{i,j} + \mathbf{v}_{i,j})) = \lim_{j\to\infty} m(H_{i,j}) = 1$ for all $i \in \mathbb{N}$. Given that $m(Q_i) = 1$, our desired result $m(Q_i \setminus \bigcup_{j=1}^{\infty}(F_{i,j} + \mathbf{v}_{i,j})) = 0$ is achieved.

Finally, let $\mathbf{x}_j := \mathbf{v}_{\pi^{-1}(j)}$ if $j \in \pi(\mathbb{N}^2)$ and $\mathbf{x}_j = 0$ otherwise. Then, each $E_k + \mathbf{x}_k$ corresponds exactly to some $F_{i,j} + \mathbf{v}_{i,j}$ (for $k \in \pi(\mathbb{N}^2)$). In that regard, for any $k \in \mathbb{N}$, $\bigcup_{j=k}^{\infty}(E_j + \mathbf{x}_j)$ will only exclude finitely many $F_{i,j} + \mathbf{x}_{i,j}$'s thus finitely many Q_i's will not be covered almost everywhere. However, given how the Q_i's are enumerated, each unit cube in $\mathbb{R}^n$ will still be covered almost everywhere by the remaining (infinitely many) $F_{i,j} + \mathbf{x}_{i,j}$'s. Therefore, $m(\mathbb{R}^n \setminus \bigcap_{k=1}^{\infty} \bigcup_{j-k}^{\infty}(E_j + \mathbf{x}_j)) = 0$. $\square$

Proposition 1.14. *Let $\mathcal{M}_{\mathcal{B}}$ be the Hardy-Littlewood maximal operator as defined above. Then, $\mathcal{M}_{\mathcal{B}}$ is weak-type $(1,1)$.*

PROOF. Let $\alpha > 0$, let $f \in L^1(\mathbb{R}^n)$, and let $c < \lambda_{\mathcal{M}_{\mathcal{B}} f}(\alpha)$. By Remark 1.9, we have that for each $\mathbf{x} \in (\mathcal{M}_{\mathcal{B}} f)^{-1}((\alpha, \infty])$, there exists $r_{\mathbf{x}} > 0$ such that $(\mathcal{A}_{m_{B_{r_{\mathbf{x}}}(\mathbf{0})}} f)(\mathbf{x}) > \alpha$. In particular,

$$\begin{aligned}\alpha < (\mathcal{A}_{m_{B_{r_{\mathbf{x}}}(\mathbf{0})}} f)(\mathbf{x}) &= \frac{1}{m(B_{r_{\mathbf{x}}}(\mathbf{x}))} \int_{B_{r_{\mathbf{x}}}(\mathbf{0})} |f(\mathbf{x}-\mathbf{y})| \, d\mathbf{y} && \text{(by translation invariance of } m) \\ &= \frac{1}{m(B_{r_{\mathbf{x}}}(\mathbf{x}))} \int_{B_{r_{\mathbf{x}}}(\mathbf{x})} |f(\mathbf{y})| \, d\mathbf{y}. && \text{(by symmetry of the ball)}\end{aligned}$$

Observe that $\{B_{r_{\mathbf{x}}}(\mathbf{x}) : \mathbf{x} \in (\mathcal{M}_{\mathcal{B}} f)^{-1}((\alpha, \infty])\}$ is an open cover of $(\mathcal{M}_{\mathcal{B}} f)^{-1}((\alpha, \infty])$. Hence, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_k$ such that

$$\begin{aligned} c &< 3^n \sum_{i=1}^{k} m(B_{r_{\mathbf{x}_i}}(\mathbf{x}_i)) && \text{(by the Vitali Covering Lemma (Lemma 1.12))} \\ &= \frac{3^n}{\alpha} \sum_{i=1}^{k} \alpha \cdot m(B_{r_{\mathbf{x}_i}}(\mathbf{x}_i)) \\ &\leq \frac{3^n}{\alpha} \sum_{i=1}^{k} \int_{B_{r_{\mathbf{x}_i}}(\mathbf{x}_i)} |f(\mathbf{y})| \, d\mathbf{y} && \text{(as noted in the inequality above)} \\ &\leq \frac{3^n}{\alpha} \|f\|_{L^1}. && \text{(since the balls are disjoint by the lemma)} \end{aligned}$$

Given that $c < \frac{3^n}{\alpha}\|f\|_{L^1}$ whenever $c < \lambda_{\mathcal{M}_{\mathcal{B}} f}(\alpha)$ by the arbitrariness of c, it must be the case that $\lambda_{\mathcal{M}_{\mathcal{B}} f}(\alpha) \leq \frac{3^n}{\alpha}\|f\|_{L^1}$. By the arbitrariness of α and f, we are done. □

Proposition 1.15. *Let $p \in [1, \infty)$, and let $K \subseteq \mathbb{R}^n$ be compact. Let $\mathcal{C} \subseteq \{\mu \text{ measure} : \mu(\mathbb{R}^n) = \mu(K) < \infty\}$ be such that $\mathcal{M}_{\mathcal{C}}$ is a maximal operator on some $\mathcal{V} \supseteq L^p(\mathbb{R}^n)$. Suppose for each $f \in L^p(\mathbb{R}^n)$ that $m\left((\mathcal{M}_{\mathcal{C}} f)^{-1}([0, \infty))\right) > 0$. Then, $\mathcal{M}_{\mathcal{C}}$ is weak type (p, p).*

PROOF. Define $\lambda_{g,E}(\alpha) \coloneqq m(E \cap |g|^{-1}((\alpha, \infty]))$ for measurable functions g and sets E. Let $Q \coloneqq [0, 1)^n$, and let $B \subseteq \mathbb{R}^n$ be a ball such that $Q \cup (K + \mathbf{x}) \subseteq B$ for all $\mathbf{x} \in \mathbb{R}^n$ such that $Q \cap (K + \mathbf{x}) \neq \varnothing$ (note that B exists due to boundedness of Q and K).

We will first show that $\mathcal{M}_{\mathcal{C}}$ is weak type (p, p) with respect to our modified distribution function $\lambda_{\mathcal{M}_{\mathcal{C}} f, B}$ for functions $f \in L^p(\mathbb{R}^n)$ supported on Q, and then extend the weak result to the main result. To that end, suppose on the contrary that for all $C > 0$, there exist $\alpha > 0$ and $f \in L^p(\mathbb{R}^n)$ with $\text{supp}(f) \subseteq Q$ such that $\lambda_{\mathcal{M}_{\mathcal{C}} f, B}(\alpha) > \frac{C}{\alpha^p}\|f\|_{L^p}^p$. Then, there exist $\alpha_k > 0$ and $g_k \in L^p(\mathbb{R}^n)$ with $\text{supp}(g_k) \subseteq Q$ such that $\lambda_{\mathcal{M}_{\mathcal{C}} g_k, B}(\alpha_k) > \frac{2^k}{\alpha_k^p}\|g_k\|_{L^p}^p$ for each $k \in \mathbb{N}$. Let $h_k \coloneqq \frac{k g_k}{\alpha_k}$ for each $k \in \mathbb{N}$. Then,

$$m(B) \geq \lambda_{\mathcal{M}_{\mathcal{C}} h_k, B}(k) = \lambda_{\mathcal{M}_{\mathcal{C}} g_k, B}(\alpha_k) > \frac{2^k}{\alpha_k^p}\|g_k\|_{L^p}^p = \frac{2^k}{k^p}\|h_k\|_{L^p}^p \geq 0$$

for all $k \in \mathbb{N}$. Given that $\lambda_{\mathcal{M}_{\mathcal{C}} h_k, B}(k) \in (0, m(B)]$, we have that $N_k \coloneqq \lceil \frac{1}{\lambda_{\mathcal{M}_{\mathcal{C}} h_k, B}(k)} \rceil$ is defined for each $k \in \mathbb{N}$. Let $\pi : \mathbb{N} \to \mathbb{N}$ be such that $\pi(k) = j$ whenever $\sum_{i=1}^{j-1} N_i < k \leq \sum_{i=1}^{j} N_i$. Then since $N_k \geq \frac{1}{\lambda_{\mathcal{M}_{\mathcal{C}} h_k, B}(k)}$ for all k,

$$\sum_{k=1}^{\infty} \lambda_{\mathcal{M}_{\mathcal{C}} h_{\pi(k)}, B}(\pi(k)) = \sum_{k=1}^{\infty} N_k \lambda_{\mathcal{M}_{\mathcal{C}} h_k, B}(k) \geq \sum_{k=1}^{\infty} 1 = \infty$$

Given that $\|h_k\|_{L^p}^p < \frac{k^p}{2^k} m(B)$, that $\frac{\|h_k\|_{L^p}^p}{\lambda_{\mathcal{M}_\mathcal{C} h_k, B}(k)} < \frac{k^p}{2^k}$, and that $N_k \le \frac{1}{\lambda_{\mathcal{M}_\mathcal{C} h_k, B}(k)} + 1$ for all $k \in \mathbb{N}$, we also have that

$$\sum_{k=1}^{\infty} \|h_{\pi(k)}\|_{L^p}^p = \sum_{k=1}^{\infty} N_k \|h_k\|_{L^p}^p \le \sum_{k=1}^{\infty} \left(\frac{\|h_k\|_{L^p}^p}{\lambda_{\mathcal{M}_\mathcal{C} h_k, B}(k)} + \|h_k\|_{L^p}^p \right) \le \sum_{k=1}^{\infty} \frac{(1+m(B))k^p}{2^k} < \infty.$$

Now, let $f_k \coloneqq |h_{\pi(k)}|$ and $E_k \coloneqq B \cap |\mathcal{M}_\mathcal{C} f_k|^{-1}((\pi(k), \infty])$ for all $k \in \mathbb{N}$. Note that $(\mathcal{M}_\mathcal{C} f_k)(\mathbf{x}) = \sup_{\mu \in \mathcal{C}} \int |h_{\pi(k)}(\mathbf{x}-\mathbf{y})| \ d\mu(\mathbf{y}) = (\mathcal{M}_\mathcal{C} h_{\pi(k)})(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ which means that $E_k = B \cap |\mathcal{M}_\mathcal{C} h_{\pi(k)}|^{-1}((\pi(k), \infty])$ for all $k \in \mathbb{N}$. Hence, $\sum_{k=1}^{\infty} m(E_k) = \sum_{k=1}^{\infty} \lambda_{\mathcal{M}_\mathcal{C} h_{\pi(k)}, B}(\pi(k)) = \infty$ and $\sum_{k=1}^{\infty} \|f_k\|_{L^p}^p = \sum_{k=1}^{\infty} \|h_{\pi(k)}\|_{L^p}^p < \infty$. By the Pseudo-Borel-Cantelli Lemma (Lemma 1.13), there exists $\{\mathbf{x}_j\}_{j=1}^{\infty} \subseteq \mathbb{R}^n$ such that $m(\mathbb{R}^n \setminus \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} (E_j + \mathbf{x}_j)) = 0$. Let $\tau_\mathbf{v}(\mathbf{x}) \coloneqq \mathbf{x} - \mathbf{v}$ for $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$, and let $F \coloneqq \sup_{k \in \mathbb{N}} f_k \circ \tau_{\mathbf{x}_k}$. Then,

$$\begin{aligned}
(\mathcal{M}_\mathcal{C} F)(\mathbf{x}) &= \sup_{\mu \in \mathcal{C}} \int |F(\mathbf{x}-\mathbf{y})| \ d\mu(\mathbf{y}) = \sup_{\mu \in \mathcal{C}} \int \left| \sup_{k \in \mathbb{N}} (f_k \circ \tau_{\mathbf{x}_k})(\mathbf{x}-\mathbf{y}) \right| \ d\mu(\mathbf{y}) \\
&\ge \sup_{\mu \in \mathcal{C}, k \in \mathbb{N}} \int |f_k \circ \tau_{\mathbf{x}_k}| (\mathbf{x}-\mathbf{y}) \ d\mu(\mathbf{y}) \qquad \text{(since } f_k \ge 0 \text{ for all } k) \\
&= \sup_{k \in \mathbb{N}} (\mathcal{M}_\mathcal{C}(f_k \circ \tau_{\mathbf{x}_k}))(\mathbf{x}).
\end{aligned}$$

Let $\mathbf{x} \in \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} (E_j + \mathbf{x}_j)$. Then, $\mathbf{x} \in \bigcup_{j=k}^{\infty} (E_j + \mathbf{x}_j)$ for all $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, there exists $j_k \ge k$ such that $\mathbf{x} \in E_{j_k} + \mathbf{x}_{j_k}$. Since $\mathbf{x} \in E_{j_k} + \mathbf{x}_{j_k}$, then $\mathbf{x} - \mathbf{x}_{j_k} \in E_{j_k}$ which means that $(\mathcal{M}_\mathcal{C}(f_{j_k} \circ \tau_{\mathbf{x}_{j_k}}))(\mathbf{x}) > \pi(j_k)$ for each $k \in \mathbb{N}$. Observe that as $k \to \infty$, $\pi(j_k) \to \infty$ since π and j_k are both unbounded and increasing. Hence, $(\mathcal{M}_\mathcal{C}(f_{j_k} \circ \tau_{\mathbf{x}_{j_k}}))(\mathbf{x}) \to \infty$ and $(\mathcal{M}_\mathcal{C} F)(\mathbf{x}) \ge \sup_{k \in \mathbb{N}} (\mathcal{M}_\mathcal{C}(f_k \circ \tau_{\mathbf{x}_k}))(\mathbf{x}) = \infty$. By the arbitrariness of $\mathbf{x}$, it follows that $(\mathcal{M}_\mathcal{C} F)(\mathbf{x}) = \infty$ almost everywhere on $\mathbb{R}^n$. However, $|F|^p = \sup_{k \in \mathbb{N}} |f_k \circ \tau_{\mathbf{x}_k}|^p \le \sum_{k=1}^{\infty} |f_k \circ \tau_{\mathbf{x}_k}|^p$ which implies that $F \in L^p(\mathbb{R}^n)$ as

$$\|F\|_{L^p}^p \le \sum_{k=1}^{\infty} \|f_k \circ \tau_{\mathbf{x}_k}\|_{L^p}^p = \sum_{k=1}^{\infty} \|f_k\|_{L^p}^p < \infty.$$

Given that this violates the condition that $\mathcal{M}_\mathcal{C} f$ is finite on sets of positive measure for each $f \in L^p(\mathbb{R}^n)$, we must conclude by contradiction that there exists $C > 0$ such that $\lambda_{\mathcal{M}_\mathcal{C} f, B}(\alpha) \le \frac{C}{\alpha^p} \|f\|_{L^p}^p$ for all $\alpha > 0$ and $f \in L^p(\mathbb{R}^n)$ with $\mathrm{supp}(f) \subseteq Q$.

We are now ready to prove the main result; let $Q_\mathbf{z} \coloneqq Q + \mathbf{z}$, let $B_\mathbf{z} \coloneqq B + \mathbf{z}$, and let $I_\mathbf{z} \coloneqq \{\mathbf{w} \in \mathbb{Z}^n : B_\mathbf{w} \cap B_\mathbf{z} \ne \varnothing\}$ for each $\mathbf{z} \in \mathbb{Z}^n$. Then, $\{Q_\mathbf{z}\}_{\mathbf{z} \in \mathbb{Z}^n}$ is a mutually disjoint covering of $\mathbb{R}^n$ and there exists $M \in \mathbb{N}$ such that $|I_\mathbf{z}| = M$ for all $\mathbf{z} \in \mathbb{R}^n$ since the balls have uniform size and are arranged in a lattice. Now, let $f \in L^p(\mathbb{R}^n)$. Then,

$$\begin{aligned}
(\mathcal{M}_\mathcal{C} f)(\mathbf{x}) &= \sup_{\mu \in \mathcal{C}} \int |f(\mathbf{x}-\mathbf{y})| \ d\mu(\mathbf{y}) = \sup_{\mu \in \mathcal{C}} \int \left| \sum_{\mathbf{z} \in \mathbb{Z}^n} (f \cdot \chi_{Q_\mathbf{z}})(\mathbf{x}-\mathbf{y}) \right| \ d\mu(\mathbf{y}) \\
&\le \sum_{\mathbf{z} \in \mathbb{Z}^n} \sup_{\mu \in \mathcal{C}} \int |f \cdot \chi_{Q_\mathbf{z}}| (\mathbf{x}-\mathbf{y}) \ d\mu(\mathbf{y}) = \sum_{\mathbf{z} \in \mathbb{Z}^n} (\mathcal{M}_\mathcal{C}(f \cdot \chi_{Q_\mathbf{z}}))(\mathbf{x})
\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$ for each $\mathbf{z} \in \mathbb{Z}^n$. As proven above (and by translation invariance of m), there exists $C > 0$ such that $\lambda_{\mathcal{M}_\mathcal{C}(f \cdot \chi_{Q_\mathbf{z}}), B_\mathbf{z}}(\alpha) \le \frac{C}{\alpha^p} \|f \cdot \chi_{Q_\mathbf{z}}\|_{L^p}^p$ for all $\alpha > 0$ and $\mathbf{z} \in \mathbb{Z}^n$. Note also that $\lambda_{\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{B_{\mathbf{z}}}),B_{\mathbf{w}}} \le \lambda_{\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{B_{\mathbf{z}}}),B_{\mathbf{z}}}$ for all $\mathbf{w},\mathbf{z}\in\mathbb{Z}^n$ because $\operatorname{supp}(\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{Q_{\mathbf{z}}}))\subseteq B_{\mathbf{z}}$ for each $\mathbf{z}\in\mathbb{Z}^n$. Hence,

$$\begin{aligned}
\lambda_{\mathcal{M}_{\mathcal{C}}f}(\alpha) &\le \sum_{\mathbf{z}\in\mathbb{Z}^n}\lambda_{\mathcal{M}_{\mathcal{C}}f,B_{\mathbf{z}}}(\alpha) && (\text{since } \textstyle\bigcup_{\mathbf{z}\in\mathbb{Z}^n}B_{\mathbf{z}}=\mathbb{R}^n)\\
&\le \sum_{\mathbf{z}\in\mathbb{Z}^n}\lambda_{\sum_{\mathbf{w}\in\mathbb{Z}^n}\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{Q_{\mathbf{w}}}),B_{\mathbf{z}}}(\alpha) && (\text{since } \mathcal{M}_{\mathcal{C}}f\le\textstyle\sum_{\mathbf{w}\in\mathbb{Z}^n}\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{Q_{\mathbf{z}}}))\\
&= \sum_{\mathbf{z}\in\mathbb{Z}^n}\lambda_{\sum_{\mathbf{w}\in I_{\mathbf{z}}}\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{Q_{\mathbf{w}}}),B_{\mathbf{z}}}(\alpha) && (\text{since } \operatorname{supp}(\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{Q_{\mathbf{w}}}))\subseteq B_{\mathbf{w}})\\
&\le \sum_{\mathbf{z}\in\mathbb{Z}^n}\sum_{\mathbf{w}\in I_{\mathbf{z}}}\lambda_{\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{Q_{\mathbf{w}}}),B_{\mathbf{z}}}\left(\frac{\alpha}{M}\right)\\
&\qquad (\text{since } |\textstyle\sum_{\mathbf{w}\in I_{\mathbf{z}}}\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{Q_{\mathbf{w}}})|^{-1}((\alpha,\infty])\subseteq\bigcup_{\mathbf{w}\in I_{\mathbf{z}}}|\mathcal{M}_{\mathcal{C}}(f\cdot\chi_{Q_{\mathbf{w}}})|^{-1}((\tfrac{\alpha}{M},\infty]))\\
&\le \sum_{\mathbf{z}\in\mathbb{Z}^n}\sum_{\mathbf{w}\in I_{\mathbf{z}}}\frac{C}{(\frac{\alpha}{M})^p}\|f\cdot\chi_{Q_{\mathbf{w}}}\|_{L^p}^p\\
&= \frac{CM^{p+1}}{\alpha^p}\sum_{\mathbf{z}\in\mathbb{Z}^n}\|f\cdot\chi_{Q_{\mathbf{z}}}\|_{L^p}^p && (\text{since each } \|f\cdot\chi_{Q_{\mathbf{z}}}\|_{L^p}^p \text{ is counted exactly } M \text{ times})\\
&= \frac{CM^{p+1}}{\alpha^p}\|f\|_{L^p}^p && (\text{since the } Q_{\mathbf{z}}\text{'s were mutually disjoint})
\end{aligned}$$

for all $\alpha>0$. By the arbitrariness of f, we indeed have that $[\mathcal{M}_{\mathcal{C}}f]_{L^{p,w}}\le A\|f\|_{L^p}$ for all $f\in L^p(\mathbb{R}^n)$ where $A\coloneqq\sqrt[p]{CM^{p+1}}$, so we are done. □

1.2 Introduction to Nets

Some operators are defined by evaluating limits over collections of sets. For the case of some countable collections, one may define the limit by using sequential convergence. For the case of nicely ordered shrinking sets, although not necessarily countable, one may simply adopt the ε-δ definition of convergence since the sets are indexed by real numbers $r>0$. However, there are many collections of sets that may not be indexed as nicely, so we will need a more general notion of sequential convergence. To that end, we begin by introducing **directed sets**, of which can be interpreted as the analog of $\mathbb{N}$ in the context of sequences.

