Invariant Theory and Group Coaction on Artin-Shelter Regular Algebra

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Abstract

In this thesis, we studied dual reflection groups of Artin-Schelter regular algebras. We classified the dual reflection groups of global dimension 2 Artin-Schelter regular domains. We showed that a dual reflection group of an Artin-Schelter regular algebra can be extended to a dual reflection group of an Ore extension using semidirect product. This allows us to produce more examples of dual reflection groups.

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Chapter 1

Introduction

Throughout this thesis, $\Bbbk$ is assumed to be an algebraically closed field of characteristic 0. Consider the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ and a finite group G. From [ST54] and [Che55], we have the following:

$\Bbbk[x_1, \ldots, x_n]^G$ is a polynomial ring if and only if G is generated by pseudoreflections.

This is known as the Shephard-Todd-Chevalley Theorem from classical invariant theory. A natural question arises in this context:

Is there a noncommutative analogue of the Shephard-Todd-Chevalley Theorem?

It turns out we want to study Hopf algebra (co)actions on Artin-Schelter regular algebras. In Chapter 2, we will be giving backgrounds on Artin-Schelter regularity and Hopf algebra (co)action. In Chapter 3, we will focus on group-grading of an Artin-Schelter regular algebra A with respect to a finite group G. This is equivalent to $\Bbbk G$ coacting on A as well as $(\Bbbk G)^*$ acting on A. In [KKZ17], the authors defined dual reflection groups. We showed that

$$G \text{ is a dual reflection group of } A \iff A \text{ is homogeneously } G\text{-graded with } A_g \text{ nonzero for all } g \in G \text{ and } A_e \text{ AS regular.}$$

In [Cra23], the author proved the following:

Let G be a dual reflection group of A where A is an Artin-Schelter regular algebra of global dimension 2 generated in degree 1. If G is nonabelian, then $A = \Bbbk\langle u, v\rangle/(u^2 - v^2)$ and $G = \Gamma_m$ for some m with u, v G-homogeneous.

We were able to show the following:

If G is abelian, then $A = \Bbbk\langle u, v\rangle/(uv - qvu)$ for some $q \in \Bbbk^\times$ and $G = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ for some m, n where $\deg_G(u) = (1, 0)$ and $\deg_G(v) = (0, 1)$.

These two results combined classify dual reflection groups of Artin-Schelter regular algebra of global dimension 2. In Chapter 4, we will introduce the idea of Ore extension. It is known that an Ore extension of AS regular algebra is AS regular. We were able to show that if we have a dual reflection group G of an AS regular algebra R, then $G \rtimes_\psi C_m$ is a dual reflection group of an Ore extension of R for some ψ and m.

Chapter 2

Artin-Shelter Regular Algebra and Hopf Algebra

2.1 Artin-Schelter Regular Algebra

A $\Bbbk$-algebra A is $\mathbb{N}$-***graded*** if it has a direct sum decomposition

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

where A_i is a $\Bbbk$-vector space for every $i \in \mathbb{N}$ such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$. We say A is ***connected*** if $A_0 = \Bbbk$. An element a in A is ***homogeneous*** if there exists $n \in \mathbb{N}$ such that $a \in A_n$. A left or right ideal I of A is a ***homogeneous ideal*** if I is generated by homogeneous elements. We say that A is ***finitely graded*** if A is connected $\mathbb{N}$-graded and finitely generated as a $\Bbbk$-algebra.

Example 2.1. Let $A = \Bbbk[x_1, \dots, x_n]$ be the polynomial ring over $\Bbbk$ of n variables. We can grade A by the degrees of the polynomials. For instance,

$$A = \Bbbk \oplus A_1 \oplus A_2 \oplus \cdots$$

where A_i is the $\Bbbk$-vector space of all homogeneous polynomials of degree i.

Let A be a finitely generated $\Bbbk$-algebra. We say $V \subseteq A$ is a ***generating subspace*** of A if V is a finitely generated $\Bbbk$-vector subspace of A which generates A as $\Bbbk$-algebra and $1 \in V$. The ***growth function*** associated to a generating subspace is

$$f(n) = f_V(n) := \dim_{\Bbbk} V^n.$$

We say that A has ***polynomial growth*** if there exist positive real numbers c, r such that $f(n) \leq cn^r$ for all n. We define the ***Gelfand-Kirillov(GK) dimension*** of A to be

$$\mathrm{GKdim}(A) = \limsup_{n\to\infty} \log_n(f(n)).$$

We say that A has ***exponential growth*** if

$$\limsup_{n\to\infty} (f(n))^{1/n} > 1.$$

Remark 2.2. Gelfand-Kirillov(GK) dimension measures if an algebra has polynomial growth. More precisely, A has polynomial growth if and only if $\mathrm{GKdim}(A) < \infty$. In fact, if A is a commutative finitely generated algebra, then the GK dimension of A equals to the Krull dimension of A.

Let A be an arbitrary ring. A module F over A is ***free*** if F is isomorphic to a direct sum of copies of A. A module P over A is ***projective*** if for every surjective A-module homomorphism $\pi\colon M \to N$ and every A-module homomorphism $\psi\colon P \to N$, there exists an A-module homomorphism $\phi\colon P \to M$ such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\phi"'] \arrow[d, "\psi"] \\ M \arrow[r, two heads, "\pi"'] & N \end{tikzcd}$$

A ***projective resolution*** of an A-module M is a complex of A-modules

$$\mathcal{F}\colon \qquad \cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

together with a surjective ***augmentation map*** $\epsilon\colon F_0 \to M$ such that each F_i is projective, and the following sequence is exact

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\epsilon} M \longrightarrow 0.$$

It is called a ***free resolution*** if F_i is free. We define $\mathrm{Ext}^i_A(M,N)$ to be the i-th homology group of

$$0 \longrightarrow \mathrm{Hom}_A(F_0, N) \longrightarrow \mathrm{Hom}_A(F_1, N) \longrightarrow \cdots$$

Note that $\mathrm{Ext}^i_A(M,N)$ doesn't depend on the choice of projective resolution $\mathcal{F}$.

Let A be a finitely graded $\Bbbk$-algebra while some definitions may be defined in some general ring. A left (resp. right) A-module M is ***$\mathbb{Z}$-graded*** if M has a decomposition

$$M = \bigoplus_{i\in\mathbb{Z}} M_i$$

as $\Bbbk$-vector spaces such that $A_iM_j \subseteq M_{i+j}$ (resp. $M_jA_i \subseteq M_{i+j}$) for all i, j. We define $M(d)$ to be the A-module M with degree shifted by d which means that

$$M(d)_i = M_{d+i}.$$

Let M, N be $\mathbb{Z}$-graded A-modules, an A-module homomorphism $\phi\colon M \to N$ is called ***graded*** if $\phi(M_i) \subseteq N_i$ for all i. Let $\mathrm{Hom}_{\mathrm{gr}-A}(M,N)$ be the vector space of graded A-module homomorphism from M to N. We define $\underline{\mathrm{Hom}}_A(M,N) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{gr}-A}(M, N(d))$ which is naturally a subset of $\mathrm{Hom}_A(M,N)$. A ***graded free module*** over A is a direct sum of shifted copies of A, i.e. $\bigoplus_{\alpha\in I} A(i_\alpha)$ for some index set I and integers i_α. A ***graded free resolution*** of M is a free resolution of M replacing the free module with graded free module and homomorphism with graded homomorphism. We also want to define the graded Ext function $\underline{\mathrm{Ext}}^i_A(M,N)$ by taking a graded free resolution $\mathcal{F}$ of M and taking the i-th homology of $\underline{\mathrm{Hom}}_A(\mathcal{F}, N)$. If M is both a left and a right A-module, then we write ${}_AM$ when considering M as a left A-module and M_A when considering M as a right A-module.

