Complex Analysis Qualifying Exam – Fall 2021

Name: ____________________________________

Student ID: ________________________________

Instructions:

You do not have to reprove any results from Conway. However, if using a homework problem, please make sure you reprove it.

You have 180 minutes to complete the test.

Notation: $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

Problem 1. [*10 points.*]

Compute the following integral via residues

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx.$$

Please explain the necessary estimates.

Problem 2. [*10 points; 5, 5.*]

Let $K = \{z \in \mathbb{C} : |z| \leq 3,\ |z-1| \geq 1,\ |z+1| \geq 1\}$.

(i) True/false: every holomorphic function in a neighborhood of K is the local uniform limit on K of a sequence of polynomials. Please justify your answer.

(ii) Determine, with justification, the set

$$\widehat{K} = \{z \in \mathbb{C} : |p(z)| \leq \sup_{w \in K} |p(w)| \text{ for all polynomials } p\}.$$

Problem 3. [*10 points; 5, 5.*]

Let $a, b : \mathbb{C} \to \mathbb{C}$ be entire functions. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$f(z)^2 + a(z)f(z) + b(z) = 0.$$

(i) Show that if a, b have finite order, then f is also of finite order.

(ii) Show that if a, b are polynomials, then f is also a polynomial.

Problem 4. [*10 points.*]

Let $a_n \in \Delta$ be a sequence such that $a_n \to 1$. Let $f_n : \Delta \to \Delta$ be a sequence of holomorphic functions such that $f_n(0) = a_n$. Show that $f_n \to 1$ uniformly on compact subsets of Δ.

Problem 5. [*10 points; 5, 5.*]

Let $G = \{z = x + iy : x > 0,\ y > 0,\ xy < 1\}$.

(i) Construct a biholomorphism between G and the trip $S = \{z = x + iy,\ 0 < y < 1\}$.

(ii) Construct an unbounded continuous function $u : \overline{G} \to \mathbb{R}$, harmonic in G, and such that u vanishes on ∂G.

Problem 6. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)| = 1$ for $|z| = 1$. Show that there exists $a \in \mathbb{C}$ and $n \geq 0$ such that $f(z) = az^n$.

Problem 7. [*10 points.*]

Describe all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{R}$ we have $|f(z)| = 1$.