Definition 1.16 (Directed Set)**.** *A set A endowed with a binary relation $\lesssim$ is a **directed set** iff*

1. *$\lesssim$ is reflexive, i.e. for all $\alpha\in A$, $\alpha\lesssim\alpha$,*
2. *$\lesssim$ is transitive, i.e. for all $\alpha,\beta,\gamma\in A$, if $\alpha\lesssim\beta$ and $\beta\lesssim\gamma$, then $\alpha\lesssim\gamma$, and*
3. *for all $\alpha,\beta\in A$, there exists $\gamma\in A$ such that $\alpha\lesssim\gamma$ and $\beta\lesssim\gamma$.*

We shall also write $\alpha\gtrsim\beta$ to mean $\beta\lesssim\alpha$.

Much like how $\mathbb{N}$ defines sequences by mapping the naturals to some set X, we can use directed sets to define **nets**, a more general notion of sequences, by mapping elements of directed sets to elements of X. Formally,

Definition 1.17 (Nets)**.** *Let A be a directed set and let X be a set. A* ***net*** *in X is a map $A \to X, \alpha \mapsto x_\alpha$, and it is notated as $\langle x_\alpha \rangle_{\alpha \in A}$.*

One should note that $\mathbb{N}$ itself is a directed set under the usual ordering, so a sequence is a type of net (obviously, not all nets are sequences). Now, if X is a topological space, we can thus define a more general notion of convergence using nets.

Definition 1.18 (Convergence of Nets)**.** *Let X be a topological space, and let A be a directed set. Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X. Then,*

1. *$\langle x_\alpha \rangle_{\alpha \in A}$ is* ***eventually*** *in some $E \subseteq X$ iff there exists $\alpha_0 \in A$ such that $x_\alpha \in E$ for all $\alpha \gtrsim \alpha_0$, and*
2. *$\langle x_\alpha \rangle_{\alpha \in A}$* ***converges*** *to some $x \in X$ iff for every open $U \ni x$, $\langle x_\alpha \rangle_{\alpha \in A}$ is eventually in U.*

We can also define the $\limsup$ and $\liminf$ of nets in $\mathbb{R}$ in a similar fashion to sequences.

Definition 1.19 ($\limsup$ and $\liminf$ of Nets)**.** *Let A be a directed set, and let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in $\mathbb{R}$. Then,*

$$\liminf_{\alpha \in A} x_\alpha \coloneqq \sup_{\alpha \in A} \inf_{\beta \gtrsim \alpha} x_\beta, \text{ and}$$
$$\limsup_{\alpha \in A} x_\alpha \coloneqq \inf_{\alpha \in A} \sup_{\beta \gtrsim \alpha} x_\beta.$$

Remark 1.20.

1. *Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in $\mathbb{R}$, and let $x \in \mathbb{R}$. Then, $\langle x_\alpha \rangle_{\alpha \in A}$ converges to x iff $\liminf_{\alpha \in A} x_\alpha = \limsup_{\alpha \in A} x_\alpha$.*
2. *Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in a metric space (X, d), and let $x \in X$. Then, $\langle x_\alpha \rangle_{\alpha \in A}$ converges to x iff $\limsup_{\alpha \in A} d(x_\alpha, x) = 0$.*

1.3 The Generalized Lebesgue Differentiation Theorem

Recall that the diameter of a set E in a metric space (X, d) is defined as $\operatorname{diam}(E) \coloneqq \sup_{x,y \in E} d(x, y)$, i.e. the supremum of distances between two points in E. We shall use diameters to formalize the notion of **shrinking sets** since differentiation in measure involves taking averages of a function over regions decreasing in size.

Definition 1.21 (Shrinking Sets)**.** *A collection of* ***shrinking sets*** *is a collection of measurable subsets $\mathcal{C}$ of $\mathbb{R}^n$ such that*

1. *$m(E) \in (0, \infty)$ for all $E \in \mathcal{C}$,*
2. *$E \subseteq B_{k \cdot \operatorname{diam}(E)}(\mathbf{0})$ for all $E \in \mathcal{C}$ and some $k > 0$, and*
3. *for all $r > 0$, there exists $E \in \mathcal{C}$ such that $\operatorname{diam}(E) \leq r$.*

A collection of shrinking sets is a directed set under $\lesssim$ where $E \lesssim F$ iff $\operatorname{diam}(E) \geq \operatorname{diam}(F)$ for all $E, F \subseteq \mathbb{R}^n$. Hence, if $\langle x_E \rangle_{E \in \mathcal{C}}$ is a net in a topological space X and converges to some $x \in X$, we write

$$\lim_{\operatorname{diam}(E) \to 0} x_E = x,$$

and if $\langle x_E \rangle_{E \in \mathcal{C}}$ is a net in $\mathbb{R}$, we write

$$\liminf_{\operatorname{diam}(E) \to 0} x_E := \liminf_{E \in \mathcal{C}} x_E, \text{ and}$$
$$\limsup_{\operatorname{diam}(E) \to 0} x_E := \limsup_{E \in \mathcal{C}} x_E.$$

Using the language of nets, we can generalize the notion of nicely ordered shrinking sets so that indexing by $r > 0$ is no longer necessary.

Definition 1.22 (Nicely Shrinking Sets)**.** *A collection of* ***nicely shrinking sets*** *is a collection of measurable subsets $\mathcal{C}$ of $\mathbb{R}^n$ such that*

1. *$\mathcal{C}$ is a collection of shrinking sets, and*
2. *there exists $\alpha > 0$ such that $m(E) > \alpha \cdot \operatorname{diam}(E)^n$ for all $E \in \mathcal{C}$.*

Using this definition of nicely shrinking sets, we will now prove the **Generalized Lebesgue Differentiation Theorem**. The proof is similar to Folland's proof of the Lebesgue Differentiation Theorem [1].

Theorem 1.23 (Generalized Lebesgue Differentiation Theorem)**.** *If $f \in L^1_{loc}(\mathbb{R}^n)$ and $\mathcal{C}$ is a collection of nicely shrinking sets, then*

$$\lim_{\operatorname{diam}(E) \to 0} \frac{1}{m(E)} \int_E f(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} = f(\mathbf{x})$$

for almost every $\mathbf{x} \in \mathbb{R}^n$.

Proof. Note that if $f \in L^1_{loc}(\mathbb{R}^n)$, then $f \cdot \chi_{\overline{B_N(\mathbf{0})}} \in L^1(\mathbb{R}^n)$ for all $N \in \mathbb{N}$. If we can show that the Generalized Lebesgue Differentiation Theorem holds for $L^1(\mathbb{R}^n)$ functions, then

$$\bigcap_{N=1}^{\infty} \left\{ \mathbf{x} \in \mathbb{R}^n : \lim_{\operatorname{diam}(E) \to 0} \frac{1}{m(E)} \int_E (f \cdot \chi_{\overline{B_N(\mathbf{0})}})(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} = (f \cdot \chi_{\overline{B_N(\mathbf{0})}})(\mathbf{x}) \right\}$$

has full measure since it is an intersection of full measure sets. Hence, for every $\mathbf{x}$ in the above set, i.e. almost everywhere, there exists $N \in \mathbb{N}$ such that $\mathbf{x} \in B_N(\mathbf{0})$ and

$$\begin{aligned} \lim_{\operatorname{diam}(E) \to 0} \frac{1}{m(E)} \int_E f(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} &= \lim_{\substack{\operatorname{diam}(E) \to 0 \\ \operatorname{diam}(E) < 1}} \int_E (f \cdot \chi_{\overline{B_{N+1}(\mathbf{0})}})(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \\ &\quad \text{(since } \mathbf{x} - \mathbf{y} \in B_{N+1}(\mathbf{0}) \text{ for all } \mathbf{y} \in E \subseteq B_{\operatorname{diam}(E)}(\mathbf{0}) \subseteq B_1(\mathbf{0}) \text{ with } \operatorname{diam}(E) < 1) \\ &= (f \cdot \chi_{\overline{B_{N+1}(\mathbf{0})}})(\mathbf{x}) = f(\mathbf{x}). \end{aligned}$$

In that regard, it suffices to show that the Generalized Lebesgue Differentiation Theorem holds for $f \in L^1(\mathbb{R}^n)$.

Let $\mathcal{M}_{\mathcal{B}}$ be the Hardy-Littlewood maximal operator. Recall from Proposition 1.14 that $\mathcal{M}_{\mathcal{B}}$ is weak type $(1,1)$. Now, let $f \in L^1(\mathbb{R}^n)$ and let $\varepsilon > 0$. Since $f \in L^1(\mathbb{R}^n)$, there exists a continuous $g \in L^1(\mathbb{R}^n)$ such that $\|g - f\|_{L^1} < \varepsilon$ by Remark 1.1. Note that

$$\begin{aligned}\limsup_{\operatorname{diam}(B)\to 0} (\mathcal{A}_{m_B}(g - g(\mathbf{x})))(\mathbf{x}) &\leq \limsup_{\operatorname{diam}(B)\to 0} \frac{1}{m(B)} \int_B \sup_{\mathbf{y}\in B} |g(\mathbf{x}-\mathbf{y}) - g(\mathbf{x})| \, d\mathbf{y} \\ &= \limsup_{\operatorname{diam}(B)\to 0} \sup_{\mathbf{y}\in B} |g(\mathbf{x}-\mathbf{y}) - g(\mathbf{x})| = 0 \quad \text{(by the continuity of } g)\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$ where B ranges in $\mathcal{B}$. Hence,

$$\begin{aligned}\limsup_{\operatorname{diam}(B)\to 0} (\mathcal{A}_{m_B}(f - f(\mathbf{x})))(\mathbf{x}) &\leq \limsup_{\operatorname{diam}(B)\to 0} [(\mathcal{A}_{m_B}|f-g|)(\mathbf{x}) + (\mathcal{A}_{m_B}|g - g(\mathbf{x})|)(\mathbf{x}) + |g(\mathbf{x}) - f(\mathbf{x})|] \\ &\qquad \text{(note that } \mathcal{A}_{m_B} c = c \text{ for all constants } c) \\ &\leq (\mathcal{M}_{\mathcal{B}}|f-g|)(\mathbf{x}) + |f-g|(\mathbf{x})\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$ where B ranges in $\mathcal{B}$. Let $F(\mathbf{x}) \coloneqq \limsup_{\operatorname{diam}(B)\to 0} (\mathcal{A}_{m_B}|f - f(\mathbf{x})|)(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ (we will not need to assume it is measurable). Then,

$$\begin{aligned}F^{-1}((a,\infty]) &\subseteq (\mathcal{M}_{\mathcal{B}}|f-g| + |f-g|)^{-1}((a,\infty]) \\ &\subseteq (\mathcal{M}_{\mathcal{B}}|f-g|)^{-1}\left(\left(\frac{a}{2},\infty\right]\right) \cup |f-g|^{-1}\left(\left(\frac{a}{2},\infty\right]\right)\end{aligned}$$

for all $a > 0$. Note that

$$m\left(|f-g|^{-1}\left(\left(\frac{a}{2},\infty\right]\right)\right) = \frac{2}{a}\int_{|f-g|^{-1}((\frac{a}{2},\infty])} \frac{a}{2} \, d\mathbf{x} \leq \frac{2}{a}\int_{|f-g|^{-1}((\frac{a}{2},\infty])} |f-g|(\mathbf{x}) \, d\mathbf{x} < \frac{2\varepsilon}{a},$$

and there exists $A > 0$ such that $m((\mathcal{M}_{\mathcal{B}}|f-g|)^{-1}((\frac{a}{2},\infty])) \leq \frac{2A}{a}\|f-g\|_{L^1} < \frac{2A\varepsilon}{a}$ for all $a > 0$ since $\mathcal{M}_{\mathcal{B}}$ is weak type $(1,1)$. By the arbitrariness of ε, we have that $F^{-1}((a,\infty])$ is contained in a set of measure less than $\frac{2A\varepsilon}{a} + \frac{2\varepsilon}{a}$ for all $\varepsilon > 0$ for each $a > 0$. In that regard, $F^{-1}((a,\infty])$ must be contained in a null set for each $a > 0$, so it is measurable (by completeness of the Lebesgue measure) and it has zero measure. Hence, $m(F^{-1}((0,\infty])) = m(F^{-1}(\bigcup_{k=1}^{\infty}(\frac{1}{k},\infty])) = m(\bigcup_{k=1}^{\infty} F^{-1}(\frac{1}{k},\infty])) = 0$ since unions of null sets are null sets. This means that

$$\limsup_{\operatorname{diam}(B)\to 0} (\mathcal{A}_{m_B}(f - f(\mathbf{x})))(\mathbf{x}) = 0$$

for almost every $\mathbf{x} \in \mathbb{R}^n$ where B ranges in $\mathcal{B}$. Given that $\mathcal{C}$ is a collection of nicely shrinking sets, we have that

$$\begin{aligned}\limsup_{\operatorname{diam}(E)\to 0} \left| \frac{1}{m(E)} \int_E f(\mathbf{x}-\mathbf{y}) \, d\mathbf{y} - f(\mathbf{x}) \right| &\leq \limsup_{\operatorname{diam}(E)\to 0} (\mathcal{A}_{m_E}(f - f(\mathbf{x})))(\mathbf{x}) \\ &\leq \frac{m(B_1(\mathbf{0}))}{\alpha} \limsup_{\operatorname{diam}(E)\to 0} (\mathcal{A}_{m_{B_{k\cdot\operatorname{diam}(E)}(\mathbf{0})}}(f - f(\mathbf{x})))(\mathbf{x}) \\ &\qquad \text{(by the nicely shrinking properties)}\end{aligned}$$

$$\leq \frac{m(B_1(\mathbf{0}))}{\alpha} \limsup_{\operatorname{diam}(B)\to 0} (\mathcal{A}_{m_B}(f - f(\mathbf{x})))(\mathbf{x}) = 0$$

(since the diameters of shrinking sets can get arbitrarily small)

for almost every $\mathbf{x} \in \mathbb{R}^n$ where B ranges in $\mathcal{B}$ and E ranges in $\mathcal{C}$. By the arbitrariness of f, we are done. $\square$

At this point, we will begin introducing other topics unrelated to the Lebesgue Differentiation Theorem that Kakeya sets are also applicable to. If you would like to see more immediate results about the Lebesgue Differentiation Theorem, you are welcome to skip to the next section.

1.4 Introduction to the Fourier Transform

The **Fourier transform** is an important linear operator that is used often in harmonic analysis. It is also frequently used in some areas of partial differential equations. In this thesis, we will specifically look at its interactions with **multiplier operators**. Most definitions and theorems in this subsection can be found in [1] and [2].

Notation 1.24. *We write*

$$\begin{aligned} \widehat{f} : \mathbb{R}^n &\to \mathbb{C} \\ \xi &\mapsto \int f(\mathbf{x}) e^{-2\pi i \xi \cdot \mathbf{x}}\, d\mathbf{x}, \end{aligned}$$

and similarly

$$\begin{aligned} f^{\vee} : \mathbb{R}^n &\to \mathbb{C} \\ \mathbf{x} &\mapsto \widehat{f}(-\mathbf{x}) = \int f(\xi) e^{2\pi i \mathbf{x} \cdot \xi}\, d\xi \end{aligned}$$

for some $f \in L^0(\mathbb{R}^n)$ if the integrals exist for all $\xi \in \mathbb{R}^n$.

The notations above, in essence, are the Fourier transform and its inverse respectively (although we shall later define it specifically for $L^1(\mathbb{R}^n)$ functions). We will not be able to draw out any useful properties by taking the above transform over functions without suitable restriction of the space of functions. With that being said, we will find that the above transform behaves much more nicely when acting on $L^1(\mathbb{R}^n)$ functions as hinted earlier. Before we list these properties out, we shall first introduce some new definitions.

Definition 1.25 (Translation, Dilation and Rotation)**.** *We shall later derive some identities that involve translations, dilations and rotations of $L^1(\mathbb{R}^n)$ functions and the Fourier transform. We thus define*

1. $(\tau_{\mathbf{y}} f)(\mathbf{x}) \coloneqq f(\mathbf{x} - \mathbf{y})$*, i.e. the* ***translation*** *of f by $\mathbf{y}$, and*
2. $(\delta_t f)(\mathbf{x}) \coloneqq f(t\mathbf{x})$*, i.e. the* ***dilation*** *of f by t*

3. $(\rho_O f)(\mathbf{x}) \coloneqq f(O\mathbf{x})$*, i.e. the* ***rotation*** *of* f *by* O*.*

for all $f : \mathbb{R}^n \to \mathbb{C}$*,* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*,* $t > 0$ *and orthogonal transformations* O*. Note that we previously used* τ *as a translation function on* $\mathbb{R}^n$*, but we shall now see it as a transform instead.*

Notation 1.26. *We will also often write*

$$\phi_t \coloneqq t^{-n}\delta_{t^{-1}}\phi$$

for $\phi : \mathbb{R}^n \to \mathbb{C}$ *and* $t > 0$*. One can easily check that scaling in this manner preserves the value of the integral if* ϕ *is also integrable.*

We shall now scope out some useful properties of $\widehat{f}$ for $f \in L^1(\mathbb{R}^n)$. The following lemma will help us to understand some fundamental properties of $\widehat{f}$, and give us a way to pass derivatives through integrals.

Lemma 1.27. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space, let* $f \in L^0(X \times \mathbb{R}^n)$ *be such that* $f(\cdot, \mathbf{y}) \in L^1(\mu)$ *for each* $\mathbf{y} = (y_1, \ldots, y_n) \in U$ *where* $U \subseteq \mathbb{R}^n$ *is open, and let*

$$\begin{aligned} F : U &\to \mathbb{C} \\ \mathbf{y} &\mapsto \int f(x, \mathbf{y})\ d\mu(x). \end{aligned}$$

1. *Suppose* $f(x, \cdot)$ *is continuous on* U *for all* $x \in X$ *and that there exists* $g \in L^1(\mu)$ *such that* $|f(x, \mathbf{y})| \leq g(x)$ *for all* $(x, \mathbf{y}) \in X \times U$*. Then* F *is continuous.*
2. *Suppose* $\frac{\partial f}{\partial y_j}$ *exists for some* $1 \leq j \leq n$ *and that there exists* $g \in L^1(\mu)$ *such that* $|\frac{\partial f}{\partial y_j}(x, \mathbf{y})| \leq g(x)$ *for all* $(x, \mathbf{y}) \in X \times U$*. Then,* $\frac{\partial F}{\partial y_j}$ *exists and* $\frac{\partial F}{\partial y_j}(\mathbf{y}) = \int \frac{\partial f}{\partial y_j}(x, \mathbf{y})\ d\mu(x)$ *for all* $\mathbf{y} \in U$*.*

Proof.

1. Let $\{\mathbf{y}_k\}_{k=1}^{\infty} \subseteq U$ be a sequence that converges to some $\mathbf{y} \in U$. Then,

$$\begin{aligned} \lim_{k\to\infty} F(\mathbf{y}_k) &= \int \lim_{k\to\infty} f(x, \mathbf{y}_k)\ d\mu(x) && \text{(by the Dominated Convergence Theorem)} \\ &= \int f(x, \mathbf{y})\ d\mu(x) = F(\mathbf{y}). && \text{(by continuity of } f(x,\cdot) \text{ for all } x \in X) \end{aligned}$$

   By the arbitrariness of $\{\mathbf{y}_k\}_{k=1}^{\infty}$, we indeed have that F is continuous.