Let M be a $\mathbb{Z}$-graded right A-module. We can define the ***projective dimension*** $\operatorname{proj.dim}(M)$ to be the minimal n such that there exists a graded projective resolution of M of length n. If such n doesn't exist, then we have $\operatorname{proj.dim}(M) = \infty$. The ***right (left) global dimension*** of A, $\operatorname{r.gl.dim}(A)$ ($\operatorname{l.gl.dim}(A)$), is defined to be the supremum of the projective dimensions of all $\mathbb{Z}$-graded right (left) A-modules. We have the following fact,

Proposition 2.3 ([Bel+16], Proposition 1.5.7)**.** *Let A be a finitely graded $\Bbbk$-algebra. Then*

$$\operatorname{r.gl.dim}(A) = \operatorname{proj.dim}(\Bbbk_A) = \operatorname{proj.dim}({}_A\Bbbk) = \operatorname{l.gl.dim}(A) = m$$

where m is the length of the minimal graded free resolution of k_A.

Because of this result, we may write the common value of $\operatorname{r.gl.dim}(A)$ and $\operatorname{l.gl.dim}(A)$ as $\operatorname{gl.dim}(A)$ and call this the ***global dimension*** of A.

Definition 2.4. Let A be a finitely graded $\Bbbk$-algebra. We say that A is ***Artin-Schelter regular*** or ***AS regular*** if

1. $\operatorname{gl.dim}(A) = d < \infty$;
2. $\operatorname{GKdim}(A) < \infty$;
3. $\underline{\operatorname{Ext}}^i_A(\Bbbk_A, A_A) = \begin{cases} 0 & i \neq d \\ {}_A\Bbbk(l) & i = d \end{cases}$,

for some shift $l \in \mathbb{Z}$. A finitely graded $\Bbbk$-algebra A with finite injective dimension that satisfies condition 3 is ***Artin-Schelter Gorenstein***.

Example 2.5.

1. For any set of nonzero values $q_{ij} \in \Bbbk$, the algebra

$$A = \Bbbk\langle x_1, \ldots, x_n\rangle/(x_j x_i - q_{ij} x_i x_j \mid 1 \leq i < j \leq n)$$

is called a ***quantum polynomial ring***. More specifically, the case where $n = 2$, we have

$$A = \Bbbk\langle x, y\rangle/(yx - qxy)$$

for some nonzero $q \in \Bbbk$, is called the ***quantum plane***.

2. The algebra

$$A = \Bbbk\langle x, y\rangle/(yx - xy - x^2)$$

is called the ***Jordan plane***.

Theorem 2.6 ([Bel+16], Exercise 2.4.5, Theorem 2.2.1)**.** *Let A be an AS regular $\Bbbk$-algebra of global dimension 2. Then A is isomorphic to $\Bbbk\langle x, y\rangle/(f)$ where $\deg(x) = d_1$, $\deg(y) = d_2$, and either*

(i) $f = yx - qxy$ for some nonzero $q \in \Bbbk$, or

(ii) $f = yx - xy - x^{i+1}$ for some i such that $d_1 i = d_2$.

In particular, when $\deg(x) = \deg(y) = 1$*, the* A *is either the quantum plane or the Jordan plane.*

In [AS87], Artin and Schelter classified the AS regular algebras of global dimension 3 generated in degree 1. Let A be a such algebra, then either

1. $A \cong k\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3)$ where f_i have degree 2, or
2. $A \cong k\langle x_1, x_2\rangle/(f_1, f_2)$ where f_i have degree 3.

Artin and Schelter showed that there are exactly 7 types in the first case and 6 types in the second case.

Example 2.7. The quantum polynomial ring in 3 variables

$$A = \frac{k\langle x, y, z\rangle}{\begin{pmatrix} yx - qxy \\ xz - pzx \\ yz - rzy \end{pmatrix}}$$

with q, p, r nonzero is an example of a global dimension 3 AS regular algebra with 3 generators in degree 1.

2.2 Hopf Algebra

A $\Bbbk$-***coalgebra*** is a $\Bbbk$-vector space A together with two $\Bbbk$-linear maps, ***comultiplication*** $\Delta\colon A \to A \otimes A$ and ***counit*** $\varepsilon\colon A \to \Bbbk$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes A \arrow[d, "\Delta \otimes id_A"] \\ A \otimes A \arrow[r, "id_A \otimes \Delta"] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "1\otimes -"'] \arrow[dr, "-\otimes 1"] \arrow[dd, "\Delta"] & \\ \Bbbk \otimes A & & A \otimes \Bbbk \\ & A \otimes A \arrow[ul, "\varepsilon \otimes id_A"] \arrow[ur, "id_A \otimes \varepsilon"'] & \end{tikzcd}$$

A is a $\Bbbk$-***bialgebra*** if A is both a $\Bbbk$-algebra and a $\Bbbk$-coalgebra such that Δ and ε are algebra homomorphisms.

Example 2.8. Let G be a group. We know that the group algebra $\Bbbk G$ is a $\Bbbk$-algebra via the usual multiplication map m and the unit map $u\colon \Bbbk \to \Bbbk G$ which is defined by $u(1) = 1$. Let the comultiplication and counit be defined by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$. Then $(\Bbbk G, m, u, \Delta, \varepsilon)$ is a $\Bbbk$-bialgebra.

For any $\Bbbk$-space V, let $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ denote the dual of V. If V is a $\Bbbk$-algebra, the ***finite dual*** of A is

$$A^\circ = \{f \in A^*\colon f(I) = 0 \text{ for some ideal } I \subseteq A \text{ such that } \dim A/I < \infty\}.$$

Theorem 2.9 ([Mon93], Theorem 9.1.3)**.** *If* $(A, m, u, \Delta, \varepsilon)$ *is a* $\Bbbk$*-bialgebra, then* $(A^\circ, \Delta^*, \varepsilon^*, m^*, u^*)$ *is also a* $\Bbbk$*-bialgebra.*

Remark 2.10. Let G be a group, and $\Bbbk G$ be the group algebra. We can give another description to the finite dual of $\Bbbk G$,

$$(\Bbbk G)^\circ = \{f \in (\Bbbk G)^* :\ \dim_{\Bbbk} \operatorname{span}\{G \cdot f\} < \infty\}$$

where G acts on $(\Bbbk G)^*$ by $(g \cdot f)(h) = f(hg)$ for all $g, h \in G$ and $f \in (\Bbbk G)^*$. When G is a finite group, we have $(\Bbbk G)^\circ = (\Bbbk G)^*$.

