Monodromy of a symmetric surface

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1. Introduction and Main Results

A translation surface, in the simplest meaning, is a collection of Euclidean polygons with opposite sides identified by translation. In particular, the square-tiled surfaces (origamis) are the ones that have rich algebraic, geometric, and topological structures. We investigate a quite unique surface that is tiled by 16 squares, and its Veech group, the stabilizer of $SL_2(\mathbb{Z})$-action on this surface, is precisely $SL_2(\mathbb{Z})$. In this paper, we will study the Kontsevich-Zorich monodromy group, which encodes the homological information of translation surfaces along the $SL_2(\mathbb{Z})$-orbits, of this specific surface. We shall use this powerful tool to prove the Zariski denseness of the monodromy group. Given the Zariski denseness of such a monodromy group, with a little twist on the Singh-Venkataramana criterion, we can prove the arithmeticity of this monodromy group arising from the surface.

2. Basic Properties of Origamis

This section aims to motivate and introduce the concepts of translation surfaces and monodromy.

2.1. **From Billiards to Surfaces.** [9] An interesting scenario that we always encounter while we are in the game room is seeing the cue ball rolling and hitting on the billiard table. Then, if we view this billiard table as a Euclidean polygon, a natural following question is: How can we analyze the billiard flow on this rational angle Euclidean polygons? This means, given a polygon with angles that are rational multiples of π, we consider the trajectory of an ideal point mass that moves smoothly and bounces within the boundaries and interior and polygons where the angles of incidence and reflections are equal.

2.2. **Definition and Examples.** Let us start by defining translation surfaces and origamis, i.e., square-tiled surfaces.

Definition 2.1 (Translation surfaces)**.** *A translation surface is a collection of polygons with identifications given by translations up to cut and paste.*

Definition 2.2 (Geometric definition of Square-tiled surfaces (Origamis))**.** *An origami is a finite collection of unit squares in $\mathbb{C}$ with parallel sides identified by the translation in the way that the topological surface is orientable.*

Remark 2.3. *In the previous definition, "parallel sides identified by the translation" means that a right vertical side of a square can only be glued to a left vertical side of a square. Similarly, a top horizontal side of a square can only be glued to a bottom horizontal side of a square. In particular, we will not allow, for example, a situation where a right vertical side glues to another right vertical side of a square.*

Example 2.4. *The torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ can be obtained from the unit square $[0,1] \times [0,1]$ by identifications of opposite sides.*

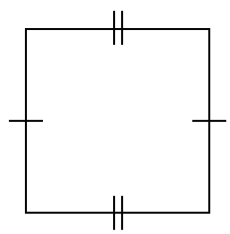

FIGURE 1. Torus

Example 2.5. *[6] Similarly, the L-shape origami in Figure 2, is obtained from a collection of 3 unit squares by identifying the opposite side with the same labels.*

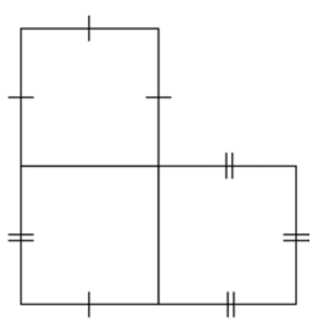

FIGURE 2. L-shape

Definition 2.6 (Holomorphic differential version of origami [6])**.** *An origami is a pair (X, ω), where X is a Riemann surface obtained as a finite cover $\pi : X \to \mathbb{T}^2 \cong \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ branched only the origin in $\mathbb{T}^2$. Moreover, this X equips with a holomorphic 1-form ω.*