2. Let $\{y_j^k\}_{k=1}^{\infty} \subseteq p_j(U)$ be a sequence that converges to some $y_j^0 \in p_j(U)$ where p_j is the projection onto the j-th coordinate. Given that p_j is an open map, there exists $\varepsilon > 0$ small enough and $N \in \mathbb{N}$ such that $y_j^k \in \overline{B_\varepsilon(y_j^0)} \subseteq p_j(U)$ whenever $k \geq N$. On that note, we have that

$$\left| \frac{f(x, y_1, \ldots, y_j^k, \ldots y_n) - f(x, y_1, \ldots, y_j^0, \ldots, y_n)}{y_j^k - y_j^0} \right| \leq \sup_{y_j \in \overline{B_\varepsilon(y_j^0)}} \left| \frac{\partial f}{\partial y_j}(x, \mathbf{y}) \right| \leq g(x)$$

by the mean value theorem whenever $k \geq N$. Hence,

$$\begin{aligned}\frac{\partial F}{\partial y_j}(y_1, \ldots, y_j^0, \ldots, y_n) &= \lim_{k\to\infty} \frac{F(x, y_1, \ldots, y_j^k, \ldots y_n) - F(x, y_1, \ldots, y_j^0, \ldots, y_n)}{y_j^k - y_j^0} \\ &= \int \lim_{k\to\infty} \frac{f(x, y_1, \ldots, y_j^k, \ldots y_n) - f(x, y_1, \ldots, y_j^0, \ldots, y_n)}{y_j^k - y_j^0}\, d\mu(x) \\ &\qquad\text{(by the Dominated Convergence Theorem)} \\ &= \int \frac{\partial f}{\partial y_j}(x, y_1, \ldots, y_j^0, \ldots, y_n)\, d\mu(x).\end{aligned}$$

By the arbitrariness of $\{y_j^k\}_{k=1}^{\infty}$, we see that the derivative commutes with the integral.

□

Corollary 1.28. *Let $f \in L^1(\mathbb{R}^n)$. Then, $\widehat{f} \in C_b(\mathbb{R}^n)$ where $C_b(\mathbb{R}^n)$ is the space of bounded and continuous functions on $\mathbb{R}^n$. Furthermore, $\|\widehat{f}\|_u \leq \|f\|_{L^1}$ where $\|\cdot\|_u$ is the uniform norm.*

Proof. It is clear that $\xi \mapsto f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}}$ is continuous for all $\mathbf{x} \in \mathbb{R}^n$, and that $|f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}}| = |f(\mathbf{x})| \in L^1(\mathbb{R}^n)$ for all $(\mathbf{x}, \xi) \in \mathbb{R}^{2n}$. Hence, $\widehat{f}$ is continuous on $\mathbb{R}^n$ by Lemma 1.27.

Finally, $\|\widehat{f}\|_u := \sup_{\xi\in\mathbb{R}^n} |\widehat{f}(\xi)| \leq \int |f(\mathbf{x})|\, d\mathbf{x} = \|f\|_{L^1} < \infty$ so $\widehat{f}$ is also bounded. □

We will later prove a slightly stronger result than Corollary 1.28. For now, we have enough tools to derive some basic properties of taking the transform in Notation 1.24 on $L^1(\mathbb{R}^n)$ functions.

Lemma 1.29. *Let $f, g \in L^1(\mathbb{R}^n)$, let p_j be the projection onto the j-th coordinate for $1 \leq j \leq n$, and let id be the identity map on $\mathbb{R}^n$. Then,*

1. $\widehat{\tau_{\mathbf{y}} f} = e^{-2\pi i\mathbf{y}\cdot\mathrm{id}}\widehat{f}$ *and* $\tau_{\mathbf{y}}\widehat{f} = (\widehat{e^{2\pi i\mathbf{y}\cdot\mathrm{id}}f})$ *for all* $\mathbf{y} \in \mathbb{R}^n$.
2. $\widehat{\delta_t f} = (\widehat{f})_t$ *for all* $t > 0$ *where* $(\widehat{f})_t$ *is as in Notation 1.26.*
3. $\widehat{\rho_O f} = \rho_O \widehat{f}$ *for all orthogonal transformations* O.
4. $\widehat{f * g} = \widehat{f}\widehat{g}$.
5. $\int \widehat{f} g\, dm = \int f\widehat{g}\, dm$.
6. *If* $p_j f \in L^1(\mathbb{R}^n)$ *for some* $1 \leq j \leq n$, *then* $\frac{\partial \widehat{f}}{\partial \xi_j}$ *exists and* $\frac{\partial \widehat{f}}{\partial \xi_j} = -2\pi i\widehat{p_j f}$.
7. *If* $f \in C_0(\mathbb{R}^n)$ *and* $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$ *for some* $1 \leq j \leq n$, *then* $\widehat{\frac{\partial f}{\partial x_j}} = 2\pi i p_j \widehat{f}$.

Proof.

1. Observe that

$$(\widehat{\tau_{\mathbf{y}} f})(\xi) = \int f(\mathbf{x} - \mathbf{y})e^{-2\pi i\xi\cdot\mathbf{x}}\, d\mathbf{x}$$

$$= \int f(\mathbf{x}) e^{-2\pi i \xi \cdot (\mathbf{x}+\mathbf{y})}\, d\mathbf{x} \qquad \text{(by translation invariance of } m\text{)}$$

$$= e^{-2\pi i \xi \cdot \mathbf{y}} \widehat{f}(\xi),$$

and that

$$\begin{aligned}(\tau_{\mathbf{y}} \widehat{f})(\xi) &= \int f(\mathbf{x}) e^{-2\pi i (\xi - \mathbf{y}) \cdot \mathbf{x}}\, d\mathbf{x} \\ &= \int f(\mathbf{x}) e^{2\pi i \mathbf{y} \cdot \mathbf{x}} e^{-2\pi i \xi \cdot \mathbf{x}}\, d\mathbf{x} \\ &= (\widehat{e^{2\pi i \mathbf{y} \cdot \mathrm{id}} f})(\xi)\end{aligned}$$

for all $\mathbf{y}, \xi \in \mathbb{R}^n$.

2. Observe that

$$\begin{aligned}(\widehat{\delta_t f})(\xi) &= \int f(t\mathbf{x}) e^{-2\pi i \xi \cdot \mathbf{x}}\, d\mathbf{x} \\ &= t^{-n} \int f(\mathbf{u}) e^{-2\pi i \frac{\xi}{t} \cdot \mathbf{u}}\, d\mathbf{u} && \text{(by scaling with } \mathbf{x} = \tfrac{\mathbf{u}}{t}\text{)} \\ &= (\widehat{f})_t(\xi)\end{aligned}$$

for all $\xi \in \mathbb{R}^n$ and $t > 0$.

3. Observe that for all orthogonal transformations O,

$$\begin{aligned}(\widehat{\rho_O f})(\xi) &= \int f(O\mathbf{x}) e^{-2\pi i \xi \cdot \mathbf{x}}\, d\mathbf{x} \\ &= \int f(\mathbf{u}) e^{-2\pi i \xi \cdot O^T \mathbf{u}}\, d\mathbf{u} && \text{(by letting } \mathbf{u} = O\mathbf{x}\text{)} \\ &= \int f(\mathbf{u}) e^{-2\pi i O\xi \cdot \mathbf{u}}\, d\mathbf{u} && \text{(since } O \text{ preserves dot products)} \\ &= \rho_O \widehat{f}(\xi)\end{aligned}$$

for all $\xi \in \mathbb{R}^n$.

4. Observe that

$$\begin{aligned}(\widehat{f * g})(\xi) &= \int \left(\int f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})\, d\mathbf{y} \right) e^{-2\pi i \xi \cdot \mathbf{x}}\, d\mathbf{x} \\ &= \int \left(\int f(\mathbf{x} - \mathbf{y}) e^{-2\pi i \xi \cdot (\mathbf{x} - \mathbf{y})}\, d\mathbf{x} \right) g(\mathbf{y}) e^{-2\pi i \xi \cdot \mathbf{y}}\, d\mathbf{y} && \text{(by Fubini's Theorem)} \\ &= \int \widehat{f}(\xi) g(\mathbf{y}) e^{-2\pi i \xi \cdot \mathbf{y}}\, d\mathbf{y} = \widehat{f}(\xi) \widehat{g}(\xi)\end{aligned}$$

for all $\xi \in \mathbb{R}^n$.

5. Observe that

$$\begin{aligned}\int \widehat{f}(\xi) g(\xi)\, d\xi &= \int \left(\int f(\mathbf{x}) e^{-2\pi i \xi \cdot \mathbf{x}}\, d\mathbf{x} \right) g(\xi)\, d\xi \\ &= \int f(\mathbf{x}) \left(\int g(\xi) e^{-2\pi i \xi \cdot \mathbf{x}}\, d\xi \right) d\mathbf{x} && \text{(by Fubini's Theorem)} \\ &= \int f(\mathbf{x}) \widehat{g}(\mathbf{x})\, d\mathbf{x}.\end{aligned}$$

6. Note that $|\frac{\partial}{\partial \xi_j}(f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}})| = |-2\pi i x_j f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}}| = 2\pi|x_j f(\mathbf{x})|$ for all $(\mathbf{x},\xi) \in \mathbb{R}^{2n}$ and $2\pi|p_j f| \in L^1(\mathbb{R}^n)$ by the assumption. Hence, by Lemma 1.27,

$$\begin{aligned}\frac{\partial \widehat{f}}{\partial \xi_j}(\xi) &= \int \frac{\partial}{\partial \xi_j}\left(f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}}\right)\,d\mathbf{x}\\ &= -2\pi i\int x_j f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}}\,d\mathbf{x}\\ &= -2\pi i\widehat{p_j f}(\xi)\end{aligned}$$

for all $\xi \in \mathbb{R}^n$.

7. Observe that

$$\begin{aligned}\widehat{\frac{\partial f}{\partial x_j}}(\xi) &= \iint \frac{\partial f}{\partial x_j}e^{-2\pi i\xi\cdot\mathbf{x}}\,dx_j\,d(x_k)_{k\neq j} && \text{(by Fubini's Theorem)}\\ &= \int\left(\left[f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}}\right]_{x_j\to-\infty}^{x_j\to\infty} + 2\pi i\xi_j\int f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}}\,dx_j\right)\,d(x_k)_{k\neq j}\\ & && \text{(by integration by parts)}\\ &= 2\pi i\xi_j\iint f(\mathbf{x})e^{-2\pi i\xi\cdot\mathbf{x}}\,dx_j\,d(x_k)_{k\neq j} && \text{(since } f\in C_0(\mathbb{R}^n))\\ &= 2\pi i\xi_j\widehat{f}(\xi) && \text{(by Fubini's Theorem)}\end{aligned}$$

for all $\xi \in \mathbb{R}^n$.

□

The last part of Lemma 1.29 gives us a clue as to how we can tighten the space of $\widehat{f}$ for $f \in L^1(\mathbb{R}^n)$. This is indeed captured by the Riemann-Lebesgue Lemma, of which is stated and proved below.

Lemma 1.30 (Riemann-Lebesgue Lemma). *Let $f \in L^1(\mathbb{R}^n)$. Then, $\widehat{f} \in C_0(\mathbb{R}^n)$.*

Proof. There exist $f_k \in C_c^\infty(\mathbb{R}^n)$ with $\|f_k - f\|_{L^1} < \frac{1}{k}$ for all $k \in \mathbb{N}$ by Remark 1.1. Observe that

$$\begin{aligned}|\widehat{f_k}(\xi)| = \frac{1}{|\xi|}|\xi\widehat{f_k}(\xi)| &\leq \frac{1}{|\xi|}\sum_{j=1}^n|\xi_j\widehat{f_k}(\xi)| && \text{(since } |\xi|\leq \textstyle\sum_{j=1}^n|\xi_j|)\\ &= \frac{1}{|\xi|}\sum_{j=1}^n\left|\frac{1}{2\pi i}\widehat{\frac{\partial f_k}{\partial x_j}}(\xi)\right| && \text{(by Lemma 1.29.7)}\\ &\leq \frac{1}{2\pi|\xi|}\sum_{j=1}^n\left\|\frac{\partial f_k}{\partial x_j}\right\|_{L^1} && \text{(by Corollary 1.28)}\end{aligned}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $k \in \mathbb{N}$. Hence, we indeed see that $\widehat{f_k} \in C_0(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ since we already know that $\widehat{f_k} \in C_b(\mathbb{R}^n)$. Note also from Corollary 1.28 that $\|\widehat{f_k} - \widehat{f}\|_u = \|\widehat{f_k - f}\|_u \leq \|f_k - f\|_{L^1}$ for all $k \in \mathbb{N}$, so by the squeeze theorem we have that $\widehat{f_k} \to \widehat{f}$ in the uniform norm. Since $C_0(\mathbb{R}^n)$ is closed under the uniform norm, we conclude that $\widehat{f} \in C_0(\mathbb{R}^n)$. □

The Riemann-Lebesgue Lemma thus defines the **Fourier transform** and its "inverse":

Definition 1.31 (Fourier Transform and Inverse)**.** *We define*

$$\begin{aligned} \mathcal{F} : L^1(\mathbb{R}^n) &\to C_0(\mathbb{R}^n) \\ f &\mapsto \widehat{f} \end{aligned}$$

to be the ***Fourier transform****. Similarly, we define*

$$\begin{aligned} \mathcal{F}^{-1} : L^1(\mathbb{R}^n) &\to C_0(\mathbb{R}^n) \\ f &\mapsto f^\vee \end{aligned}$$

to be the ***Inverse Fourier transform****. Note that the Inverse Fourier transform is not actually the inverse of the Fourier transform.*

More care is needed to check when the Inverse Fourier transform actually behaves like an inverse. Before that discussion, we shall first identify a function that remains fixed under the Fourier transform. This property will be useful for us later.

Lemma 1.32. *Let $\phi(\mathbf{x}) \coloneqq e^{-\pi|\mathbf{x}|^2}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then, $\widehat{\phi} = \phi$.*

Proof. Let p_j be the projection onto the j-th coordinate for $1 \le j \le n$. Clearly, $\phi, p_j\phi, \frac{\partial \phi}{\partial x_j} \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ for all $1 \le j \le n$. Note also that $\frac{1}{\phi(\mathbf{x})} = e^{\pi|\mathbf{x}|^2}$ for all $\mathbf{x} \in \mathbb{R}^n$. Hence,

$$\begin{aligned} \frac{\partial}{\partial \xi_j}\left(\frac{\widehat{\phi}(\xi)}{\phi(\xi)}\right) &= 2\pi\xi_j \frac{\widehat{\phi}(\xi)}{\phi(\xi)} + \frac{1}{\phi(\xi)}\frac{\partial \widehat{\phi}}{\partial \xi_j}(\xi) && \text{(by the product rule)} \\ &= 2\pi\xi_j \frac{\widehat{\phi}(\xi)}{\phi(\xi)} - \frac{2\pi i}{\phi(\xi)}\widehat{p_j\phi}(\xi) && \text{(by Lemma 1.29.6)} \\ &= 2\pi\xi_j \frac{\widehat{\phi}(\xi)}{\phi(\xi)} + \frac{i}{\phi(\xi)}\widehat{\frac{\partial \phi}{\partial x_j}}(\xi) && \text{(since } \tfrac{\partial \phi}{\partial x_j} = -2\pi a p_j \phi\text{)} \\ &= 2\pi\xi_j \frac{\widehat{\phi}(\xi)}{\phi(\xi)} - \frac{2\pi}{\phi(\xi)}\xi_j\widehat{\phi}(\xi) = 0 && \text{(by Lemma 1.29.7)} \end{aligned}$$

for all $\xi \in \mathbb{R}^n$ and for all $1 \le j \le n$. Since all partial derivatives of $\frac{1}{\phi}\widehat{\phi}$ are zero, it must be the case it is constant. In that regard,

$$\frac{1}{\phi}\widehat{\phi} = \frac{\widehat{\phi}(\mathbf{0})}{\phi(\mathbf{0})} = \widehat{\phi}(\mathbf{0}) = \int e^{-\pi|\mathbf{x}|^2}\, d\mathbf{x} = \prod_{j=1}^{n}\int e^{-\pi x_j^2}\, dx_j = \prod_{j=1}^{n} 1 = 1$$

by Fubini's Theorem. Rearranging the above gives our desired result, so we are done. □

The following lemma gives a useful way to approximate functions via convolutions with functions that progressively focuses their 'masses' at the origin. We shall see later that approximating in this manner will allow us to verify the invertibility of the Fourier transform under some nice conditions.

Lemma 1.33. *Let $p \in [1, \infty)$, and let $f \in L^p(\mathbb{R}^n)$. Let $\phi \in L^1(\mathbb{R}^n)$ be such that $\int \phi \, dm = c \in \mathbb{C}$. Then,*

$$\lim_{t \to 0} \|f * \phi_t - cf\|_{L^p} = 0.$$

PROOF. Observe that

$$\begin{aligned}
f * \phi_t(\mathbf{x}) - cf(\mathbf{x}) &= \int f(\mathbf{x} - \mathbf{y})\phi_t(\mathbf{y}) \, d\mathbf{y} - f(\mathbf{x}) \int \phi(\mathbf{y}) \, d\mathbf{y} \\
&= \int (f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x}))\phi_t(\mathbf{y}) \, d\mathbf{y} \qquad \text{(since } \textstyle\int \phi \, dm = \int \phi_t \, dm) \\
&= \int (f(\mathbf{x} - t\mathbf{u}) - f(\mathbf{x}))\phi(\mathbf{u}) \, d\mathbf{u} \\
&\qquad \text{(by scaling with } \mathbf{y} = t\mathbf{u}\text{; note that } t^n \text{ from scaling cancels out } t^{-n} \text{ from } \phi_t) \\
&= \int ((\tau_{t\mathbf{u}} f)(\mathbf{x}) - f(\mathbf{x}))\phi(\mathbf{u}) \, d\mathbf{u}
\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$. Hence, by Minkowski's inequality for integrals, we have that

$$\|f * \phi_t - cf\|_{L^p} \le \int \|\tau_{t\mathbf{u}} f - f\|_{L^p} |\phi(\mathbf{u})| \, d\mathbf{u}.$$

We will now show that $\|\tau_{t\mathbf{u}} f - f\|_{L^p}$ converges to 0 for any $\mathbf{u} \in \mathbb{R}^n$. Let $\varepsilon > 0$. Given that $f \in L^p(\mathbb{R}^n)$, there exists $g \in C_c(\mathbb{R}^n)$ such that $\|g - f\|_{L^p} < \frac{\varepsilon}{3}$ by Remark 1.1. Note then that if $B_r(\mathbf{0})$ contains the support of g (by boundedness of its support), then $B_{r+t|\mathbf{u}|}(\mathbf{0})$ also contains the support of $\tau_{t\mathbf{u}} g$ for all $t \ge 0$. Hence, $K := \overline{B_{r+|\mathbf{u}|}(\mathbf{0})} \supseteq \bigcup_{t \in [0,1]} B_{r+t|\mathbf{u}|}(\mathbf{0})$ contains the support of $\tau_{t\mathbf{u}} g - g$ whenever $t \in [0, 1]$. On that note,

$$\begin{aligned}
0 \le \|\tau_{t\mathbf{u}} g - g\|_{L^p}^p &= \int |\tau_{t\mathbf{u}} g - g|^p \, dm \\
&\le \|\tau_{t\mathbf{u}} g - g\|_u^p m(K)
\end{aligned}$$

for all $t \in [0, 1]$. Hence, by the squeeze theorem and uniform convergence of $\tau_{t\mathbf{u}} g$ to g, we have that $\lim_{t \to 0} \|\tau_{t\mathbf{u}} g - g\|_{L^p} = 0$. In that regard, there exists $t \in (0, 1]$ small enough such that $\|\tau_{t\mathbf{u}} g - g\|_{L^p} < \frac{\varepsilon}{3}$. Thus,

$$\|\tau_{t\mathbf{u}} f - f\|_{L^p} \le \|\tau_{t\mathbf{u}}(f - g)\|_{L^p} + \|\tau_{t\mathbf{u}} g - g\|_{L^p} + \|g - f\|_{L^p} < \varepsilon.$$

Therefore, by the arbitrariness of ε and $\mathbf{u}$, we indeed have that $\lim_{t \to 0} \|\tau_{t\mathbf{u}} f - f\|_{L^p} = 0$ for all $\mathbf{u} \in \mathbb{R}^n$.

Now, note that $\|\tau_{t\mathbf{u}} f - f\|_{L^p} |\phi(\mathbf{u})| \le 2\|f\|_{L^p} |\phi(\mathbf{u})|$ for all $\mathbf{u} \in \mathbb{R}^n$ and that $2\|f\|_{L^p} |\phi| \in L^1(\mathbb{R}^n)$. Therefore, by the Dominated Convergence Theorem,

$$\lim_{t \to 0} \|f * \phi_t - cf\|_{L^p} = \int \lim_{t \to 0} \|\tau_{t\mathbf{u}} f - f\|_{L^p} |\phi(\mathbf{u})| \, d\mathbf{u} = 0.$$

$\square$

We are now ready to invert the Fourier transform.