Let (A, Δ, ε) be a $\Bbbk$-coalgebra and (B, m, u) a $\Bbbk$-algebra. For every $f, g \in \operatorname{Hom}_{\Bbbk}(A, B)$, define the ***convolution product***

$$(f * g)(a) = m \circ (f \otimes g)(\Delta(a)) \text{ for all } a \in A.$$

Note that the identity element of $\operatorname{Hom}_{\Bbbk}(A, B)$ is $u\varepsilon$.

Notation 2.11. Let (A, Δ, ε) be a $\Bbbk$-coalgebra. For any $a \in A$, we write

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}.$$

This is called the Sweedler notation.

Example 2.12. Let (A, Δ, ε) be a $\Bbbk$-coalgebra and (B, m, u) a $\Bbbk$-algebra. Let $f, g \in \operatorname{Hom}_{\Bbbk}(A, B)$ and $a \in A$.

1. Since A is a $\Bbbk$-coalgebra, then

$$1 \otimes a = (\varepsilon \otimes id_A)(\Delta(a)) = \sum \varepsilon(a_{(1)}) \otimes a_{(2)} = \sum a_{(1)} \otimes \varepsilon(a_{(2)}).$$

2. We consider the convolution product

$$(f * g)(a) = \sum f(a_{(1)}) g(a_{(2)}).$$

Definition 2.13. Let $(H, m, u, \Delta, \varepsilon)$ be a $\Bbbk$-bialgebra. $S \in \operatorname{Hom}_{\Bbbk}(H, H)$ is called an ***antipode*** for H if S is an inverse to id_H under the convolution product, i.e.

$$\sum (Sh_1)h_2 = \sum h_1(Sh_2) = u(\varepsilon(h)) = \varepsilon(h)1_H.$$

If such S exists, then H is a ***Hopf algebra***. An ideal I of H is a ***Hopf ideal*** if $\varepsilon(I) = 0$, $\Delta(I) \subseteq I \otimes H + H \otimes I$, and $SI \subseteq I$.

Example 2.14. Let G be a group, and $H = \Bbbk G$. Let $S \in \operatorname{Hom}_{\Bbbk}(H, H)$ be defined by

$$Sg = g^{-1} \text{ for every } g \in G.$$

Notice that H is a Hopf algebra with the antipode S.

Theorem 2.15 ([Mon93], Theorem 9.1.3)**.** *If $(H, m, u, \Delta, \varepsilon, S)$ is a Hopf algebra, then $(H^\circ, \Delta^*, \varepsilon^*, m^*, u^*, S^*)$ is also a Hopf algebra.*

Example 2.16. Let G be a finite group. We showed earlier that $(\Bbbk G, m, u, \Delta, \epsilon, S)$ is a Hopf algebra. By Theorem 2.15, $((\Bbbk G)^*, \Delta^*, \varepsilon^*, m^*, u^*, S^*)$ is also a Hopf algebra.

2.3 Hopf Action

Let's start with a more general context. For a $\Bbbk$-coalgebra (A, Δ, ε), a (right) ***A*-comodule** M is a $\Bbbk$-vector space with a $\Bbbk$-linear map $\rho\colon M \to M \otimes A$ such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes A \arrow[d, "id\otimes\Delta"] \\ M \otimes A \arrow[r, "\rho\otimes id"] & M \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "-\otimes 1"'] & M \otimes A \arrow[d, "id\otimes\varepsilon"] \\ & M \otimes \Bbbk \end{tikzcd}$$

Lemma 2.17 ([Mon93], Lemma 1.6.4)**.** *If M is a right C-comodule, then M is a left C^*-module. If M is a left A-module, then M is a right A°-comodule if and only if $\{A \cdot m\}$ is finite dimensional for all $m \in M$.*

Now let H, K be Hopf algebras and A a $\Bbbk$-algebra. We say that H ***acts on*** A from the left, or A is a ***left H-module algebra*** if A is a left H-module, and for all $h \in H$, we have $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, and $h \cdot 1_A = \varepsilon(h) 1_A$. The ***invariant subring*** of this action is defined to be

$$A^H = \{a \in A \colon h \cdot a = \varepsilon(h)a, \, \forall h \in H\}.$$

We say that K ***coacts on*** A from the right, or A is a ***right K-comodule algebra*** if A is a right K-comodule with the map $\rho\colon A \to A \otimes K$ such that $\rho(1_A) = 1_A \otimes 1_K$ and $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in A$. The ***coinvariant subring*** of this coaction is defined to be

$$A^{coK} = \{a \in A \colon \rho(a) = a \otimes 1_K\}.$$

Lemma 2.18 ([Mon93], Lemma 1.7.2)**.** *Let N be a right K-comodule, then $N^{K^*} = N^{coK}$. Let M be a left H-module, then $M^H = M^{coH^\circ}$.*

Let M be a left H-module. We say that H ***acts inner faithfully*** on M if $IM \neq 0$ for every nonzero Hopf ideal I of H. Let N be a right K-comodule. We say that K ***coacts inner faithfully*** on N if for any proper Hopf subalgebra $K' \subseteq K$, $\rho(N) \nsubseteq N \otimes K'$.

Lemma 2.19 ([Cha+16], Lemma 1.6)**.** *Suppose H is a finite dimensional Hopf algebra and $K = H^\circ$. Let U be a left H-module. Then the H-action on U is inner faithful if and only if the induced K-coaction on U is inner faithful.*

Let A be a $\mathbb{N}$-graded left H-module algebra. We say that H acts on A ***homogeneously*** if $H \cdot A_n \subseteq A_n$ for all $n \in \mathbb{N}$. The following lemma will be useful later.

Lemma 2.20 ([KKZ17], Lemma 3.3)**.** *Let A be a Noetherian AS regular domain and H be a semisimple Hopf algebra acting homogeneously and inner faithfully on A. If A^H is AS regular, then* $\operatorname{gl.dim} A^H = \operatorname{gl.dim} A$.

Chapter 3

Dual Reflection Group

Before introducing dual reflection group, we want to establish some general facts about group coactions.

3.1 Group Coaction and Group Grading

Through out this section, G is assumed to be a finite group.

A $\Bbbk$-algebra A is said to be ***G-graded*** if it has a direct sum decomposition

$$A = \bigoplus_{g \in G} A_g$$

where for every $g, h \in G$, we have A_g a $\Bbbk$-vector space and $A_g A_h \subseteq A_{gh}$. If $x \in A_g$ for some $g \in G$, then x is ***G-homogeneous*** of ***G-degree*** g denoted $\deg_G(x) = g$.