Remark 2.7. *We usually denote ω as the holomorphic differential. If we have a complex manifold X, a holomorphic differential is, for every $x \in X$, the choice of a complex linear map $f(x)$ from the tangent space of X at x, mapping to $\mathbb{C}$, such that $f(x)$ depends homomorphically on x. Since our X has the complex dimensional one, the only complex linear maps from $\mathbb{C}$ to itself are the maps $z \mapsto \lambda z, \lambda \in \mathbb{C}$. We consider the identity as a holomorphic differential on $\mathbb{C}$, and we denote it as dz. Therefore, we think of the translation surfaces come from (X, ω) with an atlas of charts $X \to \mathbb{C}, z \mapsto \int_p^z \omega$ centered at $p \in X$ with $\omega(p) \neq 0$ such that the changes of coordinates are given by translations.*

The above two definitions of origamis are equivalent:

(1) Definition 2.1 $\Longrightarrow$ Definition 2.6 since a translation identification is holomorphic and the differential dz is a translation-invariant.
(2) Definition 2.6 $\Longrightarrow$ Definition 2.1 because (X, ω) can be obtained through cutting and pasting by translations of the squares given by connected components of $\pi^{-1}((0,1) \times (0,1))$.

Remark 2.8. *An origami is a special case of a translation surface, namely, origami is a collection of squares glued by translation identities.*

Definition 2.9 (Permutation-invoked definition of an origami)**.** *An origami is a pair of permutations $(h, v) \in \mathrm{Sym}_n \times \mathrm{Sym}_n$ acting transitively on $\{1, \dots, n\}$.*

Remark 2.10. *Definition 2.1 and Definition 2.9 are equivalent in the following way: If we label each square from 1 to n and use the convention of $h(i), v(i), i \in \{1, \dots, n\}$ as the next neighbor square to the right, or the next neighbor square to the top respectively of the square labeled i.*

Remark 2.11. *An action of a group on a set is transitive when the set is nonempty, and there is exactly one orbit. The action of Sym_n on $\{1, \dots, n\}$ is transitive since there is a permutation sending 1 to every other number. Here, h and v act transitively on $\{1, \dots, n\}$ is equivalent to the connectedness of the origami.*

Remark 2.12. *Since certain permutations represent corresponding origami, we are more interested in how origami preserves its structure under permutations. Thus, the pair of permutation (h, v) is usually thought equivalent up to its simultaneous conjugations, i.e., for some $\phi \in \mathrm{Sym}_n$, (h, v) and $(\phi h \phi^{-1}, \phi v \phi^{-1})$ determines the same origami.*

2.3. **Cone angles and Cone points.**

Definition 2.13. *Let p be a point on a translation surface. The cone angle at p is the number of revolutions we can take about p. We say such p to be a cone point if the cone angle at p is larger than 2π.*

Remark 2.14. *In general, the cone angle around a corner of a square of origami is a multiple of 2π, and such a point is a cone point. The way we count the cone angle is: if we start at an arbitrary corner, then we revolve in the counterclockwise orientation* *until it hits some identified side. Starting from the same side identified somewhere else on the surface, keeping revolving until it returns back to the starting corner.*

Example 2.15 (Revisited Example 2.5)**.** *The corners of all squares in the L-shape are identified into one cone point with a total cone angle* 6π *which is the summation of angles that revolve around each point.*

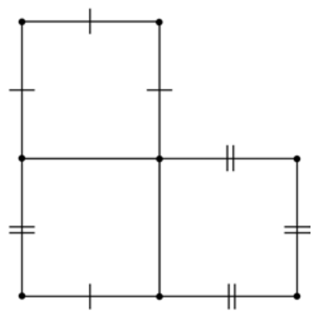

FIGURE 3. Cone points in the L-shape

Example 2.16 (Revisited Definition 2.6 [16])**.** *Given an origami* $\mathcal{O}$ *with a holomorphic 1-form* ω*, since the cone points are exactly in the integer lattice, we obtain the map*

$$\begin{aligned} \pi : \mathcal{O} &\to \mathbb{T}^2 \\ z &\mapsto \int_{p_1}^{z} \omega \, mod \, \mathbb{Z} \oplus i\mathbb{Z} \end{aligned}$$

where $\{p_1, \ldots, p_m\}$ *is the set of cone points of* $\mathcal{O}$*.*

Example 2.17. *[6]* π *of the given example in Figure 4 has the following properties:*