Theorem 1.34 (Fourier Inversion Theorem). *Let $f \in L^1(\mathbb{R}^n)$ be such that $\widehat{f} \in L^1(\mathbb{R}^n)$. Then, $f(\mathbf{x}) = (\widehat{f})^\vee(\mathbf{x})$ for almost every $\mathbf{x} \in \mathbb{R}^n$ and $(\widehat{f})^\vee = \widehat{f^\vee}$.*

Proof. Let ϕ be the Gaussian function as in Lemma 1.32. Note that $\int \phi\ dm = 1$ as noted in the end of the proof of that proposition. Let ϕ_t be as in Notation 1.26. Let $g_{\mathbf{x},t}(\xi) \coloneqq e^{2\pi i \xi\cdot\mathbf{x}}(\delta_t\phi)(\xi)$ for all $\mathbf{x}, \xi \in \mathbb{R}^n$ and $t > 0$. Then,

$$\begin{aligned}
\widehat{g_{\mathbf{x},t}}(\mathbf{y}) &= (\tau_{\mathbf{x}}\widehat{\delta_t\phi})(\mathbf{y}) && \text{(by Lemma 1.29.1)}\\
&= (\widehat{\delta_t\phi})(\mathbf{y}-\mathbf{x}) \\
&= (\widehat{\phi})_t(\mathbf{y}-\mathbf{x}) && \text{(by Lemma 1.29.2)}\\
&= \phi_t(\mathbf{y}-\mathbf{x}) = \phi_t(\mathbf{x}-\mathbf{y}) && \text{(by Lemma 1.32)}
\end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $t > 0$. Now, given that $\lim_{k\to\infty} \|f * \phi_{\frac{1}{k}} - f\|_{L^1}$ by Lemma 1.33, there exists a subsequence $\{f * \phi_{\frac{1}{k_j}}\}_{j=1}^\infty$ that converges to f almost everywhere. Note also that $|\widehat{f}(\xi)g_{\mathbf{x},t}(\xi)| = |\widehat{f}(\xi)e^{-\pi t^2|\xi|^2}| \le |\widehat{f}(\xi)| \in L^1(\mathbb{R}^n)$ for all $\mathbf{x}, \xi \in \mathbb{R}^n$ and $t > 0$ by our assumption. Hence, for almost every $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
f(\mathbf{x}) &= \lim_{j\to\infty} f * \phi_{\frac{1}{k_j}}(\mathbf{x}) \\
&= \lim_{j\to\infty} \int f(\mathbf{y})\widehat{g_{\mathbf{x},\frac{1}{k_j}}}(\mathbf{y})\ d\mathbf{y} \\
&= \lim_{j\to\infty} \int \widehat{f}(\xi) g_{\mathbf{x},\frac{1}{k_j}}(\xi)\ d\xi && \text{(by Lemma 1.29.5)}\\
&= \int \lim_{j\to\infty} \widehat{f}(\xi)e^{2\pi i\xi\cdot\mathbf{x}}e^{-\frac{\pi}{k_j^2}|\xi|^2}\ d\xi && \text{(by the Dominated Convergence Theorem)}\\
&= \int \widehat{f}(\xi)e^{2\pi i\xi\cdot\mathbf{x}}\ d\xi = (\widehat{f})^\vee(\mathbf{x}).
\end{aligned}$$

Since $f^\vee(\mathbf{x}) = \widehat{f}(-\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$, it follows that $(\widehat{f})^\vee = \widehat{f^\vee}$ so we are done. □

Remark 1.35. *Since L^p spaces are technically spaces of equivalence classes functions (so that they can be viewed as complete, normed vector spaces), we can see that $\mathcal{F}$ is an automorphism on $\{f \in L^1(\mathbb{R}^n) : \widehat{f} \in L^1(\mathbb{R}^n)\}$.*

Notation 1.36. *We write $L^1_{\mathcal{F}}(\mathbb{R}^n) \coloneqq \{f \in L^1(\mathbb{R}^n) : \widehat{f} \in L^1(\mathbb{R}^n)\}$, i.e. the space of integrable functions whose Fourier transforms are also integrable.*

We will often use $L^1_{\mathcal{F}}(\mathbb{R}^n)$ as an intermediate space to prove results. In particular, we will extend operators initially defined on $L^1_{\mathcal{F}}(\mathbb{R}^n)$ to operators on other L^p spaces. The following lemma makes this idea more concrete.

Lemma 1.37. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be complete, normed vector spaces. Let $\mathcal{V}$ be a dense subspace of $\mathcal{X}$, and let T be a bounded linear operator from $\mathcal{V}$ to $\mathcal{Y}$. Then, T extends uniquely to a bounded linear operator $\widetilde{T}$ from $\mathcal{X}$ to $\mathcal{Y}$ with the same operator norm (we will generally use $\widetilde{\cdot}$ to denote an extension of an operator unless otherwise stated).*

Furthermore, if T is a linear isometry on $\mathcal{V}$ and $T(\mathcal{V})$ is dense in $\mathcal{Y}$, then $\widetilde{T}$ is an isometric isomorphism.

Proof. Observe that for any Cauchy sequence $\{x_j\}_{j=1}^\infty \subseteq \mathcal{V}$, we have that $\|Tx_j - Tx_k\|_\mathcal{Y} \leq c\|x_j - x_k\|_\mathcal{X}$ for all $j, k \in \mathbb{N}$ for some $c > 0$ by boundedness, which means that $\{Tx_j\}_{j=1}^\infty$ is also a Cauchy sequence in $\mathcal{Y}$. By the completeness of $\mathcal{Y}$, we define $\widetilde{T}x$ to be the limit of Tx_j for any Cauchy sequence $\{x_j\}_{j=1}^\infty \subseteq \mathcal{V}$ converging to $x \in \mathcal{X}$ (it is clear that this extension is well defined). Furthermore, $\|\widetilde{T}x\|_\mathcal{Y} \leq c\|x\|_\mathcal{X}$ for all $x \in \mathcal{X}$ iff $\|Tv\|_\mathcal{Y} \leq c\|v\|_\mathcal{X}$ for all $v \in \mathcal{V}$ by density, so the operator norm remains the same. Now, suppose T' was another extension of T. Then, $T'x - \widetilde{T}x = \lim_{j\to\infty} T'x_j - \lim_{j\to\infty} \widetilde{T}x_j = \lim_{j\to\infty} Tx_j - Tx_j = 0$ for all Cauchy sequences $\{x_j\}_{j=1}^\infty \subseteq \mathcal{V}$ converging to $x \in \mathcal{X}$. Hence, the extension is indeed unique.

Suppose T is linear isometry on $\mathcal{V}$, and that $T(\mathcal{V})$ is dense in $\mathcal{Y}$. Note that T, T^{-1} are isometric isomorphisms between $\mathcal{V}$ and $T(\mathcal{V})$. Now, T extends (like in the previous paragraph) as an isometry since $\|\widetilde{T}x\|_\mathcal{Y} = \|\lim_{j\to\infty} Tx_j\|_\mathcal{Y} = \lim_{j\to\infty}\|Tx_j\|_\mathcal{Y} = \lim_{j\to\infty}\|x_j\|_\mathcal{X} = \|x\|_\mathcal{X}$ for all Cauchy sequences $\{x_j\}_{j=1}^\infty \subseteq \mathcal{V}$ converging to $x \in \mathcal{X}$ by the continuity of norms. The same applies to $T^{-1} : T(\mathcal{V}) \to \mathcal{X}$. Furthermore, $\widetilde{T^{-1}}\widetilde{T}x = \widetilde{T^{-1}}\lim_{j\to\infty} Tx_j = \lim_{j\to\infty} T^{-1}Tx_j = \lim_{j\to\infty} x_j = x$ and $\widetilde{T}\widetilde{T^{-1}}y = \lim_{j\to\infty} TT^{-1}y_j = \lim_{j\to\infty} y_j = y$ for all Cauchy sequences $\{x_j\}_{j=1}^\infty \subseteq \mathcal{V}$ and $\{y_j\}_{j=1}^\infty \subseteq T(\mathcal{V})$ converging to $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ respectively, so we are done. $\square$

As Lemma 1.37 suggests, we will need $L^1_\mathcal{F}(\mathbb{R}^n)$ to be dense in $L^p(\mathbb{R}^n)$ if we want to be able to extend any operators defined on $L^1_\mathcal{F}(\mathbb{R}^n)$.

Lemma 1.38. *$L^1_\mathcal{F}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for all $p \in [1,\infty)$ with respect to the L^p norm for all $p \in [1,\infty)$.*

Proof. Let $f \in L^1_\mathcal{F}(\mathbb{R}^n)$. Note that $C_c^\infty(\mathbb{R}^n)$ is closed under multiplication by polynomials and differentiation, and that $C_c^\infty(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$. Hence, for any $\xi \in \mathbb{R}^n \setminus B_1(\mathbf{0})$,

$$\begin{aligned}
|\widehat{f}(\xi)| &= \frac{1}{(1+|\xi|)^{n+1}}(1+|\xi|)^{n+1}|\widehat{f}(\xi)| \\
&\leq \left(\frac{2}{1+|\xi|}\right)^{n+1} |\xi|^{n+1}|\widehat{f}(\xi)| && (\text{since } |\xi| \geq 1) \\
&\leq \left(\frac{2}{1+|\xi|}\right)^{n+1} |\xi|^{n}\sum_{j_1=1}^n |\xi_{j_1}\widehat{f}(\xi)| && (\text{since } |\xi| \leq \textstyle\sum_{j_1=1}^n |\xi_{j_1}|) \\
&\leq \left(\frac{2}{1+|\xi|}\right)^{n+1} \sum_{j_1,\dots,j_{n+1}=1}^n |\xi_{j_1}\cdots\xi_{j_{n+1}}\widehat{f}(\xi)| \\
&= \left(\frac{2}{1+|\xi|}\right)^{n+1} \sum_{j_1,\dots,j_{n+1}=1}^n \left|\frac{1}{(2\pi i)^{n+1}}\frac{\widehat{\partial^{n+1} f}}{\partial x_{j_1}\cdots\partial x_{j_{n+1}}}(\xi)\right| && (\text{by Lemma 1.29.7}) \\
&\leq \frac{1}{(1+|\xi|)^{n+1}}\pi^{-n-1} \sum_{j_1,\dots,j_{n+1}=1}^n \left\|\frac{\partial^{n+1} f}{\partial x_{j_1}\cdots\partial x_{j_{n+1}}}\right\|_{L^1}. && (\text{by Corollary 1.28})
\end{aligned}$$

Let $c \coloneqq \pi^{-n-1}\sum_{j_1,\dots,j_{n+1}=1}^n \|\frac{\partial^{n+1} f}{\partial x_{j_1}\cdots\partial x_{j_{n+1}}}\|_{L^1}$. Then,

$$\int |\widehat{f}(\xi)|\, d\xi \leq \int_{B_1(\mathbf{0})} |\widehat{f}(\xi)|\, d\xi + \int_{\mathbb{R}^n \setminus B_1(\mathbf{0})} |\widehat{f}(\xi)|\, d\xi$$

$$\begin{aligned}&\leq \|\widehat{f}\|_u m(B_1(\mathbf{0})) + \int_{\mathbb{R}^n \setminus B_1(\mathbf{0})} \frac{c}{(1+|\xi|)^{n+1}}\, d\xi \\ &\leq \|f\|_{L^1} m(B_1(\mathbf{0})) + c\int \frac{1}{(1+|\xi|)^{n+1}}\, d\xi < \infty.\end{aligned}$$

Hence, $\widehat{f} \in L^1(\mathbb{R}^n)$ as well. Therefore, $C_c^\infty(\mathbb{R}^n) \subseteq L^1_{\mathcal{F}}(\mathbb{R}^n)$ by the arbitrariness of f.

Now, by the Fourier Inversion Theorem (Theorem 1.34), we have that $(\mathcal{F}^{-1}\widehat{f})(\mathbf{x}) = f(\mathbf{x})$ for almost every $\mathbf{x} \in \mathbb{R}^n$ for each $f \in L^1_{\mathcal{F}}(\mathbb{R}^n)$. Since $\mathcal{F}^{-1}\widehat{f} \in C_0$, we have that $f \in L^\infty(\mathbb{R}^n)$ for all $f \in L^1_{\mathcal{F}}(\mathbb{R}^n)$. Hence, $L^1_{\mathcal{F}}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Note also that if $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then $|f|^p = |f|^{p-1} \cdot |f| \leq \|f\|_\infty^{p-1} |f| \in L^1(\mathbb{R}^n)$. In that regard, $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$. Therefore, $C_c^\infty(\mathbb{R}^n) \subseteq L^1_{\mathcal{F}}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ which means that $L^1_{\mathcal{F}}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ with respect to the L^p norm by Remark 1.1. ☐

We are now ready to extend the Fourier Transform. One should observe that we cannot extend it from $L^1(\mathbb{R}^n)$, but we can extend it from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by using the fact that it extends from $L^1_{\mathcal{F}}(\mathbb{R}^n)$. This will essentially allow us to view the Fourier Transform as an operator on $L^2(\mathbb{R}^n)$, but we will make a small distinction since we extended it from a strict subspace of $L^1(\mathbb{R}^n)$. This extension theorem is known as the **Plancherel Theorem**.

Theorem 1.39 (Plancherel Theorem)**.** *$\mathcal{F}(L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)) \subseteq L^2(\mathbb{R}^n)$ and $\mathcal{F}|_{L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}$ extends uniquely to a unitary isomorphism $\mathcal{P}$ on $L^2(\mathbb{R}^n)$. We call $\mathcal{P}$ the **Plancherel transform**.*

Proof. Let $L^1_{\mathcal{F}}(\mathbb{R}^n)$. By Lemma 1.38, we have that $L^1_{\mathcal{F}}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ with respect to the L^2 norm. Let $f, g \in L^1_{\mathcal{F}}(\mathbb{R}^n)$. By the Fourier Inversion Theorem (Theorem 1.34), we have that

$$\widehat{\overline{\widehat{g}}}(\xi) = \int \overline{\widehat{g}(\mathbf{x})} e^{-2\pi i \xi \cdot \mathbf{x}}\, d\mathbf{x} = \int \overline{\widehat{g}(\mathbf{x}) e^{2\pi i \xi\cdot \mathbf{x}}}\, d\mathbf{x} = \overline{(\widehat{g})^\vee(\xi)} = \overline{g(\xi)}$$

for almost every $\xi \in \mathbb{R}^n$. Hence, by Lemma 1.29.5, we have that

$$\int f\overline{g}\, dm = \int f \widehat{\overline{\widehat{g}}}\, dm = \int \widehat{f}\overline{\widehat{g}}\, dm$$

which means that $\mathcal{F}|_{L^1_{\mathcal{F}}(\mathbb{R}^n)}$ preserves the L^2 inner product on $\mathcal{X}$ by the arbitrariness of f, g. In that regard, $\mathcal{F}|_{L^1_{\mathcal{F}}(\mathbb{R}^n)}$ is an isometry. Furthermore, since $\mathcal{F}|_{L^1_{\mathcal{F}}(\mathbb{R}^n)}(L^1_{\mathcal{F}}(\mathbb{R}^n)) = L^1_{\mathcal{F}}(\mathbb{R}^n)$ by Theorem 1.34, it follows that $\mathcal{F}|_{L^1_{\mathcal{F}}(\mathbb{R}^n)}$ extends uniquely to an isometric isomorphism $\widetilde{\mathcal{F}|_{L^1_{\mathcal{F}}(\mathbb{R}^n)}}$ on $L^2(\mathbb{R}^n)$ by Lemma 1.37. In particular, $\widetilde{\mathcal{F}|_{L^1_{\mathcal{F}}(\mathbb{R}^n)}}$ is unitary on $L^2(\mathbb{R}^n)$ by the continuity of inner products.

Now, since $L^1_{\mathcal{F}}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we will need to show that $\widetilde{\mathcal{F}|_{L^1_{\mathcal{F}}(\mathbb{R}^n)}}|_{L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)} = \mathcal{F}|_{L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}$ to conclude that $\mathcal{P}$ is indeed a unitary isomorphism on $L^2(\mathbb{R}^n)$ since $\mathcal{P}$ is supposed to be the extension of $\mathcal{F}|_{L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}$. In that regard, let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and let ϕ be the Gaussian function as in Lemma 1.32. Then, $f * \phi_t \in L^1_{\mathcal{F}}(\mathbb{R}^n)$ for all $t > 0$ where ϕ_t is as in Notation 1.26 because $f * \phi_t \in L^1(\mathbb{R}^n)$ by Young's inequality, and $\widehat{f * \phi_t} = \widehat{f}\widehat{\phi_t} = t^{-n}\widehat{f}\widehat{\delta_{t^{-1}}\phi} = t^{-n}\widehat{f}(\widehat{\phi})_{\frac{1}{t}} = t^{-n}\widehat{f}\phi_{\frac{1}{t}} = \widehat{f}\delta_t\phi \in L^1(\mathbb{R}^n)$ by Lemmas 1.29.2, 4 and 1.32 and the boundedness of $\widehat{f}$. Hence, $\widetilde{\mathcal{F}|_{L^1_{\mathcal{F}}(\mathbb{R}^n)}} f = \lim_{t\to 0} \widehat{f * \phi_t} = (\widehat{\lim_{t\to 0} f * \phi_t}) = \widehat{f}$ (with respect to the L^2 norm) by Lemma 1.33 and continuity of the Fourier transform. By the arbitrariness of f, we are done. ☐

Remark 1.40. *Many authors initially define the Fourier transform on $L^1_{\mathcal{F}}(\mathbb{R}^n)$ (or on a finer space of functions called the Schwartz space) instead of on $L^1(\mathbb{R}^n)$ like above. If $\mathcal{F}$ was defined on $L^1_{\mathcal{F}}(\mathbb{R}^n)$ instead, then there would be no need to introduce the Plancherel transform $\mathcal{P}$. Note that by our current definition of $\mathcal{F}$ and $\mathcal{P}$, $\mathcal{F}f$ is defined and $\mathcal{P}f$ is not defined when $f \in L^1(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$.*

1.5 The Ball Multiplier

Recall from Notation 1.36 that $L^1_{\mathcal{F}}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \widehat{f} \in L^1(\mathbb{R}^n)\}$. This makes the following operator well defined.

Definition 1.41 (Multiplier Operator)**.** *Let $m \in L^\infty(\mathbb{R}^n)$. Then,*

$$\begin{aligned} T_m : L^1_{\mathcal{F}}(\mathbb{R}^n) &\to L^2(\mathbb{R}^n) \\ f &\mapsto (m\widehat{f})^\vee \end{aligned}$$

is the ***multiplier operator with symbol*** *m, and m is a* ***multiplier****.*

From the definition above, we can see the multipliers are essentially bounded functions that perturb the frequency space of a function. As we did for the Fourier Transform, we can also extend multiplier operators to L^2 bounded linear operators.

Remark 1.42. *Note that by the Plancherel Theorem (Theorem 1.39) and Hölder's inequality, $\|T_m f\|_{L^2} = \|m\widehat{f}\|_{L^2} \le \|m\|_{L^\infty}\|\widehat{f}\|_{L^2} = \|m\|_{L^\infty}\|f\|_{L^2}$ for all $f \in L^1_{\mathcal{F}}(\mathbb{R}^n)$. Hence, T_m always extends to an $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ bounded linear operator by Lemmas 1.37 and 1.38 (note also that $T_m f \in L^2(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ for all $f \in L^1_{\mathcal{F}}(\mathbb{R}^n)$).*

In that regard, we will, from now on, identify the extension $\widetilde{T_m}$ with the multiplier operator T_m.