Lemma 3.1 ([CM84], Proposition 1.3)**.** *Suppose A is a $\Bbbk$-algebra, then*

$$\begin{aligned} A \text{ is } G\text{-graded} &\iff A \text{ is a left } (\Bbbk G)^*\text{-module algebra} \\ &\iff A \text{ is a right } \Bbbk G\text{-comodule algebra.} \end{aligned}$$

Explicitly, let $\{p_g \colon g \in G\}$ be the dual basis for $(\Bbbk G)^*$. For every element $a \in A = \bigoplus_{g \in G} A_g$, we can write $a = \sum_{g \in G} a_g$. The left $(\Bbbk G)^*$ action is given by $p_g \cdot a = a_g$. The right $\Bbbk G$ coaction is given by $\rho(a) = \sum_{g \in G} a_g \otimes g$. We say that the G-grading on A is inner faithful if the induced (co)action is inner faithful. We say that A is homogeneously G-graded if the induced (co)action on A is homogeneous.

Lemma 3.2. *Suppose the $\Bbbk$-algebra A is G-graded, then*

$$A_{e_G} = A^{(\Bbbk G)^*} = A^{co(\Bbbk G)}.$$

We denote this subring by A^G.

Proof. By Lemma 2.18, the second equality is clear. Notice that

$$A_{e_G} = \{a \in A \colon \rho(a) = a \otimes e_G = a \otimes 1_{\Bbbk G}\} = A^{co(\Bbbk G)}.$$

□

Proposition 3.3. *Suppose A is a finitely graded $\Bbbk$-algebra such that A is also G-graded. If A_g is nonzero for every $g \in G$, then G-grading on A is inner faithful. The converse is true if A is a domain.*

Proof. $\Longrightarrow$ By Lemma 2.19, it suffices to show that the induced $\Bbbk G$ coaction on A, from Lemma 3.1, is inner faithful. Let $\rho\colon A \to A \otimes \Bbbk G$ be the $\Bbbk$-linear map defined by the coaction. For every $g \in G$, since A_g is nonzero, then there exists $0 \neq x \in A_g$ such that $\rho(x) = x \otimes g$. If there exists a Hopf subalgebra $K' \subseteq \Bbbk G$ such that $\rho(A) \subseteq A \otimes K'$, then $g \in K'$ for every $g \in G$. Hence, $K' = \Bbbk G$ which shows that the coaction is inner faithful.
$\Longleftarrow$ Suppose A is a domain. Let $G' = \{g \in G \colon A_g \neq 0\}$. We want to show that G' is a subgroup of G. For every $g, h \in G'$, we have $A_g, A_h \neq 0$. Let $x \in A_g$ and $y \in A_h$ where $x, y \neq 0$, then $xy \neq 0$ because A is a domain. This shows that $A_{gh} \neq 0$ which implies that $gh \in G'$. Fix an element $g \in G$, and let $m = |g|$. We have $g^{m-1} = g^{-1} \in G'$ and $g^m = e \in G'$. This shows that G' is a subgroup of G. Now suppose $A_h = 0$ for some $h \in G$, then G' is a proper subgroup of G. This implies that $\Bbbk G'$ is a proper Hopf subalgebra of $\Bbbk G$. This shows that the G-grading on A is not inner faithful because $\rho(A) \subseteq A \otimes \Bbbk G'$ which is a contradiction. $\square$

For the rest of this paper, we want to further assume that if A is G-graded, then A is homogeneously G-graded with $A_g \neq 0$ for all $g \in G$.

Lemma 3.4. *Suppose the $\Bbbk$-algebra A is G-graded. If we have $x, y, x+y \in A$ G-homogeneous and $x, y, x+y \neq 0$, then they are G-homogeneous of the same degree.*

Proof. Since $x, y, x+y \neq 0$ and G-homogeneous, then $\deg_G(x) = g$, $\deg_G(y) = h$, $\deg_G(x+y) = f$ for some $g, h, f \in G$. Let $\rho\colon A \to A \otimes \Bbbk G$ be the induced coaction. We have

$$x \otimes g + y \otimes h = \rho(x) + \rho(y) = \rho(x+y) = x \otimes f + y \otimes f.$$

This implies that $g = f = h$. $\square$

For the rest of this section, we are going to discuss a general approach of determining if a $\Bbbk$-algebra A is G-graded. Suppose A is finitely graded and G-graded for some finite group G. We can write $A = \Bbbk\langle x_1, \dots, x_n\rangle / I$ for some positive integer n and an (2-sided) ideal I generated by the relations of A. If A is assumed to be generated in degree 1, we may assume that the generators $x_1, \dots, x_n$ are G-homogeneous. If not, we can always apply a linear change of coordinates so that it is true. This forces I to be a G-homogeneous ideal. Furthermore, we can check if A is H-graded for some finite group H via some assignment of H-degree by checking if the ideal generated by the relations is H-homogeneous. By applying this method, we have this following result. Proposition 3.5 is also proven in [Cra23] independent of our work.

Proposition 3.5. *Let $A = \Bbbk\langle x, y\rangle/(r)$ be an AS regular algebra of global dimension 2 generated in degree 1 and G be a finite group such that A is G-graded with $u = x + ay$ and $v = y + bx$ where $ab \neq 1$ and $u \in A_g, v \in A_h$ for some $g, h \in G$. The following holds:*

1. *If A is the Jordan plane, then $G = \langle g, h \mid g^m = gh^{-1} = e\rangle = \langle g \mid g^m = e\rangle$ for some m.*

2. *If A is the quantum plane with $a = b = 0$ or $q = 1$, then G is a quotient group of $\langle g, h \mid gh = hg\rangle$.*

3. *If A is the quantum plane with $q = -1$ and either a or b is nonzero, then the following holds:*

   (a) *If $ab \neq -1$, then $G = \langle g, h \mid g^m = gh^{-1} = e\rangle = \langle g \mid g^m = e\rangle$ for some m.*

   (b) *If $ab = -1$, then G is a quotient group of $\langle g, h \mid g^2 = h^2\rangle$.*

4. *If A is the quantum plane with $q \neq \pm 1$ and either a or b is nonzero, then $G = \langle g, h \mid g^m = gh^{-1} = e\rangle = \langle g \mid g^m = e\rangle$ for some m.*

Proof. Note that $\{g, h\}$ generates G because $\{u, v\}$ generates A.

1. Suppose $A = \Bbbk\langle x, y\rangle/(yx - xy - x^2)$ is the Jordan plane. After this change of coordinates from x, y to u, v, we have

$$A = \frac{\Bbbk\langle u, v\rangle}{(-u^2 - a^2v^2 + (ab - 1 + a)uv + (1 - ab + a)vu)}.$$

   Since A is G-graded, then the generator of the relations $-u^2 - a^2v^2 + (ab - 1 + a)uv + (1 - ab + a)vu$ is G-homogeneous. If $ab - 1 + a = 1 - ab + a$, then $ab = 1$ is a contradiction, so $ab - 1 + a, 1 - ab + a$ can't both be 0. Without the loss of generality, assume that $ab-1+a$ is nonzero. If $a, b \neq 0$, then by Lemma 3.4 we know that $g^2 = h^2 = gh$ which implies that $g = h$. If $a = 0$, then we have the relation $-u^2 - uv + vu$. By Lemma 3.4, we have $g^2 = gh = hg$ which implies that $g = h$. If $b = 0$ and $a \neq 0$, then we either have $g^2 = h^2 = gh$ or $g^2 = h^2 = hg$ which implies that $g = h$. We can conclude that $G = \langle g, h \mid g^m = gh^{-1} = e\rangle = \langle g \mid g^m = e\rangle$ for some m.

