Fall Applied Algebra Qualifying Exam: Part A

5:00pm–8:00pm (PDT), via Zoom. Meeting ID: 943 0514 4675
Tuesday September 7th, 2021

- Write your name and student PID at the top right corner of each page of your submission.
- Do all four problems. Show your work.
- This part of the exam will represent 40% of the total score.
- Your completed examination must be uploaded to Gradescope while you are connected to Zoom. You may leave the meeting once the Proctor has checked that your exam has been uploaded.
- It is your responsibility to check that any uploaded material is both complete and legible.
- By participating in this exam you are agreeing to abide by the UCSD Policy on Academic Integrity. The instructors reserve the right to require a follow-up oral examination.
- This is a closed-book examination. No cell-phone or Internet aids.
- Please keep your camera turned on throughout the exam.
- Notation:
  - $\mathcal{M}_{m,n}$ denotes the set of $m \times n$ matrices with complex components.
  - $\mathcal{M}_n$ denotes the set $\mathcal{M}_{m,n}$ with $m = n$.
  - $\mathbb{C}^n$ is the set of column vectors with n complex components.
  - x^H is the Hermitian transpose of a vector or matrix x.
  - $\text{eig}(A)$ is the set of eigenvalues of the matrix A (counting multiplicities).
  - $\text{Re}(\alpha)$ is the real part of the complex scalar α.
  - $\text{Im}(\alpha)$ is the imaginary part of the complex scalar α.

Question 1.

(a) (3 points) State, *but do not prove*, the Schur decomposition theorem for a matrix $A \in M_n$.

(b) (12 points) Prove that for A, $B \in \mathcal{M}_n$, if $x^H Ax = x^H Bx$ for all $x \in \mathbb{C}^n$, then $A = B$. Give an example for which $x^T Ax = x^T Bx$ for all $x \in \mathbb{C}^n$ but $A \neq B$.

(c) (10 points) Prove that A is an orthogonal projection if and only if A is Hermitian, i.e., $A = A^H$.

Question 2. Assume that the eigenvalues of a Hermitian matrix $A \in \mathcal{M}_n$ are arranged in the order

$$\lambda_n(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A).$$

(a) (5 points.) Let $A \in \mathcal{M}_n$ be Hermitian. Prove that

$$\lambda_n = \min_{x \neq 0} \frac{x^H A x}{x^H x}.$$

(b) (8 points) Prove that every $A \in \mathcal{M}_n$ may be written uniquely as $A = S + iT$, where S and T are Hermitian.

(c) (12 points) For any $A \in \mathcal{M}_n$, consider the unique expansion $A = S + iT$, where S and T are Hermitian. Prove that for any $\lambda \in \operatorname{eig}(A)$, it holds that

$$\lambda_n(S) \leq \operatorname{Re}(\lambda) \leq \lambda_1(S) \quad \text{and} \quad \lambda_n(T) \leq \operatorname{Im}(\lambda) \leq \lambda_1(T).$$

Question 3.

(a) (2 points) Define $A^{\frac{1}{2}}$ for a positive semidefinite $A \in \mathcal{M}_n$.

(b) (2 points) Define $|A|$ for any $A \in \mathcal{M}_{m,n}$.

(c) (7 points) Prove that the eigenvalues of $|A|$ are the singular values of A.

(d) (7 points) Prove that A is positive semidefinite if and only if $|A| = A$.

(e) (7 points) Prove that $|A|$ and $|A^H|$ are similar.

Question 4.

(a) (4 points.) Define the p-norm $\|A\|_p$ and Frobenius norm $\|A\|_F$ of a matrix $A \in \mathcal{M}_{m,n}$.

(b) (6 points.) For every $A \in \mathcal{M}_{m,n}$, establish the following identities:

(i) $\|A^H\|_2 = \|A\|_2$.

(ii) $\|A^H A\|_2 = \|A^H\|_2 \, \|A\|_2$.

(c) (6 points.) Given two n-vectors x and y and the matrix $Z = xy^H$, show that

$$\|Z\|_2 = \|Z\|_F = \|x\|_2 \|y\|_2.$$

(d) (9 points.) Prove that the Frobenius norm and the matrix two-norm are invariant under unitary transformations, i.e., show that if P and Q are unitary matrices of suitable dimension, then

$$\|A\|_2 = \|PAQ\|_2 \;\text{ and }\; \|A\|_F = \|PAQ\|_F.$$

Applied Algebra Qualifying Exam: Part B
Fall 2021

Instructions: Do all problems. All problems are weighted equally. You are not allowed to consult any external resource during this exam. Good luck!

Problem 1: Let G be a finite group and let V be an irreducible complex representation of G. If $g \in G$ lies in the center of G, show that there exists $c \in \mathbb{C}$ with

$$g \cdot v = cv$$

for all $v \in V$.

Problem 2: Let $\mathbb{Z}$ be the additive group of integers. Is every indecomposable $\mathbb{Z}$-module over the complex numbers irreducible?

Problem 3: Write down the character table of the symmetric group S_4. If we let

$$X := \{\text{all 2-element subsets of } \{1, 2, 3, 4\}\}$$

then X carries a natural permutation action of S_4. Find the decomposition of $\mathbb{C}[X]$ into irreducibles.

Problem 4: Find the character table of the dihedral group D_4 of symmetries of a square. The group algebra of D_4 is isomorphic to a direct sum

$$\mathbb{C}[D_4] \cong \mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_r}(\mathbb{C})$$

of matrix algebras over $\mathbb{C}$. Determine r and the numbers $n_1, \ldots, n_r > 0$. (Hint: Try showing that $\mathbb{C}[D_4] \cong \mathrm{End}_{D_4} \mathbb{C}[D_4]$ as algebras. How does the endomorphism ring of $\mathbb{C}[D_4]$ decompose?)

Applied Algebra Qualifying Exam: Part C

5:00pm–8:00pm (PDT), via Zoom.
Tuesday September 7th, 2021

- Write your name and student PID at the top right corner of each page of your submission.
- Do both problems. Show your work.
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Question 1.

(a) (2 points) Let $\mathrm{C}(d)$ be the group generated by the cyclic permutation $\gamma = (1\ 2\ \ldots\ d)$ in the symmetric group $\mathrm{S}(d)$. Explicitly describe the dual group of $\mathrm{C}(d)$.

(b) (8 points) State the definition of the Cayley graph of C(d), and find its eigenvalues and eigenvectors.

Question 2. Let γ

(a) (2 points.) Given a Young diagram $\alpha \vdash d$, identity the corresponding conjugacy class $C_\alpha \subset \mathrm{S}(d)$ with the formal sum of its elements, so that it becomes an element of the group algebra $\mathbb{C}\mathrm{S}(d)$. Given another Young diagram $\lambda \vdash d$, show that C_α acts in the corresponding irreducible representation V^λ of $\mathbb{C}\mathrm{S}(d)$ as multiplication by a scalar, ω_α^λ, and express this number in terms of the character of V^λ.

(b) (8 points) Compute the ω_α^λ explicitly in the case that $\alpha = (d)$ is the Young diagram consisting of a single row of d cells.