Lemma 1.43. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be complete, normed vector spaces. Let $\mathcal{V}$ be a dense subspace of $\mathcal{X}$, and let T be a bounded linear operator from $\mathcal{V}$ to $\mathcal{Y}$. Let R be a bounded linear operator from $\mathcal{Y}$ to $\mathcal{Y}$, and let S be a bounded linear operator from $\mathcal{X}$ to $\mathcal{X}$ such that $S(\mathcal{V}) = \mathcal{V}$. Then, $\widetilde{RTS} = R\widetilde{T}S$ where $\widetilde{\cdot}$ has the meaning in Lemma 1.37.*

Proof. Observe that $\widetilde{RTS}x = \lim_{j\to\infty} RTSx_j = R\lim_{j\to\infty} TSx_j = R\widetilde{T}Sx$ for all Cauchy sequences $\{x_j\}_{j=1}^\infty \subseteq \mathcal{V}$ converging to $x \in \mathcal{X}$. □

Corollary 1.44. *$T_m = \mathcal{P}^{-1}m\mathcal{P}$ where $\mathcal{P}$ is the Plancherel transform as defined in the Plancherel Theorem (Theorem 1.39).*

Proof. By the Plancherel Theorem (Theorem 1.39), we have that $\mathcal{P}^{-1}m\mathcal{P}f = \mathcal{P}^{-1}(m\widehat{f}) = (m\widehat{f})^\vee$ for all $f \in L^1_{\mathcal{F}}(\mathbb{R}^n)$ (note that $m\widehat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ since $\widehat{f} \in L^1(\mathbb{R}^n)$ and by Remark 1.42). Given that $\mathcal{P}(L^1_{\mathcal{F}}(\mathbb{R}^n)) = L^1_{\mathcal{F}}(\mathbb{R}^n)$, it follows from Lemma 1.43 that $T_m = \mathcal{P}^{-1}m\mathcal{P}$. □

Multipliers help enrich our understanding of the Fourier transform by serving as tools to make sense of how we may extend the Fourier Inversion Theorem (Theorem 1.34) to functions in $L^p(\mathbb{R}^n)$. Consider the **Bochner-Riesz operators** given by

$$(S_r^\delta f)(\mathbf{x}) \coloneqq \int_{B_r(\mathbf{0})} \widehat{f}(\xi) \left(1 + \frac{|\xi|^2}{r^2}\right)^\delta e^{2\pi i \xi \cdot \mathbf{x}} \, d\xi$$

for $\delta \geq 0$, $r > 0$ and $f \in L^1_{\mathcal{F}}(\mathbb{R}^n)$. From [3], if we want $f = \lim_{r \to \infty} S_r^\delta f$ to hold in the L^p sense when $n \geq 2$ (such convergence holds for all $p \in (1, \infty)$ when $n = 1$), then it suffices to make sure S_1^δ can be extended to a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. If $\delta > 0$, it is necessary that $\delta > \max\{n \cdot |\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$ for S_1^δ to be L^p bounded. However, it is not known yet whether the condition is sufficient when $n \geq 3$. This is known as the **Bochner-Riesz conjecture**. On the other hand, if $\delta = 0$, then S_1^0 can only be bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. The proof of this can be found in Section 4. Note that S_0^1 is essentially the multiplier operator whose symbol is the characteristic function of a unit ball.

We will now pay special attention to multipliers that are characteristic functions, as these shall later help us prove that S_0^1 cannot be extended. These operators essentially cut out portions of the frequency space of functions, and they are notated as follows.

Notation 1.45. *If the multiplier is a characteristic function χ_E for some measurable $E \subseteq \mathbb{R}^n$, we write $S_E \coloneqq T_{\chi_E}$.*

Remark 1.46 (Ball Multiplier Operator)**.** $S_{B_1(\mathbf{0})} = S_0^1$ *is the* ***ball multiplier operator****.*

We will now derive some basic properties of these multiplier operators whose symbols are characteristic functions to conclude this section.

Lemma 1.47. *Let $E \subseteq \mathbb{R}^n$ be measurable and let $f \in L^2(\mathbb{R}^n)$. Then,*

1. *$S_E = \tau_{-\mathbf{y}} S_E \tau_{\mathbf{y}} f = S_E$ and $S_{E+\mathbf{y}} = e^{2\pi i \mathbf{y} \cdot \mathrm{id}} S_E(e^{-2\pi i \mathbf{y} \cdot \mathrm{id}} f)$ for all $\mathbf{y} \in \mathbb{R}^n$, where $E + \mathbf{y}$ is a translation of E by $\mathbf{y}$ (note that the first equality implies that translation commutes with S_E).*
2. *$S_{tE} = \delta_t S_E \delta_{t^{-1}}$ for all $t > 0$, where tE is a scaling of E by t.*
3. *$S_{OE} = \rho_{O^T} S_E \rho_O$ for all orthogonal transformations O, where OE is a rotation of E by O.*
4. *If $\{E_j\}_{j=1}^\infty$ is an increasing collection of subsets in $\mathbb{R}^n$, i.e. $E_1 \subseteq E_2 \subseteq \dots$, such that $E = \bigcup_{j=1}^\infty E_j$, then $S_E f = \lim_{j \to \infty} S_{E_j} f$ with respect to the L^2 norm.*
5. *If $E = E_1 \times E_2$ and $f = f_1 f_2$ where $E_1 \subseteq \mathbb{R}^{n_1}$, $E_2 \subseteq \mathbb{R}^{n_2}$ are measurable and $f_1 \in L^2(\mathbb{R}^{n_1})$, $f_2 \in L^2(\mathbb{R}^{n_2})$ (with $n_1 + n_2 = n$), then $S_E f = S_{E_1} f_1 \cdot S_{E_2} f_2$.*

Proof. By Lemma 1.43, it suffices to check that the operators in 1,2 and 3 are equal on $L^1_{\mathcal{F}}(\mathbb{R}^n)$.

1. Observe that

$$\begin{aligned}
(\tau_{-\mathbf{y}} S_E \tau_{\mathbf{y}} f)(\mathbf{x}) &= \int_E \widehat{\tau_{\mathbf{y}} f}(\xi) e^{2\pi i \xi\cdot(\mathbf{x}+\mathbf{y})}\ d\xi \\
&= \int_E e^{-2\pi i \xi\cdot\mathbf{y}} \widehat{f}(\xi) e^{2\pi i \xi\cdot(\mathbf{x}+\mathbf{y})}\ d\xi && \text{(by Lemma 1.29.1)} \\
&= (S_E f)(\mathbf{x}),
\end{aligned}$$

and that

$$\begin{aligned}
(S_{E+\mathbf{y}} g)(\mathbf{x}) &= \int_{E+\mathbf{y}} \widehat{g}(\xi) e^{2\pi i \xi\cdot\mathbf{x}}\ d\xi \\
&= \int_E \tau_{-\mathbf{y}} \widehat{g}(\xi) e^{2\pi i (\xi+\mathbf{y})\cdot\mathbf{x}}\ d\xi && \text{(by translation invariance of } m) \\
&= e^{2\pi i \mathbf{y}\cdot\mathbf{x}} \int_E (\widehat{e^{-2\pi i \mathbf{y}\cdot \mathrm{id}} g})(\xi) e^{2\pi i \xi\cdot\mathbf{x}}\ d\xi \\
&= e^{2\pi i \mathbf{y}\cdot\mathbf{x}} (S_E(e^{-2\pi i \mathbf{y}\cdot\mathrm{id}} g))(\mathbf{x})
\end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $g \in L^1_{\mathcal{F}}(\mathbb{R}^n)$.

2. Observe that

$$\begin{aligned}
(S_{tE} g)(\mathbf{x}) &= \int_{tE} \widehat{g}(\xi) e^{2\pi i \xi\cdot\mathbf{x}}\ d\xi \\
&= \int_E (\widehat{g})_{\frac{1}{t}}(\mathbf{u}) e^{2\pi i \mathbf{u}\cdot t\mathbf{x}}\ d\mathbf{u} \\
&\qquad \text{(by scaling with } \xi = t\mathbf{u}\text{; } (\widehat{g})_{\frac{1}{t}} \text{ has the meaning in Notation 1.26)} \\
&= \int_E \widehat{\delta_{\frac{1}{t}} g}(\mathbf{u}) e^{2\pi i \mathbf{u}\cdot t\mathbf{x}}\ d\mathbf{u} && \text{(by Lemma 1.29.2)} \\
&= (\delta_t S_E \delta_{t^{-1}} g)(\mathbf{x})
\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$, $g \in L^1_{\mathcal{F}}(\mathbb{R}^n)$ and $t > 0$.

3. Observe that for all orthogonal transformations O,

$$\begin{aligned}
(S_{OE} g)(\mathbf{x}) &= \int_{OE} \widehat{g}(\xi) e^{2\pi i \xi\cdot\mathbf{x}}\ d\xi \\
&= \int_E \widehat{g}(O\mathbf{u}) e^{2\pi i O\mathbf{u}\cdot\mathbf{x}}\ d\mathbf{u} && \text{(by letting } \mathbf{u} = O^T\xi) \\
&= \int_E \widehat{\rho_O g}(\mathbf{u}) e^{2\pi i \mathbf{u}\cdot O^T\mathbf{x}}\ d\mathbf{u} \\
&\qquad \text{(By Lemma 1.29.3 and since } O \text{ preserves dot products)} \\
&= (\rho_{O^T} S_E \rho_O g)(\mathbf{x})
\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $g \in L^1_{\mathcal{F}}(\mathbb{R}^n)$.

4. Let $\mu(F) \coloneqq \int_F |\mathcal{P} f|^2\ dm$ for all measurable $F \subseteq \mathbb{R}^n$. Note that μ is a finite measure since $\mathcal{P} f \in L^2(\mathbb{R}^n)$. In that regard,

$$\lim_{j\to\infty} \|S_E f - S_{E_j} f\|_{L^2} = \lim_{j\to\infty} \|\mathcal{P}^{-1}(\chi_{E\setminus E_j} \mathcal{P} f)\|_{L^2} \qquad \text{(by Corollary 1.44)}$$

$$= \lim_{j\to\infty} \|\chi_{E\setminus E_j}\mathcal{P}f\|_{L^2}$$

(by the Plancherel Theorem (Theorem 1.39))

$$= \lim_{j\to\infty} \sqrt{\mu(E\setminus E_j)} = 0.$$

(by continuity from above and since μ is finite)

5. By Lemma 1.38, there exist sequences $\{g_k\}_{k=1}^{\infty} \subseteq L^1_{\mathcal{F}}(\mathbb{R}^{n_1})$ and $\{h_k\}_{k=1}^{\infty} \subseteq L^1_{\mathcal{F}}(\mathbb{R}^{n_2})$ such that $f_1 = \lim_{k\to\infty} g_k$ and $f_2 = \lim_{k\to\infty} h_k$ with respect to the L^2 norm. Hence,

$$\begin{aligned}\|f_1 f_2 - g_k h_k\|_{L^2} &= \|f_1(f_2 - h_k) + h_k(f_1 - g_k)\|_{L^2} \\ &\le \|f_1(f_2 - h_k)\|_{L^2} + \|h_k(f_1 - g_k)\|_{L^2} \\ &= \|f_1\|_{L^2}\|f_2 - h_k\|_{L^2} + \|h_k\|_{L^2}\|f_1 - g_k\|_{L^2}. \qquad \text{(by the Tonelli Theorem)}\end{aligned}$$

by L^2 convergence of g_k and h_k, it is clear that the right hand side converges to 0 (one can easily check that $\|h_k\|_{L^2} \to \|f_2\|_{L^2}$ by L^2 convergence as well). Hence, $f = \lim_{k\to\infty} g_k h_k$ with respect to the L^2 norm by the squeeze theorem (note also that $g_k h_k \in L^1_{\mathcal{F}}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ by Fubini's theorem; in particular, $\widehat{g_k h_k} = \widehat{g_k}\widehat{h_k}$). On that note,

$$\begin{aligned}S_E f &= \lim_{k\to\infty} S_E(g_k h_k) \\ &= \lim_{k\to\infty} \int_{E_1\times E_2} \widehat{g_k}(\xi_1)\widehat{h_k}(\xi_2) e^{2\pi i \xi\cdot \mathrm{id}}\, d\xi \qquad \text{(where } \xi = (\xi_1, \xi_2)\text{)} \\ &= \lim_{k\to\infty} \int_{E_1} \widehat{g_k}(\xi_1) e^{2\pi i \xi_1 \cdot \mathrm{id}_1}\, d\xi_1 \int_{E_2} \widehat{h_k}(\xi_2) e^{2\pi i \xi_2\cdot \mathrm{id}_2}\, d\xi_2 \\ &\qquad \text{(by Fubini's theorem; note that id} = \mathrm{id}_1 \times \mathrm{id}_2\text{)} \\ &= \lim_{k\to\infty} S_{E_1} g_k \cdot S_{E_2} h_k \\ &= S_{E_1} f_1 \cdot S_{E_2} f_2\end{aligned}$$

so we are done.

□

2 The Kakeya Sets

To understand the importance of the conditions for nicely shrinking sets, we first turn our attention to the Kakeya Needle Problem. A **Kakeya set** or **Besicovitch set** is a set in $\mathbb{R}^n$ such that it contains a unit line segment in any orientation, and the Kakeya Needle Problem asks whether there is a Kakeya set of minimum area in $\mathbb{R}^n$. As we shall see later, there is indeed no minimal Kakeya set, and the infimum of measures of all Kakeya sets is 0. In this section, we only construct Kakeya sets in $\mathbb{R}^2$. In particular, these constructions can have arbitrarily small measures. The first construction involves partitioning a triangle into subtriangles, and then translating the subtriangles to maximize the degree of overlap

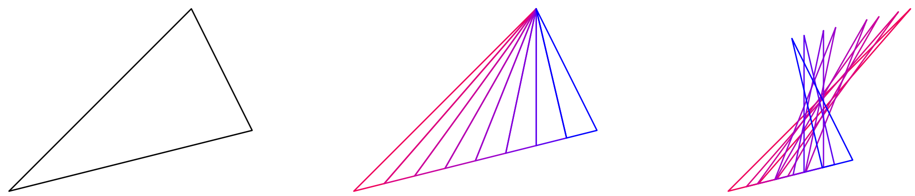


Figure 2.1: The triangle (left) is partitioned into subtriangles (middle) to produce the Perron Tree (right).

between them. This construction is called a **Perron Tree**, and an example is provided in Figure 2.1.

The second construction involves placing rectangles into each subtriangle of the Perron Tree. The union of these rectangles also form a Besicovitch set. In that regard, we shall call it a **Rectangular Besicovitch set**. Although the Perron Tree is easier to describe, the Rectangular Besicovitch set is more useful for proving other results. Both constructions can be found in [3], but I have included more detail on how the sets can be constructed. Before proceeding with the formalization of the constructions, we shall first familiarize ourselves with some notation:

Notation 2.1.

1. *$\Delta ABC \subseteq \mathbb{R}^2$ is a (solid) triangle with vertices at $A, B, C \in \mathbb{R}^2$.*
2. *$\overline{AB} \subseteq \mathbb{R}^n$ is a line segment between A and B.*
3. *$\overrightarrow{AB} \coloneqq B - A$ is the direction vector from A to B in Euclidean space.*
4. *$\|\cdot\|$ is the usual Euclidean norm.*

Definition 2.2 (Perron Tree)**.** *Let $N \in \mathbb{N}$ and $\alpha \in (\frac{1}{2}, 1)$. A (N, α)-**Perron Tree** $\mathbf{P}_{N,\alpha}(\Delta ABC) \subseteq \mathbb{R}^2$ generated by a triangle ΔABC is constructed in the following manner:*

Let $t_0, t_1, \ldots, t_{2^N} \in \overline{AB}$ evenly partition the line segment $\overline{AB}$ with $t_0 = A$ and $t_{2^N} = B$, and let $A_{0,j} \coloneqq t_{j-1}$, $B_{0,j} \coloneqq t_j$ and $C_{0,j} \coloneqq C$ for each $1 \leq j \leq 2^N$. We then inductively define the following for each $1 \leq i \leq N$ and $1 \leq j \leq 2^N$:

$$\begin{aligned}
c_{i,j} &\coloneqq \left\lfloor \frac{j + 2^{i-1} - 1}{2^i} \right\rfloor, \\
\mathbf{v}_i &\coloneqq (1-\alpha)\overrightarrow{B_{i-1,2^i}A}, \\
A_{i,j} &\coloneqq A_{i-1,j} + c_{i,j}\mathbf{v}_i, \\
B_{i,j} &\coloneqq B_{i-1,j} + c_{i,j}\mathbf{v}_i, \\
C_{i,j} &\coloneqq C_{i-1,j} + c_{i,j}\mathbf{v}_i,
\end{aligned}$$

where $\lfloor\cdot\rfloor$ is the floor function. From this, we make the following definitions for each $0 \leq i \leq N$ and $1 \leq j \leq 2^N$:

1. *$\mathbf{T}^{i,j}_{N,\alpha}(\Delta ABC) \coloneqq \Delta A_{i,j}B_{i,j}C_{i,j}$ is the (i,j)-**th subtriangle of** $\mathbf{P}_{N,\alpha}(\Delta ABC)$.*

2. $\mathbf{S}^i_{N,\alpha}(\Delta ABC) \coloneqq \bigcup_{j=1}^{2^N} \mathbf{T}^{i,j}_{N,\alpha}(\Delta ABC)$ *is the* i***-th stage of*** $\mathbf{P}_{N,\alpha}(\Delta ABC)$.
3. D_i *is the intersection of the line segments* $\overline{AC}$ *and* $\overline{B_{i,2^N}C_{i,2^N}}$.
4. $\mathbf{H}^i_{N,\alpha}(\Delta ABC) \coloneqq \Delta AB_{i,2^N}D_i$ *is the* i***-th heart of*** $\mathbf{P}_{N,\alpha}(\Delta ABC)$.

The (N,α)***-Perron Tree*** *is thus* $\mathbf{P}_{N,\alpha}(\Delta ABC) \coloneqq \mathbf{S}^N_{N,\alpha}(\Delta ABC)$. *When* $N = 1$, *we also define* $\mathbf{E}_\alpha(\Delta ABC) \coloneqq \mathbf{P}_{1,\alpha}(\Delta ABC) \smallsetminus \mathbf{H}^1_{1,\alpha}(\Delta ABC)$ *to be the* α***-ears of*** ΔABC.

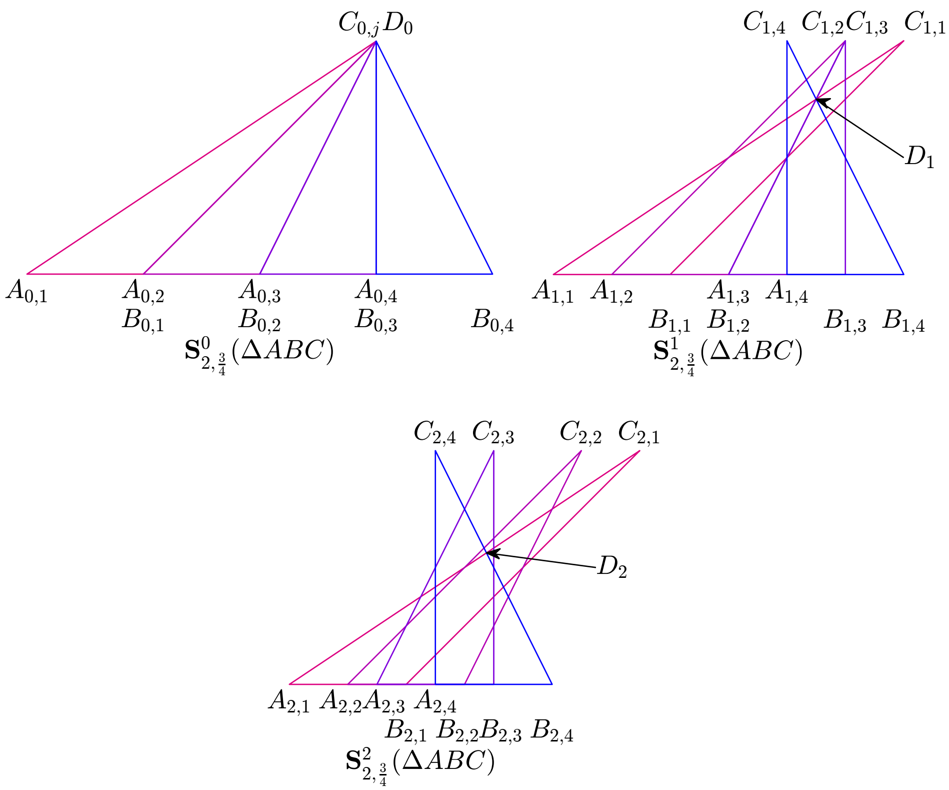


Figure 2.2: Three stages of $\mathbf{P}_{2,\frac{3}{4}}(\Delta ABC)$ with $A = (0,0)$, $B = (4,0)$ and $C = (3,2)$.

Remark 2.3. *One can easily deduce with reference to Figure 2.2 that*

1. $\Delta ABC = \mathbf{S}^0_{N,\alpha}(\Delta ABC) = \mathbf{H}^0_{N,\alpha}(\Delta ABC)$.
2. $A_{i,1} = A$ *and* $C_{i,1} = C$ *and* $B_{i,1} = B_{0,1}$ *for all* $0 \le i \le N$.
3. $\mathbf{H}^i_{N,\alpha}(\Delta ABC)$ *is similar to* ΔABC *for each* $0 \le i \le N$.

To prove that the measures of Perron Trees can get arbitrarily small, we first need to calculate the measures of certain components of the Perron Trees. These calculations are highlighted in the following two lemmas.