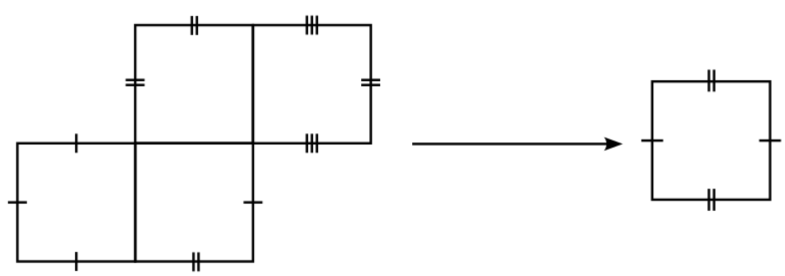

FIGURE 4. The map π from $\mathcal{O}$ to $\mathbb{T}^2$

(1) $\forall t \in \mathbb{T}^2 - origin, |\pi^{-1}(t)| = 4.$
(2) Since π *is holomorphic and surjective, it is a ramified covering where the ramification points are precisely the cone points projected to the origin. Therefore,* π *is branched at the origin means that the origin is the only point with a different preimage cardinality, i.e.* $|\pi^{-1}(origin)| = |\{p_1, \ldots, p_m\}| =$ *number of cone points.*

Borrowing the convention in Definition 2.9, from the combinatorial point of view [12], we can turn a square i around its left-bottom point by 2π by the commutator $[h, v] = vhv^{-1}h^{-1}$.

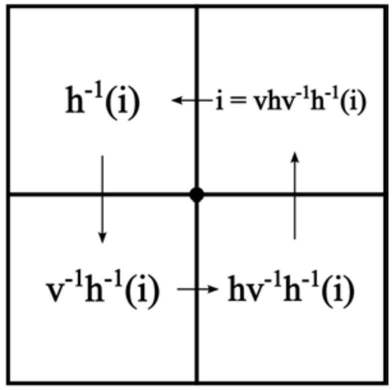

FIGURE 5. Turning by 2π around a corner by the commutator

Therefore, in an alternative point of view, a cone point corresponds to a certain non-trivial cycle c of $[h, v]$, and the corresponding cone angle of such cone point is (length of c)$\cdot 2\pi$.

Example 2.18. *The L-shape origami associates to the permutations* $h = (1\ 2)(3)$ *and* $v = (1\ 3)(2)$.

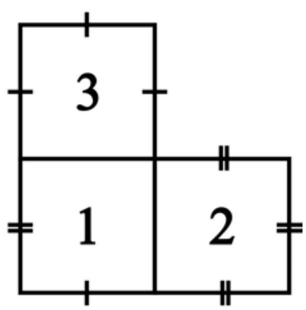

FIGURE 6. Labels of the squares in L-shape

Since the commutator $[h, v] = vhv^{-1}h^{-1} = (1\ 3\ 2)$, *the L-shape has a unique cone point of cone angle* $3 \cdot 2\pi = 6\pi$.

2.4. **Genus.** In general, Euler-Poincaré formula allows us to calculate the genus of an origami.

Theorem 2.19 (Euler-Poincaré characteristic)**.** *If a polygon has F faces, E edges, and V vertices defining a translation surface, and its corresponding torus has genus g, then*

$$2 - 2g = F - E + V$$

Alternatively, through triangulations of an origami $\mathcal{O}$, we have the following proposition:

Proposition 2.20. *[3] Suppose the set of cone points is $\{p_1, \dots, p_m\}$ and each cone point has corresponding cone angle $\{k_1, \dots, k_m\}$ and $p_i = (k_i + 1)2\pi$, the genus g can be expressed as*

$$2g - 2 = \sum_i k_i$$

Proof. Let $\mathcal{O}$ be an origami with cone points $\{p_1, \dots, p_m\}$. By Example 2.16 and Example 2.17, since origami endows a flat metric, the cone points are ramification points over the origin of $\mathbb{T}^2$. Then, given a small distance $\epsilon > 0$, a small circle with length $\epsilon \cdot 2\pi$ around the origin of the torus $\mathbb{T}^2$ lifts to a closed curve of length $(k_i + 1)\epsilon \cdot 2\pi$ around the i^{th} ramification point of $\mathcal{O}$ with $k_i \in \mathbb{Z}_{\geq 0}$. Therefore, by Gauss-Bonnet formula, we have

$$k_1 + \cdots + k_m = \sum_{i=1}^{m} k_i = 2g - 2.$$

□

This convention will provide us more insights on categorizing origamis.

Example 2.21. *In Figure 3, the L-shape has a unique cone point with cone angle 6π. Hence, by Proposition 2.20, the genus of L-shape is given by the formula*

$$2g - 2 = 2 \implies g = 2,$$

which matches with what we can deduce directly from Euler's formula.

2.5. **Stratum and moduli spaces.**

Definition 2.22. *We say an origami $\mathcal{O}$ lives in the stratum $\mathcal{H}(k_1, \dots, k_m)$ when the cone angles of the corresponding cone points are $(k_i + 1)2\pi, i = 1, \dots, m$. Usually, we omit the zero entries k_j for some $j \in \{1, \dots, m\}$.*

Example 2.23. *Since the L-shape has a unique cone point with cone angle $6\pi = (2 + 1)2\pi$, the L-shape lives in the stratun $\mathcal{H}(2)$.*

Proposition 2.24. *Suppose an origami $\mathcal{O}$ is in the stratum $\mathcal{H}(k_1, \dots, k_m)$, then $\mathcal{O}$ is tiled by at least $\sum\limits_{i=1}^{m}(k_i + 1)$ unit squares.*

Proof. By the nature of permutation and its commutator, we see that an origami $\mathcal{O}$ in $\mathcal{H}(k_1, \dots, k_m)$ is given by a pair of permutations $(h, v) \in \mathrm{Sym}_n \times \mathrm{Sym}_n$, and the commutator $[h, v] \in \mathrm{Sym}_n$ has m non-trivial cycles with length $k_i + 1, i = 1, \dots, m$. Since each non-trivial corresponds to a cone point with its neighborhood tiled by squares, we deduce that

$$N \geq \sum_{i=1}^{m}(k_i + 1)$$

to be the upper bound of the numbers of tiled squares. □

Example 2.25. *From Figure 3, L-shape is tiled by exactly 3 squares.*

Example 2.26. *The Swiss cross has only one cone point with cone angle* 6π. *Thus, it lives in* $\mathcal{H}(2)$. *However, the Swiss cross is tiled by 5 squares.*

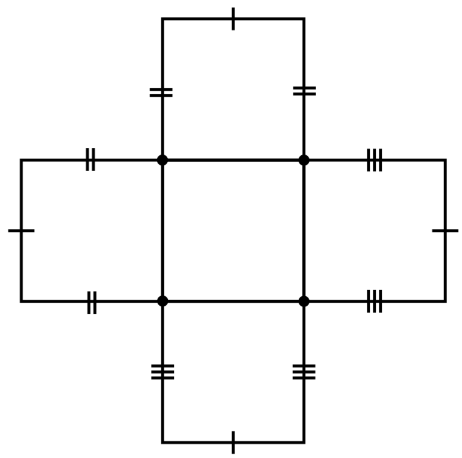

FIGURE 7. Swiss cross

Remark 2.27. *Proposition 2.24 tells us that origami in* $\mathcal{H}(2)$ *has at least 3 unit squares tiling it. Yet, in Example 2.5, the L-shape is tiled by exactly 3 unit squares, where such origami is one of the smallest possible ones living in* $\mathcal{H}(2)$.