2. Suppose $A = \Bbbk\langle x, y\rangle/(yx - qxy)$ is the quantum plane. After this change of coordinates from x, y to u, v, we have

$$A = \frac{\Bbbk\langle u, v\rangle}{((q - 1)bu^2 + (q - 1)av^2 + (ab - q)uv + (1 - qab)vu)}.$$

   Either $a = b = 0$ or $q = 1$ implies that $gh = hg$, which forces G to be a quotient group of $\langle g, h \mid gh = hg\rangle$.

3. Suppose $A = \Bbbk\langle x, y\rangle/(yx+xy)$ is the quantum plane with $q = -1$. After this change of coordinates, we have

$$A = \frac{\Bbbk\langle u, v\rangle}{(-2bu^2 - 2av^2 + (ab + 1)uv + (1 + ab)vu)}.$$

   (a) If $ab \neq -1$. Since either a or b is nonzero, we have $g = h$ which forces $G = \langle g \mid g^m = e\rangle$ for some m.

   (b) If $ab = -1$, we have the relation $g^2 = h^2$, so G is a quotient group of

$$\langle g, h \mid g^2 = h^2\rangle.$$

4. Suppose $A = \Bbbk\langle x, y\rangle/(yx - qxy)$ is the quantum plane with $q \neq \pm 1$. After this change of coordinates from x, y to u, v, we have

$$A = \frac{\Bbbk\langle u, v\rangle}{((q-1)bu^2 + (q-1)av^2 + (ab-q)uv + (1-qab)vu)}.$$

Since A is G-graded, then the generator of the relations $(q-1)bu^2 + (q-1)av^2 + (ab-q)uv + (1-qab)vu$ is G-homogeneous. Since $q \neq \pm 1$ and $ab \neq 1$, then $ab - q$ and $1 - qab$ can't both be 0. Since either a or b is nonzero and $q \neq \pm 1$, then either $(q-1)b$ or $(q-1)a$ is nonzero. Then, we have $g = h$ which forces $G = \langle g \mid g^m = e\rangle$ for some m.

□

However, this approach is very difficult to apply to AS regular algebra of global dimension 3. For example, if we have

$$A = \frac{k\langle x, y, z\rangle}{\begin{pmatrix} yx - qxy \\ xz - pzx \\ yz - rzy \end{pmatrix}},$$

the quantum polynomial ring in 3 variables. Let G be a finite group such that A is G-graded. After some change of coordinates, we have $A = \Bbbk\langle u, v, w\rangle/I$ for some G-homogeneous ideal I and G-homogeneous generators u, v, w. Unlike the global dimension 2 case, the ideal I is not principal. This leads to messy computations. However, we expect very few number of cases where A can be graded by a nonabelian group just like in Proposition 3.5.

3.2 Dual Reflection Groups

Definition 3.6 ([KKZ17]). A finite group G is called a ***dual reflection group*** if the Hopf algebra $(\Bbbk G)^*$ acts homogeneously and inner faithfully on a Noetherian AS regular domain A generated in degree 1 such that the fixed subring $A^G = A^{(\Bbbk G)^*}$ is again AS regular.

By Proposition 3.3, G is a dual reflection group of A if and only if A is homogeneously G-graded with A_g nonzero for every $g \in G$ such that $A^G = A_{e_G}$ is AS regular.

Theorem 3.7 ([KKZ17], Theorem 0.3). *Let A be a Noetherian AS regular domain generated in degree 1. Let G be a finite group such that it is a dual reflection group of $A = \bigoplus_{g\in G} A_g$. Then the following holds.*

1. *There is a set of G-homogeneous elements $\{f_g : g \in G\} \subseteq A$ with $f_e = 1$ such that $A_g = f_g \cdot A^G = A^G \cdot f_g$ for all $g \in G$.*

2. *There is a generating subset $\mathfrak{R}$ of G satisfying $e \notin \mathfrak{R}$ such that*

$$A_1 = \bigoplus_{g\in\mathfrak{R}} \Bbbk f_g \oplus (A_1 \cap A^G).$$

Corollary 3.8. *Let A be a Noetherian AS regular domain generated in degree 1. Let G be a finite group such that it is a dual reflection group of $A = \bigoplus_{g \in G} A_g$. If $x_1, \dots, x_n$ is a minimal set of G-homogeneous generators of A such that $\deg(x_i) = 1$, then $\deg_G(x_i) \neq \deg_G(x_j)$ for $i \neq j$ unless $\deg_G(x_i) = \deg_G(x_j) = e$.*

Proof. By Theorem 3.7, there exists a set of G-homogeneous elements $\{f_g : g \in G\}$ and a generating subset $\mathfrak{R}$ of G satisfying $e \notin \mathfrak{R}$ such that $A_1 = \bigoplus_{g \in \mathfrak{R}} \Bbbk f_g \oplus (A_1 \cap A^G)$. Let $i \neq j$, if we have $\deg_G(x_i) = \deg_G(x_j) = e$, then we are done. We can assume $\deg_G(x_i), \deg_G(x_j) \neq e$. Suppose $\deg_G(x_i) = \deg_G(x_j) = g \neq e$ for some $g \in G$. Then there exists $a, b \in \Bbbk^\times$ such that $\deg_G(x_i) = af_g$ and $\deg_G(x_j) = bf_g$ which is a contradiction with $\{x_1, \dots, x_n\}$ being a minimal generating set. □

In [Vas16], the author checked for a number of finite groups whether they are dual reflection groups for some AS regular domains. For example, the author showed that the SmallGroup$(16, 3)$ given by the presentation

$$G = \langle a, b, c, d \mid a^2 = d, b^2 = c^2 = d^2 = e, a^{-1}ba = bc, a^{-1}ca = c,$$
$$b^{-1}cb = c, a^{-1}da = d, b^{-1}db = d, c^{-1}dc = d \rangle$$

is a dual reflection group of the AS regular algebra

$$A = \frac{\Bbbk\langle x, y, z \rangle}{(yx + xy, zx - yz, zy - xz)}$$

via the G-grading

$$\deg_G(x) = b, \deg_G(y) = bc, \deg_G(z) = a.$$

Let $A = \Bbbk\langle x, y\rangle/(yx + xy)$ be the quantum plane with $q = -1$. Assume A is G-graded with $u' = x + ay$ and $v = y + bx$ G-homogeneous for some finite group G and $ab = -1$. After the change of coordinates, we have $A = \Bbbk\langle u', v\rangle/(b^2(u')^2 - v^2)$. Let $u = bu'$, we get $A = \Bbbk\langle u, v\rangle/(u^2 - v^2)$. Let

$$\Gamma_m \coloneqq \langle g, h \mid g^2 = h^2, g^{4m} = h^{4m} = (gh)^m = (hg)^m = e\rangle.$$

We have the following

Theorem 3.9 ([Cra23], Theorem 5.5)**.** *G is a dual reflection group of $A = \Bbbk\langle u, v\rangle/(u^2 - v^2)$ if and only if $G = \Gamma_m$ for some m.*

Proposition 3.10. *Let A be an AS regular algebra of global dimension 2 generated in degree 1 and G be a finite group such that A is G-graded. If G is a dual reflection group of A, then we have*

$$A = \begin{cases} \Bbbk\langle u, v\rangle/(uv - qvu) \text{ for some } q \in \Bbbk^\times & \text{if } G \text{ is abelian} \\ \Bbbk\langle u, v\rangle/(u^2 - v^2) & \text{if } G \text{ is nonabelian} \end{cases}$$

for u, v G-homogeneous. Moreover, if G is abelian, then

$$G = \langle g, h \mid g^m = h^n = e, gh = hg\rangle \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$$

for some m, n.