Lemma 2.4. *Let* $\Delta ABC \subseteq \mathbb{R}^2$, $N \in \mathbb{N}$ *and* $\alpha \in (\frac{1}{2}, 1)$. *Then,* $m(\mathbf{H}^i_{N,\alpha}(\Delta ABC)) = \alpha^{2i} \cdot m(\Delta ABC)$ *for each heart of* $\mathbf{P}_{N,\alpha}(\Delta ABC)$.

Proof. Note that for any $1 \le i \le N$, $c_{i,2^N} = \lfloor \frac{2^N+2^{i-1}-1}{2^i} \rfloor = \lfloor 2^{N-i} + \frac{1}{2} - 2^{-i} \rfloor = 2^{N-i}$, and that $2^{N-i} \cdot \|\overrightarrow{B_{i-1,2^i}A}\| = \|\overrightarrow{B_{i-1,2^N}A}\|$ as can be observed from the above example. This fact combined with the inductive definitions of the vertices, gives us $B_{i,2^N} = B_{i-1,2^N} + 2^{N-i}(1-\alpha)\overrightarrow{B_{i-1,2^i}A} = B_{i-1,2^N} + (1-\alpha)\overrightarrow{B_{i-1,2^N}A}$ which means that $\overrightarrow{B_{i,2^N}A} = (1-(1-\alpha))\overrightarrow{B_{i-1,2^N}A} = \alpha \cdot \overrightarrow{B_{i-1,2^N}A}$ and so $\|\overrightarrow{B_{i,2^N}A}\| = \alpha \cdot \|\overrightarrow{B_{i-1,2^N}A}\|$. Given that each $\mathbf{H}^i_{N,\alpha}(\Delta ABC)$ is similar to ΔABC, it follows that $m(\mathbf{H}^i_{N,\alpha}(\Delta ABC)) = \alpha^2 \cdot m(\mathbf{H}^{i-1}_{N,\alpha}(\Delta ABC))$. Since $\mathbf{H}^0_{N,\alpha}(\Delta ABC) = \Delta ABC$, we conclude that $m(\mathbf{H}^i_{N,\alpha}(\Delta ABC)) = \alpha^{2i} \cdot m(\Delta ABC)$ for each heart of $\mathbf{P}_{N,\alpha}(\Delta ABC)$. □

Lemma 2.5. *Let $\Delta ABC \subseteq \mathbb{R}^2$ and $\alpha \in (\frac{1}{2}, 1)$. Then, $m(\mathbf{E}_\alpha(\Delta ABC)) = 2(1-\alpha)^2 \cdot m(\Delta ABC)$ and $m(\mathbf{P}_{1,\alpha}(\Delta ABC)) = (\alpha^2 + 2(1-\alpha)^2) \cdot m(\Delta ABC)$.*

Proof. Let ℓ be the line parallel to $\overline{AB}$ passing through D_1. Let M_1 be the intersection of ℓ and $\overline{A_{1,2}C_{1,2}}$, and let M_2 be the intersection of ℓ and $\overline{B_{1,1}C_{1,1}}$. Then, $\Delta D_1M_2C_{1,1}$ and $\Delta M_1D_1C_{1,2}$ are similar to $\Delta A_{1,1}B_{1,1}C_{1,1}$ and $\Delta A_{1,2}B_{1,2}C_{1,2}$ respectively. Note that $\|\overrightarrow{D_1M_2}\| = (1-\alpha) \cdot \|\overrightarrow{A_{1,1}B_{1,1}}\|$ and $\|\overrightarrow{M_1D_1}\| = (1-\alpha) \cdot \|\overrightarrow{A_{1,2}B_{1,2}}\|$. Hence, $m(\Delta D_1M_2C_{1,1}) = (1-\alpha)^2 \cdot m(\Delta A_{1,1}B_{1,1}C_{1,1})$ and $m(\Delta M_1D_1C_{1,2}) = (1-\alpha)^2 \cdot m(\Delta A_{1,2}B_{1,2}C_{1,2})$. Given that $m(\Delta A_{1,1}B_{1,1}C_{1,1}) = m(\Delta A_{1,2}B_{1,2}C_{1,2}) = \frac{m(\Delta ABC)}{2}$, then $m(\Delta D_1M_2C_{1,1} \cup \Delta M_1D_1C_{1,2}) = (1-\alpha)^2 \cdot m(\Delta ABC)$.

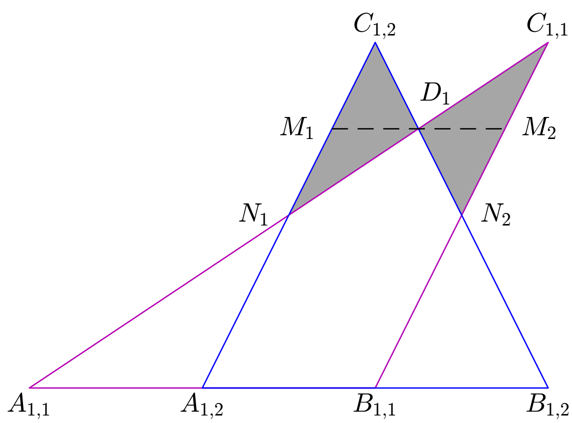


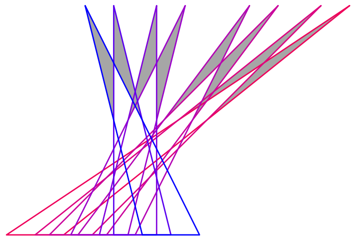
Figure 2.3: $\mathbf{P}_{1,\frac{3}{4}}(\Delta ABC)$ with $A = (0,0)$, $B = (4,0)$ and $C = (3,2)$. The shaded region is $\mathbf{E}_{\frac{3}{4}}(\Delta ABC)$.

Let N_1 be the intersection of $\overline{A_{1,1}C_{1,1}}$ and $\overline{A_{1,2}C_{1,2}}$, and let N_2 be the intersection of $\overline{B_{1,2}C_{1,2}}$ and $\overline{B_{1,1}C_{1,1}}$. Observe that $\Delta M_1D_1C_{1,2}$ and $\Delta D_1M_2C_{1,1}$ are congruent to $\Delta D_1M_2N_2$ and $\Delta M_1D_1N_1$ respectively. Therefore, $m(\mathbf{E}_\alpha(\Delta ABC)) = m(\Delta D_1M_2C_{1,1} \cup \Delta M_1D_1C_{1,2}) + m(\Delta M_1D_1N_1 \cup \Delta D_1M_2N_2) = 2(1-\alpha)^2 \cdot m(\Delta ABC)$.

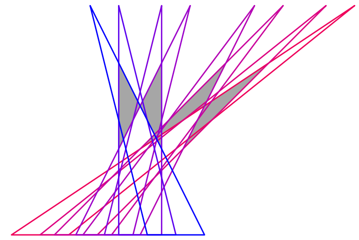
Finally, $m(\mathbf{P}_{1,\alpha}(\Delta ABC)) = m(\mathbf{H}^1_{1,\alpha}(\Delta ABC)) + m(\mathbf{E}_\alpha(\Delta ABC)) = (\alpha^2 + 2(1-\alpha)^2) \cdot m(\Delta ABC)$ by Lemma 2.4. □

Theorem 2.6 (Perron Tree Estimate)**.** *Let $N \in \mathbb{N}$, $\alpha \in (\frac{1}{2}, 1)$ and $\Delta ABC \subseteq \mathbb{R}^2$. Then, $m(\mathbf{P}_{N,\alpha}(\Delta ABC)) \le (\alpha^{2N} + 2(1-\alpha)) \cdot m(\Delta ABC)$.*

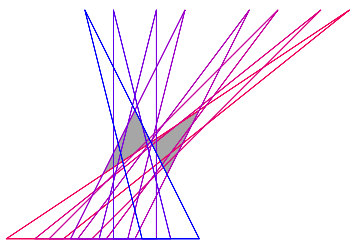
Proof. We can view $\mathbf{P}_{N,\alpha}(\Delta ABC)$ as a union of $\mathbf{H}^N_{N,\alpha}(\Delta ABC)$ and ears generated by 2^i-tuples of subtriangles. An example is provided in Figure 2.4.



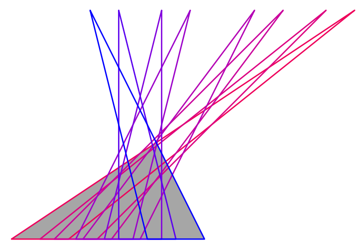
$\leq \sum_{j=1}^{4} m(\mathbf{E}_{\frac{3}{4}}(\mathbf{H}^0_{1,\frac{3}{4}}(\Delta A_{0,2(j-1)+1}B_{0,2j}C)))$



$\leq \sum_{j=1}^{2} m(\mathbf{E}_{\frac{3}{4}}(\mathbf{H}^1_{2,\frac{3}{4}}(\Delta A_{0,4(j-1)+1}B_{0,4j}C)))$



$= m(\mathbf{E}_{\frac{3}{4}}(\mathbf{H}^2_{3,\frac{3}{4}}(\Delta ABC)))$



$= m(\mathbf{H}^3_{3,\frac{3}{4}}(\Delta ABC))$

Figure 2.4: $\mathbf{P}_{3,\frac{3}{4}}(\Delta ABC)$ with $A = (0,0)$, $B = (4,0)$ and $C = (3,2)$. The shaded regions are bounded or equal to the correspoinding measures.

Hence, by subadditivity and translation invariance of m, we have that

$$\begin{aligned}
m(\mathbf{P}_{N,\alpha}(\Delta ABC)) &\leq m(\mathbf{H}^N_{N,\alpha}(\Delta ABC)) + \sum_{i=1}^{N}\sum_{j=1}^{2^{N-i}} m(\mathbf{E}_\alpha(\mathbf{H}^{i-1}_{i,\alpha}(\Delta A_{0,2^i(j-1)+1}B_{0,2^i j}C))) \\
&= \alpha^{2N}\cdot m(\Delta ABC) + \sum_{i=1}^{N}\sum_{j=1}^{2^{N-i}} 2(1-\alpha)^2\cdot\alpha^{2i-2}\cdot m(\Delta A_{0,2^i(j-1)+1}B_{0,2^i j}C) \\
&\qquad\qquad \text{(by Lemmas 2.4 and 2.5)} \\
&= \alpha^{2N}\cdot m(\Delta ABC) + 2(1-\alpha)^2\sum_{i=1}^{N}\alpha^{2i-2}\cdot 2^{N-i}\cdot m(\Delta AB_{0,2^i}C) \\
&\qquad \text{(since } \|\overrightarrow{AB_{0,2^i}}\| = \|\overrightarrow{A_{0,2^i(j-1)+1}B_{0,2^i j}}\| \text{ for all } 1\leq j\leq 2^{N-i} \text{ for all } 1\leq i\leq N) \\
&= \alpha^{2N}\cdot m(\Delta ABC) + 2(1-\alpha)^2\sum_{i=1}^{N}\alpha^{2i-2}\cdot m(\Delta ABC) \\
&\qquad\qquad \text{(since } \|\overrightarrow{AB_{0,2^i}}\| = \tfrac{\|\overrightarrow{AB}\|}{2^{N-i}}) \\
&\leq \left(\alpha^{2N} + 2(1-\alpha)^2\sum_{i=1}^{\infty}\alpha^{2i-2}\right)\cdot m(\Delta ABC) \\
&= \left(\alpha^{2N} + \frac{2(1-\alpha)^2}{1-\alpha^2}\right)\cdot m(\Delta ABC) = \left(\alpha^{2N} + \frac{2(1-\alpha)}{1+\alpha}\right)\cdot m(\Delta ABC)
\end{aligned}$$

$$\leq (\alpha^{2N} + 2(1-\alpha)) \cdot m(\Delta ABC). \qquad \text{(since } \alpha > 0\text{)}$$

$\square$

Notation 2.7. *For any $l, w > 0$, $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{v} \in S^1 \subseteq \mathbb{R}^2$ (with S^1 being the unit circle), we define $\mathbf{R}^{\mathbf{v}}_{l,w}(\mathbf{x}) \subseteq \mathbb{R}^2$ to be a rectangle with side lengths l and w centered at $\mathbf{x}$, and with the sides of length l parallel to $\overline{\mathbf{0v}}$.*

Theorem 2.8 (Existence of the Rectangular Besicovitch set)**.** *For all $\varepsilon > 0$, there exist $L \in \mathbb{N}$, $\mathbf{x}_i \in \mathbb{R}^2$ and $\mathbf{v}_i \in S^1$ for $1 \leq i \leq 2^L$ such that*

1. $m(\bigcup_{i=1}^{2^L} \mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i)) < \varepsilon$, *and*
2. $m(\bigcup_{i=1}^{2^L} \mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i + 2\mathbf{v}_i)) = 1$.

*The set $\bigcup_{i=1}^{2^L} \mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i)$ is the **Rectangular Besicovitch set**.*

Proof. The main idea of the proof is to strategically place the rectangles into each subtriangle of the Perron tree. With that in mind, let $A \coloneqq (0,0)$ $B \coloneqq (\frac{4}{\sqrt{3}}, 0)$ and $C \coloneqq (\frac{2}{\sqrt{3}}, 2)$. Then, ΔABC is an equilateral triangle with height 2 and $m(\Delta ABC) = \frac{4}{\sqrt{3}}$. Given that $\lim_{\alpha\to 1} 2(1-\alpha) = 0$ and $\lim_{N\to\infty} \alpha^{2N} = 0$ for all $\frac{1}{2} < \alpha < 1$, there exists $N \in \mathbb{N}$ large enough and α very close to 1 such that $m(\mathbf{P}_{N,\alpha}(\Delta ABC)) < \frac{\varepsilon}{4}$ by Theorem 2.6.

At this point, we want to show that rectangles can fit in the subtriangles by adjusting the length of the shorter sides of the rectangles by a constant factor. Let $L \coloneqq N + 2$, let $M_j \in \overline{A_{N,j}B_{N,j}}$ be such that $\overline{C_{N,j}M_j}$ bisects $\angle B_{N,j}C_{N,j}A_{N,j}$, let $\mathbf{x}_j \in \overline{C_{N,j}M_j}$ be such that $\|\overrightarrow{\mathbf{x}_jC_{N,j}}\| = 1$, and let $\mathbf{v}_j \coloneqq \frac{\overrightarrow{M_jC_{N,j}}}{\|\overrightarrow{M_jC_{N,j}}\|}$ for each $1 \leq j \leq 2^N$. Suppose for contradiction that there exists j_0 such that $\mathbf{R}^{\mathbf{v}_{j_0}}_{1,2^{-L}}(\mathbf{x}_{j_0}) \not\subseteq \mathbf{T}^{N,j_0}_{N,\alpha}(\Delta ABC)$. Then, the shorter edge of $\mathbf{R}^{\mathbf{v}_{j_0}}_{1,2^{-L}}(\mathbf{x}_{j_0})$ that is closer to C_{N,j_0} must protrude from $\mathbf{T}^{N,j_0}_{N,\alpha}(\Delta ABC)$, and intersect the subtriangle at some points $P_1 \in \overline{A_{N,j_0}C_{N,j_0}}$ and $P_2 \in \overline{B_{N,j_0}C_{N,j_0}}$. In that regard, $\|\overrightarrow{P_1P_2}\| \leq 2^{-L}$.

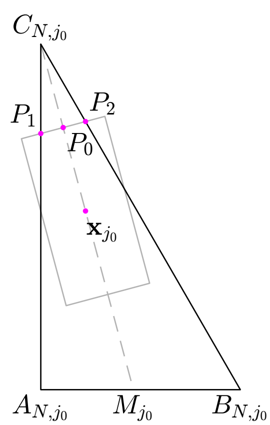


Figure 2.5: $\mathbf{T}^{N,j_0}_{N,\alpha}(\Delta ABC)$ with the grey rectangle $\mathbf{R}^{\mathbf{v}_{j_0}}_{1,2^{-L}}(\mathbf{x}_{j_0})$ assumed to not fit inside.

Note that $\overline{P_1P_2}$ is perpendicular to $\overline{C_{N,j_0}M_{j_0}}$, and that the midpoint of $\overline{P_1P_2}$ is equal to the intersection of $\overline{C_{N,j_0}M_{j_0}}$ and $\overline{P_1P_2}$. We shall call this point P_0 as seen in Figure 2.5.

Now, $\sin(\angle A_{N,j_0}B_{N,j_0}C_{N,j_0}) \in [\frac{\sqrt{3}}{2}, 1]$ because $\angle A_{N,j_0}B_{N,j_0}C_{N,j_0} \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, and $\|\overrightarrow{A_{N,j_0}C_{N,j_0}}\| \in [2, \frac{4}{\sqrt{3}}]$ by how we defined ΔABC. By the sine rule, we at least know that

$$\sin(\angle B_{N,j_0}C_{N,j_0}A_{N,j_0}) = 2^{-N} \cdot \frac{4}{\sqrt{3}} \cdot \frac{\sin(\angle A_{N,j_0}B_{N,j_0}C_{N,j_0})}{\|\overrightarrow{A_{N,j_0}C_{N,j_0}}\|} \geq 2^{-N-1} \cdot \sqrt{3} > 2^{-N-1}.$$

Hence,

$$\begin{aligned}
\|\overrightarrow{P_1P_2}\| &= \|\overrightarrow{P_1P_0}\| + \|\overrightarrow{P_0P_2}\| \\
&= \frac{1}{2} \cdot (\tan(\angle P_0C_{N,j_0}P_1) + \tan(\angle P_2C_{N,j_0}P_0)) \\
&\qquad\qquad \text{(by simple trigonometry and the fact that } \|\overrightarrow{P_0C_{N,j_0}}\| = \tfrac{1}{2}) \\
&= \tan\left(\frac{\angle B_{N,j_0}C_{N,j_0}A_{N,j_0}}{2}\right) \qquad \text{(since } \angle P_0C_{N,j_0}P_1 = \angle P_2C_{N,j_0}P_0 = \tfrac{\angle B_{N,j_0}C_{N,j_0}A_{N,j_0}}{2}) \\
&\geq \sin\left(\frac{\angle B_{N,j_0}C_{N,j_0}A_{N,j_0}}{2}\right) \geq \frac{\sin(\angle B_{N,j_0}C_{N,j_0}A_{N,j_0})}{2} > 2^{-N-2} = 2^{-L}.
\end{aligned}$$

However, this contradicts with the fact that $\|\overrightarrow{P_1P_2}\| \leq 2^{-L}$ so it follows that $\mathbf{R}^{\mathbf{v}_j}_{1,2^{-L}}(\mathbf{x}_j) \subseteq \mathbf{T}^{N,j}_{N,\alpha}(\Delta ABC)$ for all $1 \leq j \leq 2^N$. Note that $\{\mathbf{R}^{\mathbf{v}_j}_{1,2^{-L}}(\mathbf{x}_j + 2\mathbf{v}_j)\}_{j=1}^{2^N}$ is a mutually disjoint collection because the collection of the reflections of the (N,j)-subtriangles through $C_{N,j}$ is also mutually disjoint (and each translated rectangle is contained in its respective reflected subtriangle).

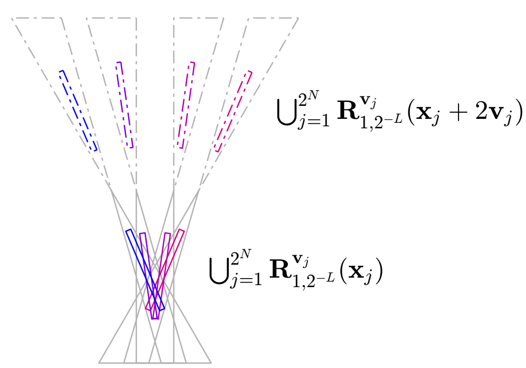


Figure 2.6: $\bigcup_{j=1}^{2^N} \mathbf{R}^{\mathbf{v}_j}_{1,2^{-L}}(\mathbf{x}_j)$ and $\bigcup_{j=1}^{2^N} \mathbf{R}^{\mathbf{v}_j}_{1,2^{-L}}(\mathbf{x}_j + 2\mathbf{v}_j)$ with $N = 2$ and $\alpha = \frac{3}{4}$. The dotted grey triangles are the reflections through the corresponding $C_{N,j}$'s.

It follows then that $m(\bigcup_{j=1}^{2^N} \mathbf{R}^{\mathbf{v}_j}_{1,2^{-L}}(\mathbf{x}_j)) \leq m(\mathbf{P}_{N,\alpha}(\Delta ABC)) < \frac{\varepsilon}{4}$ and $m(\bigcup_{j=1}^{2^N} \mathbf{R}^{\mathbf{v}_j}_{1,2^{-L}}(\mathbf{x}_j + 2\mathbf{v}_j)) = 2^N \cdot 2^{-L} = \frac{1}{4}$.