Not only for origamis, but the idea of stratum also applies to the general settings of translation surfaces. Stratum relates to the moduli spaces of translation surfaces of genus g through fixing cone angles around cone points. The idea of constructing the moduli spaces of translation surfaces is the following: two translation surfaces can transform to one another by simply cutting and pasting by translations as Definition 2.1 suggests.

Example 2.28. *By cutting and pasting by translation in Figure 8, we see that* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbb{T}^2 = \mathbb{T}^2$ *by the means of moduli space.*

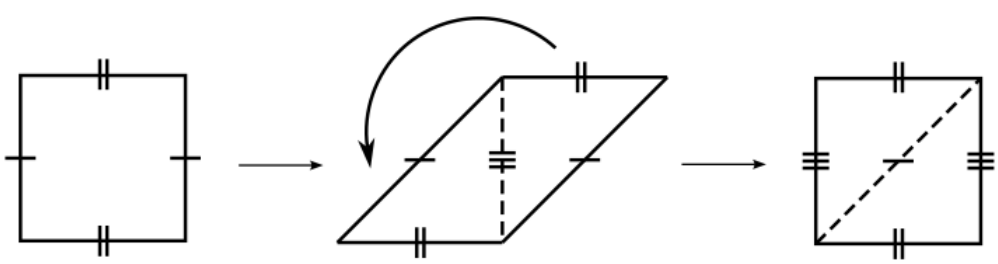

FIGURE 8. Cutting and pasting after shearing the Torus

2.6. **Saddle Connection and Holonomy Vector.**

Definition 2.29. *A saddle connection, denoted by γ, is a straight trajectory that starts at a cone point and ends at a cone point with no other cone points visited on the interior of the translation surface.*

Example 2.30. *γ (green) and δ (red) are saddle connections, but ϵ (blue) is not a saddle connection since the interior of such a path has a cone point.*

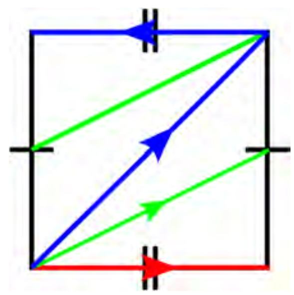

FIGURE 9. Examples of saddle connections and not saddle connections

Definition 2.31. *The holonomy vector of a saddle connection, denoted by* $\mathrm{hol}(\gamma)$*, is a vector in $\mathbb{C}$ that records the horizontal and vertical displacement of γ. The set of all holonomy vectors of a translation surface X is denoted by $\Lambda_X \subseteq \mathbb{C}$.*

Example 2.32. *In Figure 9,* $\mathrm{hol}(\gamma) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\mathrm{hol}(\delta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then, let us show two important results from Masur and Veech about the properties of saddle connections and holonomy vectors.

Theorem 2.33 (Masur [7], [9])**.** *Saddle connection directions are everywhere dense on a unit circle.*

Proof. Recall that a subset A of a space X is said to be a dense subset of X if, for every $x \in X$, every neighborhood U of x intersects A; that is, $U \cap A \neq \emptyset$. Let the set of saddle connection directions to be $\bar{\Lambda}_X = \left\{ \frac{v}{|v|} \mid v \in \Lambda_X \right\}$. Towards contradiction, suppose $\bar{\Lambda}_X$ is not dense in $\mathbb{S}^1$, then there exists $u \in \mathbb{S}^1$ s.t. for some $\delta > 0$, $B_\delta(u) \cap \bar{\Lambda}_X = \emptyset$, where u is not a saddle connection. Let A be a cone point with the trajectory L in the direction of u, and I is a line segment that is perpendicular to L which intersects at A.