Proof. For the case where G is nonabelian, by Proposition 3.5, we know that A can only be $\Bbbk\langle u,v\rangle/(u^2-v^2)$ with u,v G-homogeneous.
For the case where G is abelian, suppose $\deg_G(u)=g$ and $\deg_G(v)=h$ for some $g,h\in G$. By Corollary 3.8, G is a dual reflection group implies that $g\neq h$. By Proposition 3.5, we know that A can only be $\Bbbk\langle u,v\rangle/(uv-qvu)$ for some $q\in\Bbbk^\times$.
Now suppose that $G=\langle g,h \mid g^m=h^n=e, gh=hg, g^i=h^j\rangle$ for some m,n, $1\leq i\leq m$, and $1\leq j\leq n$. Suppose the relation $g^i=h^j$ is irredundant, then A^G is generated by more than 2 minimal generators, i.e. $x^m, y^n, x^iy^{n-j},\ldots$. By Lemma 2.20, $\operatorname{gl.dim} A^G=\operatorname{gl.dim} A=2$. By Theorem 2.6, A^G is not AS regular which is a contradiction. $\square$

Note that if $G=\mathbb{Z}/m\mathbb{Z}\oplus\mathbb{Z}/n\mathbb{Z}$ and $A=\Bbbk\langle u,v\rangle/(uv-qvu)$ with $q\in\Bbbk^\times$, then A is G-graded with $\deg_G(u)=(1,0)$ and $\deg_G(v)=(0,1)$. In fact, G is a dual reflection group of A because $A^G=\Bbbk\langle u^m,v^n\rangle/(u^mv^n-q^{mn}v^nu^m)$ is AS regular. This means that Theorem 3.9 and Proposition 3.10 classifies all dual reflection groups of AS regular algebra of global dimension 2 generated in degree 1.

Chapter 4

Ore Extension

4.1 Ore Extension

In this section, we want to introduce a construction that will be useful for studying AS regular algebras.

Definition 4.1. Let R be a ring and σ be an automorphism of R. The **Ore Extension** is defined by

$$R[x;\sigma] = \frac{R\langle x\rangle}{(xr - \sigma(r)x \colon r \in R)}.$$

We want to state some important results from [RRZ14], [LWW14], and [RR22] without actually stating what it means for an algebra to be (graded) twisted Calabi-Yau.

Theorem 4.2 ([LWW14], Theorem 2)**.** *Let A be a $\Bbbk$-algebra, $E = A[x;\sigma]$ be an Ore extension for some automorphism σ. If A is twisted Calabi-Yau of dimension d, then E is twisted Calabi-Yau of dimension $d+1$.*

Lemma 4.3 ([RRZ14], Lemma 1.2)**.** *Let A be a connected $\mathbb{N}$-graded $\Bbbk$-algebra. Then A is graded twisted Calabi-Yau if and only if A is AS regular.*

Theorem 4.4 ([RR22], Theorem 4.2)**.** *Let A be an $\mathbb{N}$-graded $\Bbbk$-algebra. Then A is twisted Calabi-Yau of dimension d if and only if A is graded twisted Calabi-Yau of dimension d.*

Lemma 4.5. *An Ore extension of R is AS regular if R is AS regular.*

Proof. By Theorem 4.4 and Lemma 4.3, if R is AS regular, then R is twisted Calabi-Yau. By Theorem 4.2, an Ore extension A of R is twisted Calabi-Yau. Apply Theorem 4.4 and Lemma 4.3 again, we have A AS regular. □

Example 4.6. Let $R = \Bbbk\langle x, y\rangle/(yx + xy)$ be the quantum plane with $q = -1$. Let σ be defined by $\sigma(x) = y$ and $\sigma(y) = x$. We have

$$A = R[z;\sigma] = \frac{R\langle z\rangle}{(zr - \sigma(r)z \colon r \in R)} = \frac{\Bbbk\langle x, y, z\rangle}{(yx + xy, zx - yz, zy - xz)}.$$

Let $G = D_8 = \langle r, \rho \mid r^2 = \rho^4 = e, \rho r = r\rho^3\rangle$. A can be graded by G by setting

$$\deg_G(x) = r, \deg_G(y) = r\rho^2, \deg_G(z) = r\rho.$$

We can check that
$$A^G = \Bbbk[x^2, y^2, z^2]$$
which is AS regular. This shows that G is a dual reflection group of A.

4.2 Dual Reflection Group induced by Ore extension

The goal of this section is to see how a dual reflection group of the ambient algebra can be extended to the Ore extension. This allows us to contruct more examples of dual reflection group via Ore extension.

Lemma 4.7. *Let G be a finite group, A be a G-graded $\Bbbk$-algebra, and σ is an automorphism of A. Define $\phi\colon G \to G$ by $\phi(g) = \deg_G(\sigma(x))$ for some $x \in A$ with $\deg_G(x) = g$. Then ϕ is a group homomorphism if and only if σ satisfies the following:*

$$\text{for every } G\text{-homogeneous element } x \in A,\ \sigma(x) \text{ is } G\text{-homogeneous.} \qquad (*)$$

Proof. $\Longrightarrow$ Suppose ϕ is a well defined group homomorphism. Let $x \in A$ be any G-homogeneous element with $g = \deg_G(x)$. We know that $\deg_G(\sigma(x)) = \phi(g) \in G$. Hence, $\sigma(x)$ is G-homogeneous.
$\Longleftarrow$ First, we want to show that ϕ is well defined. Take $x, y \in A$ such that $\deg_G(x) = \deg_G(y) = g$. We want to show that $\deg_G(\sigma(x)) = \deg_G(\sigma(y))$, i.e. the value $\phi(g)$ is independent of the choice of elements with G-degree g. Since $\deg_G(x) = \deg_G(y) = g$, then $\deg_G(x+y) = g$. By assumption, $x+y$ is G-homogeneous implies that $\sigma(x+y)$ is also G-homogeneous. There exists $h \in G$ such that $h = \deg_G(\sigma(x+y)) = \deg_G(\sigma(x)+\sigma(y))$. By Lemma 3.4, we have $\deg_G(\sigma(x)) = \deg_G(\sigma(y)) = h$.
Now let $g, h \in G$ and $x, y \in A$ such that $\deg_G(x) = g$ and $\deg_G(y) = h$, then $\deg_G(xy) = gh$. We have

$$\phi(gh) = \deg_G(\sigma(xy)) = \deg_G(\sigma(x)\sigma(y)) = \deg_G(\sigma(x))\deg_G(\sigma(y)) = \phi(x)\phi(y).$$