Notice that we have essentially proven the theorem except the estimates of the measures are quartered. Hence, to complete the proof, we will make 3 more 'sufficiently' disjoint copies of the first 2^N rectangles by letting $\mathbf{x}_i \coloneqq \mathbf{x}_{i-2^N} + (0,2)$ and $\mathbf{v}_i \coloneqq \mathbf{v}_{i-2^N}$ for all $2^N < i \leq 2^L$. $\square$

3 Counterexample for the Lebesgue Differentiation Theorem without nicely shrinking sets

An example of a collection of sets that causes the Lebesgue Differentiation Theorem to fail is the collection of rectangles centered at the origin. One can easily verify that the rectangles are not nicely shrinking, which means that the Generalized Lebesgue Differentiation Theorem would not be applicable. The following theorem is taken from [3] with some added detail in the proof.

Theorem 3.1. *Let $p \in [1, \infty)$, and let $\mathcal{R}$ be the collection of rectangles centered at $\mathbf{0} \in \mathbb{R}^2$. Then, there exists a real valued $f \in L^p(\mathbb{R}^2)$ such that*

$$\limsup_{\mathrm{diam}(R) \to 0} \frac{1}{m(R)} \int_R f(\mathbf{x} - \mathbf{y}) \; d\mathbf{y} = \infty$$

for almost every $\mathbf{x} \in \mathbb{R}^2$ with R ranging in $\mathcal{R}$.

Proof. Let $\mathcal{R}_\delta \coloneqq \{R \in \mathcal{R} : \mathrm{diam}(R) \leq \delta\}$ for all $\delta > 0$. Note that $\mathrm{diam}(R)$ is precisely the length of the diagonal of R, and that for all nets $\langle x_R \rangle_{R \in \mathcal{R}}$ in $\mathbb{R}$,

$$\limsup_{diam(R) \to 0} x_R = \inf_{R \in \mathcal{R}} \sup_{\mathrm{diam}(S) \leq \mathrm{diam}(R)} x_S = \inf_{\delta > 0} \sup_{R \in \mathcal{R}_\delta} x_R.$$

Hence, it suffices to show that there is a real valued $f \in L^p(\mathbb{R}^2)$ such that for almost every $\mathbf{x} \in \mathbb{R}^2$, $\sup_{R \in \mathcal{R}_\delta} \frac{1}{m(R)} \int_R f(\mathbf{x} - \mathbf{y}) \; d\mathbf{y} = \infty$ for all $\delta > 0$.

Now, recall the definition of the conditional Lebesgue measure, and let $\mathcal{C}_\delta \coloneqq \{m_R : R \in \mathcal{R}_\delta\}$ for each $\delta > 0$. Then, $\mathcal{C}_\delta$ is a collection of finite positive measures supported in a fixed compact set, and $\mathcal{M}_{\mathcal{C}_\delta}$ is a maximal operator on $L^1_{loc}(\mathbb{R}^2)$ by Proposition 1.11 for each $\delta > 0$. Note that $L^p(\mathbb{R}^2) \subseteq L^1_{loc}(\mathbb{R}^2)$ for all $p \in [1, \infty)$ by Hölder's inequality.

We will now show that $\mathcal{M}_{\mathcal{C}_\delta}$ is not weak type (p, p) for each $\delta > 0$ so that we can use Proposition 1.15 to construct a real valued f as above. To that end, fix $\delta > 0$, let $A > 0$, and let $\alpha \coloneqq \frac{1}{24}$. Then, there exists a Rectangular Besicovitch set $E \coloneqq \bigcup_{i=1}^{2^L} \mathbf{R}^{\mathbf{v}_i}_{1, 2^{-L}}(\mathbf{x}_i)$ such that $m(E) < \frac{144^{\frac{p-1}{p}} \alpha}{\delta^{\frac{2p-2}{p}} A^{\frac{1}{p}}}$ by Theorem 2.8. Hence, $\|\chi_{\frac{\delta}{12} E}\|^p_{L^p} = m(\frac{\delta}{12} E)^p = \frac{\delta^{2p}}{144^p} m(E)^p < \frac{\delta^2 \alpha^p}{144 A}$ (note that $\frac{\delta}{12} E$ is a scaling of E by $\frac{\delta}{12}$).

Suppose $\mathbf{x} \in \frac{\delta}{12} \mathbf{R}^{\mathbf{v}_i}_{1, 2^{-L}}(\mathbf{x}_i + 2\mathbf{v}_i)$ for some $1 \leq i \leq 2^L$. Let $R \coloneqq \frac{\delta}{12} \mathbf{R}^{\mathbf{v}_i}_{6, 2^{-L+1}}(\mathbf{0}) = \mathbf{R}^{\mathbf{v}_i}_{\frac{\delta}{2}, \frac{2^{-L-1}}{3}\delta}(\mathbf{0}) \in \mathcal{R}_\delta$. Then,

$$\begin{aligned}
(\mathcal{M}_{\mathcal{C}_\delta} \chi_{\frac{\delta}{12} E})(\mathbf{x}) &\geq \int \chi_{\frac{\delta}{12} E}(\mathbf{x} - \mathbf{y}) \; dm_R(\mathbf{y}) = \frac{1}{m(R)} \int_R \chi_{\frac{\delta}{12} E}(\mathbf{x} - \mathbf{y}) \; d\mathbf{y} \\
&= \frac{1}{m(R)} \int_{\frac{\delta}{12} E} \chi_R(\mathbf{x} - \mathbf{y}) \; d\mathbf{y} \\
&\qquad \text{(by commutativity of convolutions of } L^1 \text{ functions)} \\
&\geq \frac{1}{m(R)} \int_{\frac{\delta}{12} \mathbf{R}^{\mathbf{v}_i}_{1, 2^{-L}}(\mathbf{x}_i)} \chi_R(\mathbf{x} - \mathbf{y}) \; d\mathbf{y}
\end{aligned}$$

$$= \frac{m\left(\frac{\delta}{12}\mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i) \cap (\mathbf{x} - R)\right)}{m(R)}$$

$$= \frac{m\left(\frac{\delta}{12}\mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i) \cap (R + \mathbf{x})\right)}{m(R)} \qquad \text{(since } R = -R \text{ by symmetry)}$$

$$= \frac{m\left(\frac{\delta}{12}\mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i) \cap \frac{\delta}{12}\mathbf{R}^{\mathbf{v}_i}_{6,2^{-L+1}}\left(\frac{12\mathbf{x}}{\delta}\right)\right)}{m(R)}$$

$$= \frac{\delta^2 \cdot m\left(\mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i) \cap \mathbf{R}^{\mathbf{v}_i}_{6,2^{-L+1}}\left(\frac{12\mathbf{x}}{\delta}\right)\right)}{144m(R)} \qquad \text{(by scaling by } \tfrac{\delta}{12}\text{)}$$

$$= \frac{\delta^2 \cdot m\left(\mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i)\right)}{144m(R)}$$

(since $\frac{12\mathbf{x}}{\delta} \in \mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i + 2\mathbf{v}_i)$ so $\mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i) \subseteq \mathbf{R}^{\mathbf{v}_i}_{6,2^{-L+1}}(\frac{12\mathbf{x}}{\delta})$ (refer to Figure 3.1))

$$= \frac{1}{12}. \qquad \text{(since the ratio of measures evaluates to } \tfrac{1 \cdot 2^{-L}}{\frac{\delta}{2} \cdot \frac{2^{-L-1}\delta}{3}} = \tfrac{12}{\delta^2}\text{)}$$

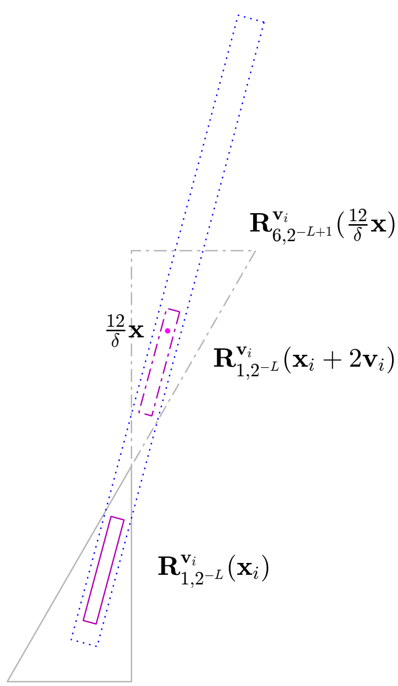


Figure 3.1: $\mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i) \cup \mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i + 2\mathbf{v}_i) \subseteq \mathbf{R}^{\mathbf{v}_i}_{6,2^{-L+1}}(\frac{12}{\delta}\mathbf{x})$,

By the arbitrariness of $\mathbf{x}$, we have that

$$\lambda_{\mathcal{M}_{\mathcal{C}_\delta}\chi_{\frac{\delta}{12}E}}(\alpha) \geq m\left(\frac{\delta}{12}\bigcup_{i=1}^{2^L}\mathbf{R}^{\mathbf{v}_i}_{1,2^{-L}}(\mathbf{x}_i + 2\mathbf{v}_i)\right) \qquad \text{(since } \alpha < \tfrac{1}{12}\text{)}$$

$$= \frac{\delta^2}{144} = \frac{A}{\alpha^p} \cdot \frac{\delta^2\alpha^p}{144A} \qquad \text{(by Theorem 2.8)}$$

$$> \frac{A}{\alpha^p}\|\chi_{\frac{\delta}{12}E}\|_{L^p}^p.$$

By the arbitrariness of A and δ, we indeed have that $\mathcal{M}_{\mathcal{C}_\delta}$ is not weak type (p,p) for all $\delta > 0$. In that regard, there exists $g_k \in L^p(\mathbb{R}^2)$ such that $(\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}} g_k)(\mathbf{x}) = \infty$ for almost every $\mathbf{x} \in \mathbb{R}^2$ for each $k \in \mathbb{N}$ by Proposition 1.15 (note that $\|g_k\|_{L^p} > 0$ otherwise $g_k = 0$ almost everywhere which would result in the contradicting statement that $\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}} g_k = 0$ almost everywhere).

We will now complete the proof of the theorem by constructing a real valued $f \in L^p(\mathbb{R}^2)$ that satisfies the hypothesis. Let $f_k \coloneqq \frac{|g_k|}{2^k\|g_k\|_{L^p}}$ for each $k \in \mathbb{N}$, and let $f \coloneqq \sum_{k=1}^\infty f_k$. Then, $f_k \geq 0$, $\|f_k\|_{L^p} = 2^{-k}\|\frac{|g_k|}{\|g_k\|_{L^p}}\|_{L^p} = 2^{-k}$ so $\|f\|_{L^p} \leq \sum_{k=1}^\infty \|f_k\|_{L^p} = 1$, and $\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}} f_k = \frac{2^{-k}}{\|g_k\|_{L^p}}\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}} g_k$ by sublinearity of $\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}}$ for each $k \in \mathbb{N}$. In that regard, we also have that $(\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}} f_k)(\mathbf{x}) = \infty$ for almost every $\mathbf{x} \in \mathbb{R}^2$ for each $k \in \mathbb{N}$.

Now, let $F \coloneqq \bigcap_{k=1}^\infty (\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}} f_k)^{-1}(\{\infty\})$. It is clear that $m(\mathbb{R}^2 \setminus F) = 0$ since the intersection of full measure sets has full measure. Hence, for any $\delta > 0$, there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$ which means that

$$\begin{aligned}
\sup_{R \in \mathcal{R}_\delta} \frac{1}{m(R)} \int_R f(\mathbf{x}-\mathbf{y})\, d\mathbf{y} &\geq (\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}} f)(\mathbf{x}) \\
&\text{(since } f \text{ is positive and the supremum is taken over a smaller collection)} \\
&\geq (\mathcal{M}_{\mathcal{C}_{\frac{1}{k}}} f_k)(\mathbf{x}) = \infty
\end{aligned}$$

for any $\mathbf{x} \in F$ so we are done. □

4 Counterexample for the Ball Multiplier

As hinted in an earlier section, the ball multiplier operator cannot be extended to an L^p bounded operator. The following results are from [3], but I have some added some more detail in the proofs.

Notation 4.1. *We write $H_{\mathbf{u}} \coloneqq \{\xi \in \mathbb{R}^n : \xi \cdot \mathbf{u} > 0\}$ to be the half open space in $\mathbb{R}^n$ with normal vector $\mathbf{u}$ (the dimension of $H_{\mathbf{u}}$ should be obvious from context).*

Lemma 4.2. *Let $p \in [1,\infty]$. Suppose there exists $A_p > 0$ such that*

$$\|S_{B_1(\mathbf{0})} f\|_{L^p} \leq A_p \|f\|_{L^p}$$

for all $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Let $M \in \mathbb{N}$ and let $f_1, \ldots, f_M \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Then,

$$\left\| \sqrt{\sum_{j=1}^M |S_{B_r(\mathbf{y}_j)} f_j|^2} \right\|_{L^p} \leq A_p \left\| \sqrt{\sum_{j=1}^M |f_j|^2} \right\|_{L^p}$$

for all $r > 0$ and $\mathbf{y}_1, \ldots, \mathbf{y}_M \in \mathbb{R}^n$, and

$$\left\| \sqrt{\sum_{j=1}^M |S_{H_{\mathbf{u}_j}} f_j|^2} \right\|_{L^p} \leq A_p \left\| \sqrt{\sum_{j=1}^M |f_j|^2} \right\|_{L^p}$$

for all $\mathbf{u}_1, \ldots, \mathbf{u}_M \in S^{n-1} \subseteq \mathbb{R}^n$ *where* S^{n-1} *is the unit* $(n-1)$*-sphere.*

PROOF. Let $r > 0$ and $\mathbf{y}_1, \ldots, \mathbf{y}_M \in \mathbb{R}^n$. Suppose $p < \infty$. Then,

$$\begin{aligned}
\int |(S_{B_r(\mathbf{0})} f)(\mathbf{x})|^p \, d\mathbf{x} &= \int |(\delta_r S_{B_1(\mathbf{0})} \delta_{r^{-1}} f)(\mathbf{x})|^p \, d\mathbf{x} && \text{(by Lemma 1.47.2)} \\
&= r^{-n} \int |(S_{B_1(\mathbf{0})} \delta_{r^{-1}} f)(\mathbf{u})|^p \, d\mathbf{u} && \text{(by scaling with } \mathbf{x} = \tfrac{\mathbf{u}}{r}\text{)} \\
&\leq r^{-n} A_p^p \int |(\delta_{r^{-1}} f)(\mathbf{u})|^p \, d\mathbf{u} \\
&\qquad \text{(by the assumption; note that } \delta_{r^{-1}} f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \text{ as well)} \\
&= A_p^p \int |f(\mathbf{x})|^p \, d\mathbf{x} && \text{(by scaling back to } \mathbf{x}\text{)}
\end{aligned}$$

which would mean that $\|S_{B_r(\mathbf{0})} f\|_{L^p} \leq A_p \|f\|_{L^p}$ as well for all $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

We will now introduce some more notation. Let $S^{2M-1} \subseteq \mathbb{C}^M$ be the unit $(2M-1)$-sphere. We endow S^{2M-1} with the usual spherical measure σ. Let $\langle \cdot, \cdot \rangle$ be the usual Hermitian inner product on $\mathbb{C}^M$, i.e. $\langle \mathbf{w}, \mathbf{z} \rangle = \mathbf{w} \cdot \overline{\mathbf{z}}$ for all $\mathbf{w}, \mathbf{z} \in \mathbb{C}^M$. For any operator T on $L^2(\mathbb{R}^n)$, we extend its domain and codomain such that $T(g_1, \ldots, g_M) = (Tg_1, \ldots, Tg_M)$ for all $g_1, \ldots, g_M \in L^2(\mathbb{R}^n)$ (note that $(g_1, \ldots, g_M)$ is the map $\mathbf{x} \mapsto (g_1(\mathbf{x}), \ldots, g_M(\mathbf{x}))$). We also define $\frac{\mathbf{0}}{0} \coloneqq \mathbf{e}_1$ where $\mathbf{e}_1 \coloneqq (1, 0, \ldots, 0) \in \mathbb{C}^m$ out of convenience.

Now, let $\mathbf{f} \coloneqq (e^{-2\pi i \mathbf{y}_1 \cdot \mathrm{id}} f_1, \ldots, e^{-2\pi i \mathbf{y}_M \cdot \mathrm{id}} f_M)$. Then, $\langle \mathbf{f}, \omega \rangle = \sum_{j=1}^M \overline{\omega_j} e^{-2\pi i \mathbf{y}_j \cdot \mathrm{id}} f_j \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $\omega \in S^{2M-1}$. Observe also that

$$|S_{B_r(\mathbf{0})} \langle \mathbf{f}, \omega \rangle (\mathbf{x})| = |\langle (S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x}), \omega \rangle| = |(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})| \cdot \left| \left\langle \frac{(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})}{|(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})|}, \omega \right\rangle \right|$$

by linearity. Similarly,

$$|\langle \mathbf{f}(\mathbf{x}), \omega \rangle| = |\mathbf{f}(\mathbf{x})| \cdot \left| \left\langle \frac{\mathbf{f}(\mathbf{x})}{|\mathbf{f}(\mathbf{x})|}, \omega \right\rangle \right|.$$

Given that $\|S_{B_r(\mathbf{0})} \langle \mathbf{f}, \omega \rangle \|_{L^p} \leq A_p \|\langle \mathbf{f}, \omega \rangle\|_{L^p}$ for all $\omega \in S^{2M-1}$ by the assumption, we have that

$$\int_{S^{2M-1}} \int |S_{B_r(\mathbf{0})} \langle \mathbf{f}, \omega \rangle (\mathbf{x})|^p \, d\mathbf{x} \, d\sigma(\omega) \leq A_p^p \int_{S^{2M-1}} \int |\langle \mathbf{f}(\mathbf{x}), \omega \rangle|^p \, d\mathbf{x} \, d\sigma(\omega)$$

by monotonicity. Hence,

$$\begin{aligned}
&\int |(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})|^p \int_{S^{2M-1}} \left| \left\langle \frac{(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})}{|(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})|}, \omega \right\rangle \right|^p \, d\sigma(\omega) \, d\mathbf{x} \\
&\qquad \leq A_p^p \int |\mathbf{f}(\mathbf{x})|^p \int_{S^{2M-1}} \left| \left\langle \frac{\mathbf{f}(\mathbf{x})}{|\mathbf{f}(\mathbf{x})|}, \omega \right\rangle \right|^p \, d\sigma(\omega) \, d\mathbf{x}
\end{aligned}$$

by the Tonelli Theorem. Now, for each $\mathbf{z} \in S^{2M-1}$, there exists a unitary transformation $U_{\mathbf{z}}$ such that $U_{\mathbf{z}}^* \mathbf{z} = \mathbf{e}_1$ (think of $U_{\mathbf{z}}$ as a rotation). Given that σ is rotation invariant and $|\det(U_{\mathbf{z}})| = 1$ for all $\mathbf{z} \in S^{2M-1}$, we have that

$$\int |(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})|^p \int_{S^{2M-1}} \left| \left\langle \frac{(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})}{|(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})|}, U_{\frac{(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})}{|(S_{B_r(\mathbf{0})} \mathbf{f})(\mathbf{x})|}} \omega \right\rangle \right|^p \, d\sigma(\omega) \, d\mathbf{x}$$

$$\leq A_p^p \int |\mathbf{f}(\mathbf{x})|^p \int_{S^{2M-1}} \left| \left\langle \frac{\mathbf{f}(\mathbf{x})}{|\mathbf{f}(\mathbf{x})|}, U_{\frac{\mathbf{f}(\mathbf{x})}{|\mathbf{f}(\mathbf{x})|}} \omega \right\rangle \right|^p \, d\sigma(\omega) \, d\mathbf{x}$$

thus

$$\int |(S_{B_r(\mathbf{0})}\mathbf{f})(\mathbf{x})|^p \int_{S^{2M-1}} |\langle \mathbf{e}_1, \omega \rangle|^p \, d\sigma(\omega) \, d\mathbf{x} \leq A_p^p \int |\mathbf{f}(\mathbf{x})|^p \int_{S^{2M-1}} |\langle \mathbf{e}_1, \omega \rangle|^p \, d\sigma(\omega) \, d\mathbf{x}$$

since unitary transformations preserve inner products as well (note that we applied U^* to the terms in the inner products). Given that $|\langle \mathbf{e}_1, \omega \rangle|^p = |\overline{\omega_1}|^p > 0$ for almost every $\omega \in S^{2M-1}$, we have that $\int_{S^{2M-1}} |\langle \mathbf{e}_1, \omega \rangle|^p \, d\sigma(\omega) > 0$. Hence, by cancellation, we have that

$$\int |(S_{B_r(\mathbf{0})}\mathbf{f})(\mathbf{x})|^p \, d\mathbf{x} \leq A_p^p \int |\mathbf{f}(\mathbf{x})|^p \, d\mathbf{x}$$

which means that

$$\left\| \sqrt{\sum_{j=1}^M |S_{B_r(\mathbf{0})}(e^{-2\pi i \mathbf{y}_j \cdot \mathrm{id}} f_j)|^2} \right\|_{L^p} \leq A_p \left\| \sqrt{\sum_{j=1}^M |e^{-2\pi i \mathbf{y}_j \cdot \mathrm{id}} f_j|^2} \right\|_{L^p} = A_p \left\| \sqrt{\sum_{j=1}^M |f_j|^2} \right\|_{L^p}$$

since $|e^{i\theta}| = 1$ for all $\theta \in \mathbb{R}$. Therefore,

$$\begin{aligned} \left\| \sqrt{\sum_{j=1}^M |S_{B_r(\mathbf{y}_j)} f_j|^2} \right\|_{L^p} &= \left\| \sqrt{\sum_{j=1}^M |e^{2\pi i \mathbf{y}_j \cdot \mathrm{id}} S_{B_r(\mathbf{0})}(e^{-2\pi i \mathbf{y}_j \cdot \mathrm{id}} f_j)|^2} \right\|_{L^p} && \text{(by Lemma 1.47.1)} \\ &= \left\| \sqrt{\sum_{j=1}^M |S_{B_r(\mathbf{0})}(e^{-2\pi i \mathbf{y}_j \cdot \mathrm{id}} f_j)|^2} \right\|_{L^p} \leq A_p \left\| \sqrt{\sum_{j=1}^M |f_j|^2} \right\|_{L^p}. \end{aligned}$$