Lemma 2.34. *L intersects I infinitely many times.*

Proof. By Theorem A.4 and thickening our I, L will return to this thickened I infinitely many times. Thus, L will intersect I infinitely many times. ∎

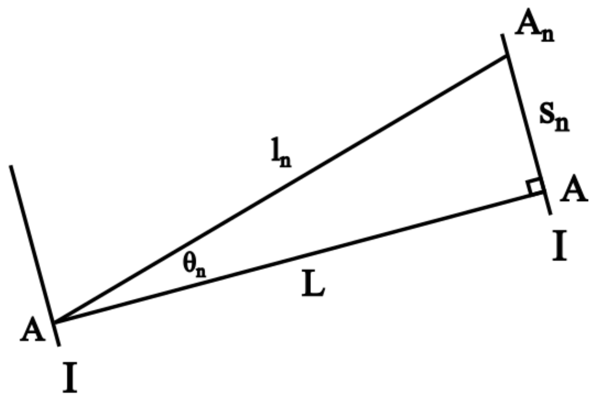

FIGURE 10. Trajectory Approximation

Then, let A_n be the point of nth intersection, and let l_n, s_n be the distances measured from A to A_n along L and along I respectively. Then, let θ_n be the angle between AA and AA_n. Since I is perpendicular to L, we have that $\tan(\theta_n) = \frac{s_n}{l_n}$.

From the claim and the assumption that u is not a saddle connection, $\lim_{n\to\infty} \tan(\theta_n) = 0$ since s_n is bounded as I is a line segment and $l_n \xrightarrow[n\to\infty]{} \infty$. Thus, $\theta_n \xrightarrow[n\to\infty]{} 0$ implies that, given $\epsilon > 0$, there exists an integer N such that for each $n > N$, $|\theta_n| < \epsilon$. Thus, for arbitrary small θ_n, the trajectory L eventually goes to the cone point A, which implies that u is a saddle connection and contradicts the assumption that u is not. Therefore, $\bar{\Lambda}_X$ is dense on a unit circle, and our theorem follows. □

Theorem 2.35 (Veech [14], [9])**.** *Holonomy vectors are discrete in* $\mathbb{R}^2$.

Proof. A set is a discrete set if each point in the set is an isolated point, i.e. a point that has a neighborhood that contains no other points of the set.
For a translation surface X, there are only finitely many cone points, then we have that, for arbitrary point p with some radius ϵ_p, there is a punctured disk of such radius centered at point p which contains no cone points. Then consider any vector $v \in \mathbb{R}^2$. For each cone point, we can generate a trajectory of v. Since there are only finitely many cone points and angles, we only have finitely many of these rays. Let $\epsilon = min(\epsilon_p)$, where p runs over all the endpoints of the paths of these rays. By construction, there is no saddle connection ending within the punctured ϵ-disk about the end point of any of the rays. This means that v cannot be the limit of holonomy vectors of saddle connections, which means there is some neighborhood of v, namely $B_{\epsilon_p}(v)$, containing no other saddle connections. Since v is chosen arbitrarily, we conclude that holonomy vectors are discrete in $\mathbb{R}^2$. □

2.7. **Reduced and Primitive origamis.** From the previous section, we introduce saddle connections and holonomy vectors. This will invoke the idea of reduced and primitive origamis.

Definition 2.36. *The period lattice $Per(\omega)$ of an origami $\mathcal{O} = (X, \omega)$ is the lattice spanned by the holonomy vectors of saddle connections γ whose endpoints are cone points of $\mathcal{O}$.*

Definition 2.37. *An origami $\mathcal{O}$ is reduced if its period lattice $Per(\omega)$ is $\mathbb{Z} \oplus i\mathbb{Z}$.*

Intuitively, reduced origami means there will not be any unnecessary squares that construct the surfaces. In other words, reduced origami always uses the least amount of squares to tile the desired surfaces. Let us take a look at the L-shape again:

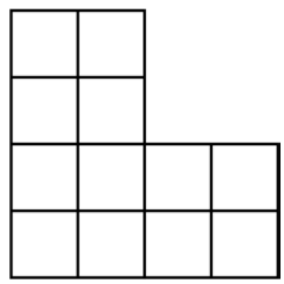

FIGURE 11. L-shape tiled by 12 squares

This L-shape tiled by 12 squares is not a reduced origami since only 3 squares are necessary for constructing such an L-shape. In other words, recall Proposition 2.24, the reduced origami has necessary the tightest upper bound for the number of tiling squares $N = \sum_{i=1}^{m}(k_i + 1)$.