Hence, ϕ is a well defined group homomorphism. $\square$

Corollary 4.8. *If σ satisfies $(*)$, then $\sigma(A_e) \subseteq A_e$.*

Proof. Let $x \in A_e$. By Lemma 4.7, we have $\deg_G(\sigma(x)) = \phi(e) = e$, so $\sigma(x) \in A_e$. $\square$

Lemma 4.9. *Let G be a finite group and A be a finitely graded $\Bbbk$-algebra which is also G-graded. Let σ be a graded automorphism of A. Let $\phi\colon G \to G$ be defined by $\phi(g) = \deg_G(\sigma(x))$ for some $x \in A_g$. If σ satisfies $(*)$ and ϕ is an automorphism of G, then σ^{-1} satisfies $(*)$.*

Proof. Fix some $g \in G$, we have $\sigma(A_g) \subseteq A_{\phi(g)}$. Since G is a finite group, then $|\phi| < \infty$, say $|\phi| = n$. We have

$$\sigma^n(A_g) \subseteq \sigma^{n-1}(A_{\phi(g)}) \subseteq \cdots \subseteq A_{\phi^n(g)} = A_g.$$

Since σ is a graded automorphism, then σ^n is also a graded automorphism. For every $i \in \mathbb{N}$, we have σ^n restricts to an automorphism of the $\Bbbk$-vector space A_i. We write $(A_g)_i$

for the i-th $\mathbb{N}$-degree part of A_g and $(A_i)_g$ for the g G-degree part of A_i. Notice that $(A_g)_i = (A_i)_g$, so we have

$$\sigma^n((A_g)_i) = \sigma^n((A_i)_g) \subseteq (A_i)_g = (A_g)_i.$$

Since A is finite graded as $\mathbb{k}$-algebra, then A_i is a finite dimensional $\mathbb{k}$-vector space, so is $(A_i)_g = (A_g)_i$. Since σ^n is injective, then $\sigma^n((A_g)_i) = (A_g)_i$ for every $i \in \mathbb{N}$. This shows that σ^n restricts to an automorphism of A_g. We have

$$A_g = \sigma^n(A_g) \subseteq \sigma^{n-1}(A_{\phi(g)}) \subseteq \cdots \subseteq A_{\phi^n(g)} = A_g.$$

This implies that $\sigma(A_g) = A_{\phi(g)}$. Since ϕ is an automorphism of G, then we have $\sigma^{-1}(A_g) = A_{\phi^{-1}(g)}$. $\square$

For the next theorem, let R be a Noetherian AS regular domain over $\mathbb{k}$ generated by $x_1, \ldots, x_n$ with $\deg(x_i) = 1$. Let G be a finite group generated by $\{g_1, \ldots, g_n\}$ such that it is a dual reflection group of R with $\deg_G(x_i) = g_i$. Let $I \subseteq \{1, \ldots, n\}$ such that $\deg_G(x_i) = e$ if and only if $i \in I$. This implies that $\{g_i \colon i \in \{1, \ldots, n\} \setminus I\}$ is also a generating subset of G. Since the generators of R all have $\mathbb{N}$-degree 1, then every graded automorphism σ can be descirbed by an invertible matrix $A = (a_{ij})_{n\times n}$ where $\sigma(x_k) = \sum_{i=1}^n a_{ik}x_i$. If σ satisfies $(*)$ with respect to G, and if $\deg_G(x_k) \neq e$, then $\sigma(x_k) = a_{lk}x_l$ for some $l \in \{1, \ldots, n\} \setminus I$ and $a_{lk} \in \mathbb{k}^\times$. If $\deg_G(x_k) = e$, then $\sigma(x_k) = \sum_{l\in I} a_{lk}x_l$ for some $a_{lk} \in \mathbb{k}$. Let $\phi \colon G \to G$ be defined by $\phi(g) = \deg_G(\sigma(x))$ for some $x \in R_g$. By Lemma 4.7, we know that ϕ is a well defined group homomorphism. Moreover, ϕ is an automorphism of G. Indeed, for every $i \in \{1, \ldots, n\} \setminus I$, we have $\phi(g_i) = g_l$ if $\sigma(x_i) = a_{li}x_l$, purmuting the generating set of G. Let $C_m = \langle a \rangle$ be the cyclic group of order m where $m \in |\phi| \cdot \mathbb{Z}_{>0}$. Let $\psi \colon C_m \to \mathrm{Aut}(G)$ be defined by $\psi(a) = \phi$. ψ is well defined because we have checked that $\phi \in \mathrm{Aut}(G)$ and the order of ϕ divides m. This also implies that this semidirect product $G \rtimes_\psi C_m$ is well defined. Now we can state our main theorem.

Theorem 4.10. *$G \rtimes_\psi C_m$ is a dual reflection group of $A = R[x; \sigma]$.*

Proof. Fix m, we want to check that $H = G \rtimes_\psi C_m$ grades A. Let $\deg_H(x_i) = (\deg_G(x_i), 0)$ and $\deg_H(x) = (0, a)$. It suffices to show that $xx_i - \sigma(x_i)x$ is homogeneous for all $i \in \{1, \ldots, n\}$. This is equivalent to

$$\deg_H(xx_i) = \deg_H(\sigma(x_i)x).$$

Indeed, we have

$$\begin{aligned}
\deg_H(xx_i) &= (0, a) \cdot (\deg_G(x_i), 0) \\
&= (\phi(g_i), a) \\
&= (\phi(g_i), 0) \cdot (0, a) \\
&= \deg_H(\sigma(x_i)x).
\end{aligned}$$

Next, we want to show that the fixed subring A^H is AS regular. By Lemma 3.2,

$$A^H = A_{e_H} = \{y \in A \colon \deg_H(y) = (e, e)\}.$$

For every $y \in A$, we can write $y = \sum_{i=0}^{k} r_i x^i$ for some $r_i \in R$. Suppose y is H-homogeneous of degree (e, e), then for $r_i \neq 0$, we have

$$\deg_H(y) = \deg_H(r_i x^i) = (\deg_G(r_i), a^i) = (e, e).$$

This implies that $r_i \in R_{e_G} = R^G$ and $m \mid i$, so $A^H \subseteq R^G[x^m; \sigma^m]$. Note that the Ore extension $R^G[x^m; \sigma^m]$ is well defined. Indeed, if σ satisfies $(*)$, then by Corollary 4.8 and Lemma 4.9, we know that $\sigma \mid_{R^G} : R^G \to R^G$ is an automorphism. Since $A^H \supseteq R^G[x^m; \sigma^m]$, then $A^H = R^G[x^m; \sigma^m]$ which is AS regular by Lemma 4.5. $\square$

