MATH 240 Qualifying Exam

September 14, 2021

Instructions: 3 hours, open book/notes (only Folland or personal lecture notes; no HW or other solutions). You may use without proofs results proved in Folland up to Section 8.3. Present your solutions clearly, with appropriate detail.

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1. (30 pts) Let $f \in C(\mathbb{R})$ and let $A \subseteq \mathbb{R}$ be a Borel set such that f is differentiable at each $x \in \mathbb{R} \setminus A$ and $f'(x) = 0$ for all such x.

(a) If A is closed and countable, show that f is constant.

(b) If A has Lebesgue measure 0, must f be constant? Prove or find a counterexample.

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2. (30 pts) Does there exist a Borel measurable function $f : \mathbb{R} \to [0, \infty)$ such that $\int_a^b f(x)\,dx = \infty$ for all real numbers $a < b$? Either find an example or show that no such f exists.

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3. (30 pts) Let $p \in (1, \infty)$, and for $f \in L^p(\mathbb{R})$ define $Tf(x) := \int_0^1 f(x + y)\,dy$.

(a) Show that $\|Tf\|_p \leq \|f\|_p$, and equality holds if and only if $f = 0$ almost everywhere.

(b) Prove that $(I - T)(L^p(\mathbb{R})) \neq L^p(\mathbb{R})$, where I is the identity map on $L^p(\mathbb{R})$.

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4. (30 pts) Let functions $f_n \in C([0, 1])$ satisfy $\sup_n |f_n(x)| < \infty$ for each $x \in [0, 1]$. Show that there are $0 \leq a < b \leq 1$ such that $\sup_n \|f_n \chi_{(a,b)}\|_u < \infty$.

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5. (20 pts) Let $f_n, f \in L^2(\mathbb{R})$ satisfy $f_n \to f$ weakly and $\|f_n\|_2 \to \|f\|_2$ as $n \to \infty$. Show that $f_n \to f$ in $L^2(\mathbb{R})$.

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6. (30 pts) Let δ_x denote the Dirac delta mass at $x \in \mathbb{R}^n$. Let $\{x_j\}_{j=1}^\infty$ be a sequence in $\mathbb{R}^n$, $\{c_j\}_{j=1}^\infty$ a sequence of positive numbers, and μ the Borel measure on $\mathbb{R}^n$ corresponding to the series $\sum_{j=1}^\infty c_j \delta_{x_j}$. Prove that μ is Radon if and only if for all convergent subsequences $\{x_{j_k}\}_{k=1}^\infty$ it holds that $\sum_{k=1}^\infty c_{j_k} < \infty$.

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7. (30 pts) For any $f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$ show that

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx \right) \left(\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2} \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^2 .$$

Here $\hat{f}$ is the Fourier transform of f and dx, $d\xi$ represent the Lebesgue measure on $\mathbb{R}$.