Remark 2.38. *From the above intuition, we can reduce an arbitrary origami by scaling the squares.*

Let us consider the set $Sq(\mathcal{O}) \cong \{1, \dots, n\}$ to be the squares that tile the origami, then, from Definition 2.9, one can encode the origami by two permutations $h, v \in \mathrm{Sym}(Sq(\mathcal{O})) \cong \mathrm{Sym}_n$. We say that $\mathcal{O}$ covers an origami $\mathcal{O}' = (X', \omega')$ if the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "\psi"] & X' \arrow[dl, "\psi'"] \\ \mathbb{T}^2 & \end{tikzcd}$$

commutes for a ramified covering ϕ [12]. Moreover, $\mathcal{O}$ is a proper ramified covering of $\mathcal{O}'$ if the degrees of ramified coverings ϕ and ψ' are greater than 1. This gives arises to the following sufficient and necessary conditions between the covering of origami and the composition of functions.

Proposition 2.39 (Zmiaikou [15]). *For two origamis $\mathcal{O}$ and $\mathcal{O}'$ represented by $h, v \in \mathrm{Sym}(Sq(\mathcal{O}))$ and $h', v' \in \mathrm{Sym}(Sq(\mathcal{O}'))$, we say $\mathcal{O}$ covers $\mathcal{O}'$ if and only if there exists a function $\pi : Sq(\mathcal{O}) \to Sq(\mathcal{O}')$ such that $\pi \circ h = h' \circ \pi$ and $\pi \circ v = v' \circ \pi$.*

Proof. ($\Longrightarrow$) Suppose the above diagram commutes for a ramified covering ϕ, and also if the right/top edge of a square i glues to the left/bottom edge of another square j correspondingly, then, since ϕ sends the squares in $\wr$ to squares in $\mathcal{O}'$, the same gluing is valid for squares $\phi(i)$ and $\phi(j)$ of $\mathcal{O}'$. Therefore, the map $\pi : i \to \phi(i), i \in Sq(\mathcal{O})$ satisfies our requirement.
($\Longleftarrow$) Suppose such function π exists, since origamis are a connected surface, π is surjective. For arbitrary squares $i, j \in Sq(\mathcal{O})$ such that $h(i) = j$, then the mapping π preserves the permutation structure, i.e. $\pi(j) = h'(\pi(i))$. Therefore, if we define the mapping ϕ that sends the square i to $\pi(i)$, then, by definition, such ϕ is a ramified covering, and the induced diagram is commutative. Hence, our proposition derives from the above arguments. □

Definition 2.40. *An origami is primitive if it is not a non-trivial cover of some other origamis.*

Remark 2.41. *A primitive origami is certainly reduced since we cannot find any covers of other origami (no extra squares needed), yet a reduced one is not necessarily primitive.*

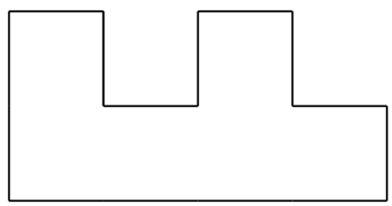

FIGURE 12. Reduced but not primitive origami

In Figure 12, the origami with opposite sides identified by translation is reduced but not primitive since it is covered by 2 L-shapes.

Let us see what this primitivity means precisely. If we define an origami $\mathcal{O}$ by permutations $h, v \in \mathrm{Sym}_n$ and denote G to be the subgroup of $\mathrm{Sym}(Sq(\mathcal{O}))$ that is generated by h and v. Then, we have the following remark.