Example 4.11. From Example 4.6, we know that D_8 is a dual reflection group of $A = R[z; \sigma]$ by assigning

$$\deg_{D_8}(x) = r, \deg_{D_8}(y) = r\rho^2, \deg_{D_8}(z) = r\rho.$$

Another way of showing this is to apply Theorem 4.10. Let $R = \Bbbk\langle x, y\rangle/(yx + xy)$ be the quantum plane with $q = -1$. Let $G = C_2 \times C_2 = \langle a\rangle \times \langle b\rangle$. We can see that R is G-graded via the grading $\deg_G(x) = a$ and $\deg_G(y) = b$. Moreover, G is a dual reflection group of R because $R^G = \Bbbk[x^2, y^2]$. Let σ be defined by $\sigma(x) = y$ and $\sigma(y) = x$. σ clearly satisfies $(*)$, so the map $\phi\colon G \to G$ defined by $\phi(a, e) = (e, b)$ and $\phi(e, b) = (a, e)$ is an automorphism of G. Let $\psi\colon C_2 = \langle c\rangle \to \mathrm{Aut}(G)$ defined by $\psi(c) = \phi$. By Theorem 4.10, $G \rtimes_\psi C_2$ is a dual reflection group of $R[z; \sigma]$. We can show that $G \rtimes_\psi C_2$ is isomorphic to D_8 via the map

$$\begin{aligned}((a, e), e) &\mapsto r\\ ((e, b), e) &\mapsto r\rho^2\\ ((e, e), c) &\mapsto r\rho.\end{aligned}$$

We need the following lemma to provide the next result.

Lemma 4.12 ([RZ12], Corollary 1.2)**.** *Let A be a connected $\mathbb{N}$-graded $\Bbbk$-algebra which is a domain. If there is a nonzero normal element $x \in A_1$, then $A/(x)$ is AS-regular if and only if A is.*

For the next result, we want to prove a weaker converse of Theorem 4.10. Let R be an AS regular domain generated in degree 1 by $x_1, \ldots, x_n$, σ be an automorphism of R, $A = R[x; \sigma]$ be the Ore extension with $\deg(x) = 1$. Let H be a finite group such that it is a dual reflection group of A with $\deg_H(x_i) = g_i$ and $\deg_H(x) = g$. We also want to assume that σ satisfies $(*)$ with respect to H.

Proposition 4.13. *If $g^j \notin G = \langle g_1, \ldots, g_n\rangle$ for all j, then G is a dual reflection group of R with $\deg_G(x_i) = g_i$.*

Proof. Since $g^j \notin G$ for all j, then we have

$$A^H = \bigoplus_{m=0}^{\infty} R^G x^{m\cdot|g|}.$$

By assumption, A^H is AS regular. This implies that $\bigoplus_m R^G x^m$ is AS regular. Since $xr = \sigma(r)x$ for all $r \in R$ and σ is an automorphism, then x is a nonzero normal element in A_1. By Lemma 4.12, $R^G = \bigoplus_m R^G x^m/(x)$ is AS regular. $\square$

4.3 Examples

In this section, we apply Proposition 3.10 and Theorem 4.10 to find dual reflection groups of an Ore extension of some global dimension 2 AS regular algebra.

1. *G is abelian:* We have

$$R = \frac{\Bbbk\langle x, y\rangle}{(yx - qxy)}, \quad G = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$$

for some $m, n \in \mathbb{Z}_{>0}$ with the G-grading

$$\deg_G(x) = (1, 0), \deg_G(y) = (0, 1).$$

Let σ be a graded automorphism of R such that σ satisfies $(*)$ with respect to G. We know that σ is determined by $\sigma(x)$ and $\sigma(y)$. Since σ satisfies $(*)$, then $\sigma(x)$ and $\sigma(y)$ are G-homogeneous. This implies that we have the following 2 cases:

(a) $\sigma(x) = ax$ and $\sigma(y) = by$ for some $a, b \in \Bbbk^\times$. We have $\phi = id\colon G \to G$. By Theorem 4.10, $H = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$ is a dual reflection group of

$$A[z; \sigma] = \frac{k\langle x, y, z\rangle}{\begin{pmatrix} yx - qxy \\ zx - axz \\ zy - byz \end{pmatrix}}$$

for all positive integer k with the H-grading

$$\deg_H(x) = (1, 0, 0), \deg_H(y) = (0, 1, 0), \deg_H(z) = (0, 0, 1).$$

(b) $\sigma(x) = ay$ and $\sigma(y) = bx$ for some $a, b \in \Bbbk^\times$. We have $\phi\colon G \to G$ defined by $(0, 1) \leftrightarrow (1, 0)$. Notice that ϕ is an automorphism of G if and only if $m = n$. Let $\psi\colon \mathbb{Z}/k\mathbb{Z} \to \mathrm{Aut}(G)$ by $1 \mapsto \phi$, we know that ψ is well defined if and only if k is a positive even integer. By Theorem 4.10, $H = (\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}) \rtimes_\psi \mathbb{Z}/k\mathbb{Z}$ is a dual reflection group of

$$A[z; \sigma] = \frac{k\langle x, y, z\rangle}{\begin{pmatrix} yx - qxy \\ zx - ayz \\ zy - bxz \end{pmatrix}}$$

with the H-grading

$$\deg_H(x) = (1, 0, 0), \deg_H(y) = (0, 1, 0), \deg_H(z) = (0, 0, 1).$$

2. *G is nonabelian:* We have

$$R = \frac{\Bbbk\langle x, y\rangle}{(x^2 - y^2)}, \quad G = \langle g, h \mid g^2 = h^2, g^{4m} = h^{4m} = (gh)^m = (hg)^m = e\rangle = \Gamma_m$$

for some $m \in \mathbb{Z}_{>0}$ with the G-grading

$$\deg_G(x) = g, \deg_G(y) = h.$$

Follow the same steps as above, we have

(a) $H = G \oplus \mathbb{Z}/k\mathbb{Z} = \langle g, h, f \mid g^2 = h^2, g^{4m} = h^{4m} = (gh)^m = (hg)^m = e = f^k = fgf^{-1}g^{-1} = fhf^{-1}h^{-1}\rangle$ is a dual reflection group of

$$A[z;\sigma] = \frac{k\langle x, y, z\rangle}{\begin{pmatrix} x^2 - y^2 \\ zx - axz \\ zy - byz \end{pmatrix}}$$

for all positive integer k with the H-grading

$$\deg_H(x) = g, \deg_H(y) = h, \deg_H(z) = f.$$

(b) $H = \langle g, h, f \mid g^2 = h^2, g^{4m} = h^{4m} = (gh)^m = (hg)^m = e = f^k = fgf^{-1}h^{-1} = fhf^{-1}g^{-1}\rangle$ is a dual reflection group of

$$A[z;\sigma] = \frac{k\langle x, y, z\rangle}{\begin{pmatrix} x^2 - y^2 \\ zx - ayz \\ zy - bxz \end{pmatrix}}$$

for all positive even integer k with the H-grading

$$\deg_H(x) = g, \deg_H(y) = h, \deg_H(z) = f.$$

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