ALGEBRA QUALIFYING EXAM, FALL 2021

All problems are worth 15 points.

1. Let G be the group given by the presentation $(a, b \mid a^8 = 1,\ ba = a^{-1}b,\ b^2 = a^4)$. Let H be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by $A = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where ζ is a primitive 8th root of 1.

Prove that $G \cong H$.

2. Let G be a group of order $2^n \cdot 11$ for some $n \geq 0$. Prove that G is solvable. (Hint: Consider the cases $n < 10$ and $n \geq 10$ separately. In the latter case, define a group homomorphism $\phi : G \to S_{11}$ and consider its kernel and image.)

3. Let $\zeta \in \mathbb{C}$ be a primitive nth root of 1 for some $n \geq 2$. Show that $\sqrt[3]{2} \notin \mathbb{Q}(\zeta)$. (Hint: use the fundamental theorem of Galois theory).

4. Let $F \subseteq K$ be an extension of fields with $[K : F] < \infty$. Show that if K is a perfect field, then so is F.

5. Suppose that A is a complex 7×7 matrix such that $A^5 = 2A^4 + A^3$. Suppose that $\operatorname{rk} A = 5$ and $\operatorname{tr} A = 4$, where rk indicates the rank and tr indicates the trace of a matrix. Find the Jordan canonical form of A.

6. Prove that in the category of commutative rings with unit, $A \otimes_{\mathbb{Z}} B$ is the coproduct of the rings A and B.

7. Let R be an integral domain. Recall that for any R-modules M, N, $\mathrm{Hom}_R(M, N)$ is also an R-module. Let M and N be torsion R-modules.

(a). Suppose that either M or N is finitely generated as an R-module. Prove that $\mathrm{Hom}_R(M, N)$ is also a torsion R-module.

(b). Give a counterexample showing that $\mathrm{Hom}_R(M, N)$ need not be torsion as an R-module if M and N are both infinitely generated. Justify your answer.