Suppose now that $p = \infty$. Then, by Lemma 1.47.2, $\|S_{B_r(\mathbf{0})} f\|_{L^\infty} = \|\delta_r S_{B_1(\mathbf{0})} \delta_{r^{-1}} f\|_{L^\infty} = \|S_{B_1(\mathbf{0})} \delta_{r^{-1}} f\|_{L^\infty} \leq A_\infty \|\delta_{r^{-1}} f\|_{L^\infty} = A_\infty \|f\|_{L^\infty}$. Note as above that $\|S_{B_r(\mathbf{0})} \langle \mathbf{f}, \omega \rangle\|_{L^\infty} \leq A_\infty \|\langle \mathbf{f}, \omega \rangle\|_{L^\infty}$ since $\langle \mathbf{f}, \omega \rangle \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Hence,

$$\sup_{|\omega|=1} \|S_{B_r(\mathbf{0})} \langle \mathbf{f}, \omega \rangle\|_{L^\infty} \leq A_\infty \sup_{|\omega|=1} \|\langle \mathbf{f}, \omega \rangle\|_{L^\infty}.$$

Since we can commute the supremum with the essential supremum, we see that

$$\left\| |S_{B_r(\mathbf{0})}\mathbf{f}| \cdot \sup_{|\omega|=1} \left| \left\langle \frac{S_{B_r(\mathbf{0})}\mathbf{f}}{|S_{B_r(\mathbf{0})}\mathbf{f}|}, \omega \right\rangle \right| \right\|_{L^\infty} \leq A_\infty \left\| |\mathbf{f}| \cdot \sup_{|\omega|=1} \left| \left\langle \frac{\mathbf{f}}{|\mathbf{f}|}, \omega \right\rangle \right| \right\|_{L^\infty}.$$

One can easily check that $\sup_{|\omega|=1} \left| \left\langle \frac{(S_{B_r(\mathbf{0})}\mathbf{f})(\mathbf{x})}{|(S_{B_r(\mathbf{0})}\mathbf{f})(\mathbf{x})|}, \omega \right\rangle \right| = \sup_{|\omega|=1} \left| \left\langle \frac{\mathbf{f}(\mathbf{x})}{|\mathbf{f}(\mathbf{x})|}, \omega \right\rangle \right| > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ since the terms in the inner product have norm 1. Therefore, by cancellation, we have that $\||S_{B_r(\mathbf{0})}\mathbf{f}|\|_{L^\infty} \leq A_\infty \||\mathbf{f}|\|_{L^\infty}$. Hence, the same argument towards the end of the case for $p < \infty$ also applies. By the arbitrariness of r and $\mathbf{y}_1, \ldots, \mathbf{y}_M$, we have proven the first part of the lemma.

Now, let $\mathbf{u}_1, \ldots, \mathbf{u}_M \in S^{n-1}$. Note that $B_N(N\mathbf{u}_j) \subseteq B_{N+1}((N+1)\mathbf{u}_j)$ for all $N \in \mathbb{N}$, and that $H_{\mathbf{u}_j} = \bigcup_{N=1}^\infty B_N(N\mathbf{u}_j)$. Hence, $\mathcal{S}_{H_{\mathbf{u}_j}} f_j = \lim_{N\to\infty} \mathcal{S}_{B_N(N\mathbf{u}_j)} f_j$ with respect to the L^2 norm for each j by Lemma 1.47.4. Since L^2 convergence implies convergence almost everywhere on a subsequence, we can inductively construct a subsequence (by building subsequences upon subsequences) such that $(S_{H_{\mathbf{u}_j}} f_j)(\mathbf{x}) = \lim_{k\to\infty}(S_{B_{N_k}(N_k\mathbf{u}_j)})(\mathbf{x})$ for almost every $\mathbf{x}\in\mathbb{R}^n$ for all $j = 1,\ldots,M$. By continuity of taking powers, sums and absolute values, it follows that $\sqrt{\sum_{j=1}^M |(S_{H_{\mathbf{u}_j}} f_j)(\mathbf{x})|^2} = \lim_{k\to\infty}\sqrt{\sum_{j=1}^M |(S_{B_{N_k}(N_k\mathbf{u}_j)} f_j)(\mathbf{x})|^2}$ for almost every $\mathbf{x}\in\mathbb{R}^n$. Therefore, by Fatou's Lemma,

$$\left\|\sqrt{\sum_{j=1}^M |S_{H_{\mathbf{u}_j}} f_j|^2}\right\|_{L^p} \le \liminf_{k\to\infty}\left\|\sqrt{\sum_{j=1}^M |S_{B_{N_k}(N_k\mathbf{u}_j)} f_j|^2}\right\|_{L^p} \le A_p\left\|\sqrt{\sum_{j=1}^M |f_j|^2}\right\|_{L^p}$$

if $p<\infty$. If $p=\infty$, then since $\sqrt{\sum_{j=1}^M |(S_{B_{N_k}(N_k\mathbf{u}_j)} f_j)(\mathbf{x})|^2} \le \left\|\sqrt{\sum_{j=1}^M |(S_{B_{N_k}(N_k\mathbf{u}_j)} f_j)(\mathbf{x})|^2}\right\|_{L^\infty}$ for almost every $\mathbf{x}\in\mathbb{R}^n$ for all $k\in\mathbb{N}$, we get the same result as above but with p replaced by ∞. By the arbitrariness of $\mathbf{u}_1,\ldots,\mathbf{u}_M$, we are done. □

Lemma 4.3. *Let $\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(\mathbf{x}_0)\subseteq\mathbb{R}^2$ where $N\in\mathbb{N}$, $\mathbf{v}\in S^1$ and $\mathbf{x}_0\in\mathbb{R}^2$. Then,*

$$\left|\left(S_{H_{\mathbf{v}}}\chi_{\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(\mathbf{x}_0)}\right)(\mathbf{x})\right| \ge \frac{1}{20\pi}\chi_{\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(\mathbf{x}_0+2\mathbf{v})}(\mathbf{x})$$

for almost every $\mathbf{x}\in\mathbb{R}^2$.

Proof. Given that $\chi_{(-\frac12,\frac12)}\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, we have that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left\|S_{(0,\infty)}\chi_{(-\frac12,\frac12)} - \int_0^\infty \widehat{\chi_{(-\frac12,\frac12)}}(\xi)e^{2\pi i(\cdot+i\varepsilon)\xi}\,d\xi\right\|_{L^2}^2\\
&= \lim_{\varepsilon\downarrow 0}\left\|\mathcal{P}^{-1}\left(\chi_{(0,\infty)}\widehat{\chi_{(-\frac12,\frac12)}}\right) - \mathcal{P}^{-1}\left(\chi_{(0,\infty)}\widehat{\chi_{(-\frac12,\frac12)}}e^{-2\pi\varepsilon\cdot\mathrm{id}}\right)\right\|_{L^2}^2\\
&\qquad\text{(by Corollary 1.44 and the Plancherel Theorem (Theorem 1.39))}\\
&= \lim_{\varepsilon\downarrow 0}\left\|\chi_{(0,\infty)}\widehat{\chi_{(-\frac12,\frac12)}}\left(1-e^{-2\pi\varepsilon\cdot\mathrm{id}}\right)\right\|_{L^2}^2 \qquad\text{(by the Plancherel Theorem again)}\\
&= \int_0^\infty \lim_{\varepsilon\downarrow 0}|\widehat{\chi_{(-\frac12,\frac12)}}(\xi)|^2(1-e^{-2\pi\varepsilon\xi})^2\,d\xi = 0.
\end{aligned}$$

(by the Dominated Convergence Theorem; note that $1-e^{-2\pi\varepsilon\cdot\mathrm{id}}\to 0$ pointwise as $\varepsilon\downarrow 0$)

Hence, $S_{(0,\infty)}\chi_{(-\frac12,\frac12)} = \lim_{\varepsilon\downarrow 0}\int_0^\infty \widehat{\chi_{(-\frac12,\frac12)}}(\xi)e^{2\pi i(\cdot+i\varepsilon)\xi}\,d\xi$ with respect to the L^2 norm. In that regard, there exists a decreasing sequence $\{\varepsilon_j\}_{j=1}^\infty\subseteq(0,\frac12)$ such that $(S_{(0,\infty)}\chi_{(-\frac12,\frac12)})(x) = \lim_{j\to\infty}\int_0^\infty \widehat{\chi_{(-\frac12,\frac12)}}(\xi)e^{2\pi i(x+i\varepsilon_j)\xi}\,d\xi$ for almost every $x\in\mathbb{R}$. Note also that for all $\varepsilon\in(0,\frac12)$ and $x\in(1,\infty)$,

$$\begin{aligned}
\left|\int_0^\infty \widehat{\chi_{(-\frac12,\frac12)}}(\xi)e^{2\pi i(x+i\varepsilon)\xi}\,d\xi\right| &= \left|\int_0^\infty\left(\int_{-\frac12}^{\frac12} e^{-2\pi i\xi y}\,dy\right)e^{2\pi i(x+i\varepsilon)\xi}\,d\xi\right|\\
&= \left|\int_{-\frac12}^{\frac12}\left(\int_0^\infty e^{2\pi i(x+i\varepsilon-y)\xi}\,d\xi\right)\,dy\right| \qquad\text{(by Fubini's Theorem)}\\
&= \frac{1}{2\pi}\left|\int_{-\frac12}^{\frac12}\frac{1}{y-x-i\varepsilon}\,dy\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{y-x}{(y-x)^2+\varepsilon^2}\ dy + i\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\varepsilon}{(y-x)^2+\varepsilon^2}\ dy\right| \\
&\geq \frac{1}{2\pi}\frac{|c_x - x|}{(c_x-x)^2+\varepsilon^2} \\
&\qquad\qquad \text{(by the mean value theorem where } c_x \in (-\tfrac{1}{2},\tfrac{1}{2})\text{)} \\
&\geq \frac{1}{4\pi}\frac{1}{x-c_x} \qquad \text{(since } x - c_x > \tfrac{1}{2} > \varepsilon\text{)} \\
&\geq \frac{1}{8\pi x}. \qquad \text{(since } x - c_x \leq x + \tfrac{1}{2} \leq 2x\text{)}
\end{aligned}
$$

In that regard, $|(S_{(0,\infty)}\chi_{(-\frac{1}{2},\frac{1}{2})})(x)| \geq \frac{1}{8\pi x}$ for almost every $x \in (1,\infty)$. Hence,

$$
\left|\left(S_{(0,\infty)}\chi_{(-\frac{1}{2},\frac{1}{2})}\right)(x)\right| \geq \frac{1}{8\pi x}\chi_{(1,\infty)}(x) \geq \frac{1}{20\pi}\chi_{(\frac{3}{2},\frac{5}{2})}(x)
$$

for almost every $x \in \mathbb{R}$. By Lemma 1.47.5, we have that

$$
\begin{aligned}
\left|\left(S_{(0,\infty)\times\mathbb{R}}\chi_{(-\frac{1}{2},\frac{1}{2})\times(-2^{-N},2^{-N})}\right)(x_1,x_2)\right| &= \left|\left(S_{(0,\infty)}\chi_{(-\frac{1}{2},\frac{1}{2})}\right)(x_1)\cdot\left(S_{\mathbb{R}}\chi_{(-2^{-N},2^{-N})}\right)(x_2)\right| \\
&\geq \frac{1}{20\pi}\chi_{(\frac{3}{2},\frac{5}{2})}(x_1)\cdot\chi_{(-2^{-N},2^{-N})}(x_2) \\
&\qquad\qquad \text{(since } S_{\mathbb{R}} = \mathcal{P}^{-1}\mathcal{P} = \mathrm{id}\text{)} \\
&= \frac{1}{20\pi}\chi_{(\frac{3}{2},\frac{5}{2})\times(-2^{-N},2^{-N})}(x_1,x_2)
\end{aligned}
$$

for almost every $(x_1,x_2) \in \mathbb{R}^2$. It remains to show that rotations and translations preserves this result (note that $(0,\infty)\times\mathbb{R} = H_{\mathbf{e}_1}$, that $(-\frac{1}{2},\frac{1}{2})\times(-2^{-N},2^{-N}) = \mathbf{R}^{\mathbf{e}_1}_{1,2^{-N}}(\mathbf{0})$, and that $(\frac{3}{2},\frac{5}{2})\times(-2^{-N},2^{-N}) = \mathbf{R}^{\mathbf{e}_1}_{1,2^{-N}}(2\mathbf{e}_1)$ where $\mathbf{e}_1 = (1,0) \in \mathbb{R}^2$). In that regard,

$$
\begin{aligned}
\left|\left(S_{H_{\mathbf{e}_1}}\chi_{\mathbf{R}^{\mathbf{e}_1}_{1,2^{-N}}(\mathbf{0})}\right)(\mathbf{x})\right| &\geq \frac{1}{20\pi}\chi_{\mathbf{R}^{\mathbf{e}_1}_{1,2^{-N}}(2\mathbf{e}_1)}(\mathbf{x}) \\
\left|\left(\rho_{O^T}S_{H_{\mathbf{e}_1}}\chi_{\mathbf{R}^{\mathbf{e}_1}_{1,2^{-N}}(\mathbf{0})}\right)(\mathbf{x})\right| &\geq \frac{1}{20\pi}\left(\rho_{O^T}\chi_{\mathbf{R}^{\mathbf{e}_1}_{1,2^{-N}}(2\mathbf{e}_1)}\right)(\mathbf{x}) \\
&\qquad \text{(where } \mathbf{v} = O\mathbf{e}_1 \text{ for some orthogonal transformation } O\text{)} \\
\left|\left(S_{H_{\mathbf{v}}}\chi_{\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(\mathbf{0})}\right)(\mathbf{x})\right| &\geq \frac{1}{20\pi}\chi_{\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(2\mathbf{v})}(\mathbf{x}) \qquad \text{(by Lemma 1.47.3)} \\
\left|\left(\tau_{\mathbf{x}_0}S_{H_{\mathbf{v}}}\chi_{\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(\mathbf{0})}\right)(\mathbf{x})\right| &\geq \frac{1}{20\pi}\left(\tau_{\mathbf{x}_0}\chi_{\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(2\mathbf{v})}\right)(\mathbf{x}) \\
\left|\left(S_{H_{\mathbf{v}}}\chi_{\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(\mathbf{x}_0)}\right)(\mathbf{x})\right| &\geq \frac{1}{20\pi}\chi_{\mathbf{R}^{\mathbf{v}}_{1,2^{-N}}(\mathbf{x}_0+2\mathbf{v})}(\mathbf{x}) \qquad \text{(by Lemma 1.47.1)}
\end{aligned}
$$

for almost every $\mathbf{x} \in \mathbb{R}^2$ so we are done. □

Theorem 4.4. *Let $n \geq 2$ and $p \in [1,\infty] \setminus \{2\}$. Then, $S_{B_1(\mathbf{0})}|_{L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)}$ is not extendable to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

Proof. Suppose the operator is extendable with A_p as the bounding constant. If we restrict the domain of the extension back to $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, we see that

$$
\|S_{B_1(\mathbf{0})}f\|_{L^p} \leq A_p\|f\|_{L^p}
$$

for all $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. We will use the Rectangular Besicovitch set to show directly that the above inequality cannot hold.

Suppose $p < 2$. Then, there exists a Rectangular Besicovitch set $E \coloneqq \bigcup_{j=1}^{2^L} \mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j)$ such that $m(E) < \left(\frac{1}{20\pi A_p}\right)^{\frac{2p}{2-p}}$ by Theorem 2.8. Hence,

$$\frac{1}{20\pi} = \frac{1}{20\pi} m\left(\bigcup_{j=1}^{2^L} \mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j + 2\mathbf{v}_j) \times [0,1]^{n-2}\right) = \left\| \sum_{j=1}^{2^L} \frac{1}{20\pi} \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j+2\mathbf{v}_j)\times[0,1]^{n-2}} \right\|_{L^p}$$

(by Theorem 2.8)

$$= \left\| \sqrt{\sum_{j=1}^{2^L} \left(\frac{1}{20\pi}\right)^2 \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j+2\mathbf{v}_j)} \chi_{[0,1]^{n-2}}} \right\|_{L^p}$$

$$\leq \left\| \sqrt{\sum_{j=1}^{2^L} \left| S_{H_{\mathbf{v}_j}} \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j)} \cdot \chi_{[0,1]^{n-2}} \right|^2} \right\|_{L^p} = \left\| \sqrt{\sum_{j=1}^{2^L} \left| S_{H_{\mathbf{v}_j}} \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j)} \cdot S_{\mathbb{R}^{n-2}} \chi_{[0,1]^{n-2}} \right|^2} \right\|_{L^p}$$

(by Lemma 4.3; note that $S_{\mathbb{R}^{n-2}} = \mathcal{P}^{-1}\mathcal{P} = \mathrm{id}$)

$$= \left\| \sqrt{\sum_{j=1}^{2^L} \left| S_{H_{(\mathbf{v}_j,\mathbf{0})}} \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j)\times[0,1]^{n-2}} \right|^2} \right\|_{L^p} \qquad \text{(by Lemma 1.47.5; note that } \mathbf{0} \in \mathbb{R}^{n-2}\text{)}$$

$$\leq A_p \left\| \sqrt{\sum_{j=1}^{2^L} \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j)\times[0,1]^{n-2}}} \right\|_{L^p} = A_p \left\| \left(\sum_{j=1}^{2^L} \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j)\times[0,1]^{n-2}}\right)^{\frac{p}{2}} \right\|_{L^1}^{\frac{1}{p}} \qquad \text{(by Lemma 4.2)}$$

$$\leq A_p \|\chi_{E\times[0,1]^{n-2}}\|_{L^{\frac{2}{2-p}}}^{\frac{1}{p}} \left\| \left(\sum_{j=1}^{2^L} \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j)\times[0,1]^{n-2}}\right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}}^{\frac{1}{p}} \qquad \text{(by Hölder's inequality)}$$

$$= A_p m(E)^{\frac{2-p}{2p}} \left\| \sum_{j=1}^{2^L} \chi_{\mathbf{R}_{1,2^{-L}}^{\mathbf{v}_j}(\mathbf{x}_j)\times[0,1]^{n-2}} \right\|_{L^1}^{\frac{1}{2}} = A_p m(E)^{\frac{2-p}{2p}} < \frac{1}{20\pi}$$

which is a contradiction. In that regard, $S_{B_1(\mathbf{0})}|_{L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)}$ is not extendable if $p < 2$.

Suppose $p > 2$. Let q be the conjugate exponent of p (i.e. $q = \frac{p}{p-1}$; note that $q \in [1,2)$). Note that operators are bounded $L^p \to L^p$ iff they are bounded $L^q \to L^q$ (see [2] Theorem 2.5.7). However, we already proved that this is not possible in the earlier case (note that $q < 2$). Therefore, by contradiction, we are done. $\square$

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