Numerical Analysis Qualifying Exam
Fall 2021

September 8, 2021

Instructions:

- There are 8 problems, worth a total of 200 points.
- The 3 hour exam starts at
  - **Wednesday, September 8 5:00 PM PST**

  and concludes at
  - **Wednesday, September 8 8:00 PM PST**.

  You then have 15 additional minutes to upload your work to Gradescope. After this, your work will be considered late and will **not be accepted**.
- You must work by yourself on these problems, with **no assistance** from other people. You are allowed all other resources that exist prior to this exam, including books and notes, online or otherwise, and calculators or computers, however, your answers should be written up in **your own words** and not copied from any source.
- Have pen or pencil ready, and enough paper. Also have ready the necessary equipment to promptly upload the finished pages onto Gradescope at the conclusion of the exam.
- Start each question on a separate page, upload clear and legible work, and properly label the page locations of each problem on Gradescope. If you do not, you risk losing points.
- You must show sufficient **detail** in your work to receive full credit.
- For questions, prior to or during the exam, email `lcheng@math.ucsd.edu`

1. (25 pts) Let $A \in \mathbf{R}^{n\times n}$. Describe a step-by-step procedure for computing, when one exists, a factorization $A = UL$, where $U \in \mathbf{R}^{n\times n}$ is upper triangular and $L \in \mathbf{R}^{n\times n}$ is lower triangular. Explain why it works and demonstrate it on the matrix

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 5 & 1 & -4 \\ -4 & 1 & 2 \end{bmatrix}.$$

2. (25 pts) Suppose $A \in \mathbf{R}^{m\times n}$, for $m > n$, has the factorization $A = UBV^T$, where $U \in \mathbf{R}^{m\times m}, V \in \mathbf{R}^{n\times n}$ are orthogonal and $B \in \mathbf{R}^{m\times n}$ satisfies

$$B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix},$$

with $B_{11} \in \mathbf{R}^{p\times p}$ nonsingular and $0 < p < n$. For given $b \in \mathbf{R}^m$, fully describe, in terms of the factorization, the set of $x \in \mathbf{R}^n$ that minimizes $\|b - Ax\|_2$.

3. (25 pts) Let $A \in \mathbf{R}^{n\times n}$ a symmetric matrix with eigenvalues λ_j, $1 \le j \le n$, satisfying

$$|\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_n|.$$

Consider the power method's iterations written in the form

$$v^{(k+1)} = \frac{Av^{(k)}}{\alpha_k},$$

for some $\alpha_k \in \mathbf{R}$, and suppose the initial guess $v^{(0)}$ satisfies $(v^{(0)})^T x_1 \neq 0$, where $x_1 \in \mathbf{R}^n$ is an eigenvector corresponding to λ_1. Prove there exists integer $M \ge 0$ and $C \in \mathbf{R}$ such that

$$\left| \frac{(v^{(k)})^T A v^{(k)}}{(v^{(k)})^T v^{(k)}} - \lambda_1 \right| \le C \left| \frac{\lambda_2}{\lambda_1} \right|^{2k},$$

for all $k \ge M$.

4. (25 pts) Consider the fixed point function

$$g(x) = \frac{x}{6} + \frac{1}{3x^2} + \frac{1}{2},$$

with fixed point at $x = 1$. Find an interval $[a, b]$, with $a < b$, such that the sequence generated by the fixed point iterations,

$$x_{k+1} = g(x_k),$$

is guaranteed converge to 1 for all initial guesses $x_0 \in [a, b]$.

5. (25 pts) Let $f \in C^2[a, b]$, with $a < b$, satisfy $f'(a) = f'(b)$ and $f''(a) = f''(b)$. Consider nodes

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let S denote a cubic spline for the data points $(x_i, f(x_i))$, $0 \leq i \leq n$, that satisfies $S'(a) = S'(b)$ and $S''(a) = S''(b)$. Prove

$$\int_a^b [S''(x)]^2 \, dx \leq \int_a^b [f''(x)]^2 \, dx.$$

6. (25 pts) Let $f(x) \in C^\infty([a, b])$, for some $a < b$, and consider evenly-spaced nodes

$$a = x_0 < x_1 < \cdots < x_n = b$$

with stepsize h. Now, for $1 \leq i \leq n - 2$, let

$$A(x_i, h) = \frac{1}{h}[\alpha f(x_{i-1}) + \beta f(x_i) + \gamma f(x_{i+2})],$$

be a second-order accurate approximation of $f'(x_i)$, for some $\alpha, \beta, \gamma \in \mathbf{R}$. For the case

$$f(x) = \cos(2x)$$

and $[a, b] = [0, \pi]$, use error bounds to find an $n \geq 3$, the smaller the better, such that

$$|A(x_i, h) - f'(x_i)| \leq 10^{-20}$$

holds for all $1 \leq i \leq n - 2$.

7. (25 pts) Consider the ODE

$$y' = f(t, y),$$

for $t \in [t_0, T]$ with $t_0 < T$ and $y : [t_0, T] \to \mathbf{R}$, and where f is continuous in

$$D = \{(t, y) \mid t \in [t_0, T], y \in (-\infty, \infty)\},$$

and Lipschitz continuous in variable y in D. Let

$$t_0 < t_1 < \cdots < t_n = T$$

be evenly-spaced nodes with stepsize h. Detail how the following Adams-Moulton iterative formula,

$$y_{i+1} = y_i + \frac{h}{12}[5f(t_{i+1}, y_{i+1}) + 8f(t_i, y_i) - f(t_{i-1}, y_{i-1})],$$

for calculating approximations y_i of $y(t_i)$, for $1 \leq i \leq n-1$, derives from the ODE and polynomial interpolation.

8. (25 pts) Consider the initial value problem with:

- ODE:

$$y' = f(t, y),$$

  for $t \in [t_0, T]$ with $t_0 < T$ and $y : [t_0, T] \to \mathbf{R}$, and where f is continuous in

$$D = \{(t, y) \mid t \in [t_0, T], y \in (-\infty, \infty)\},$$

  and Lipschitz continuous in variable y in D;
- Initial value $y(t_0) = y_0$.

Now consider applying to it the Runge-Kutta method defined by the Butcher tableau:

$$\begin{array}{c|cc} \alpha_0 & 2 & 0 \\ \alpha_1 & 1 & 2 \\ \hline & \frac{5}{2} & -\frac{3}{2} \end{array}$$

for some $\alpha_0, \alpha_1 \in \mathbf{R}$. Determine which points in the negative real axis of the complex plane,

$$\{z \in \mathbf{C} \mid z \in \mathbf{R}, z < 0\} \subset \mathbf{C},$$

are contained in the region of absolute stability of the method.