Remark 2.42. *A nonempty subset $\beta \subseteq Sq(\mathcal{O})$ is a block for the permutation group $G \subseteq \mathrm{Sym}_n$ if the following condition holds: for each $g \in G$, either $g(\beta) = \beta$ or $g(\beta) \cap \beta = \emptyset$. Then, an origami $\mathcal{O}$ is primitive if and only if G is primitive, i.e. there is no block β exists other than the singletons of $Sq(\mathcal{O})$ or the whole $Sq(\mathcal{O})$.*

3. Action of $SL_2(\mathbb{R})$ and Homology on origamis

In this section, we will explore the dynamical and topological properties of origami through its $SL_2(\mathbb{R})$ and homology $H_1(\mathcal{O})$.

3.1. Action of $SL_2(\mathbb{R})$ on origamis.

Let us consider the following two groups:

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$
$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

The linear action of elements in $SL_2(\mathbb{R})$ on $\mathbb{R}^2$ induces natural actions on origamis. We apply an arbitrary element $g \in SL_2(\mathbb{R})$ to each unit square tiling a given origami $\mathcal{O}$ and keep the opposite sides identified by translations. In particular, $g(\mathcal{O})$ is a translation surface of area n whence $\mathcal{O}$ is tiled by n units squares.

Example 3.1. *The action by the matrix* $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ *on the Torus is in Figure 13:*

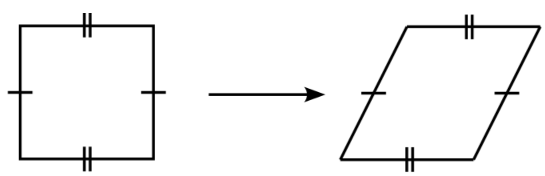

Figure 13. Example of $SL_2(\mathbb{R})$ action

Proposition 3.2. *$SL_2(\mathbb{R})$ acts equivariently on holonomy vectors of a translation surface X. That is $\Lambda_{gX} = g \cdot \Lambda_X$.*

Proof. For an arbitrary element x in Λ_{gX}, x is a holonomy vector of saddle connections in gX. That is, there exist some $x' \in X$ such that $gx' = x$, and $gx \in gX$, which implies $\Lambda_{gX} \subseteq g \cdot \Lambda_X$. Moreover, since $g \in SL_2(\mathbb{R})$ only changes the shape of a translation surface, g preserves the translation identities and geometric properties and such translation surface, i.e. g does not change the number of cone points and saddle connections. Therefore, we cannot have the strict inclusion $\Lambda_{gX} \subset g \cdot \Lambda_X$. Hence, $\Lambda_{gX} = g \cdot \Lambda_X$. □

Remark 3.3. *The translation surface $g(\mathcal{O})$ is not necessarily an origami. However, if $g \in SL_2(\mathbb{Z})$, then $g(\mathcal{O})$ is an origami.*

Example 3.4. *Torus under T and S: notice that* $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, *the generators of $SL_2(\mathbb{Z})$, preserves the structure of torus, i.e. $T(\mathbb{T}^2) = \mathbb{T}^2$ and $S(\mathbb{T}^2) = \mathbb{T}^2$.*

A natural question arises from the torus: for an arbitrary translation surface $\mathcal{O}$, what are the elements in $SL_2(\mathbb{R})$ that preserves the structure of our translation surface, how about for origamis? This question motivates the following interesting object - Veech groups.

3.2. **Veech groups.**

Definition 3.5. *The Veech group of a translation surface X, denoted as* $\mathrm{SL}(X)$*, is the set*

$$\mathrm{SL}(X) = \{g \in SL_2(\mathbb{R}) \,|\, gX = X\}.$$

Proposition 3.6. $\mathrm{SL}(X)$ *is a group under composition, that is:*

- $id_{SL_2\mathbb{R}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(X)$.
- *if* $g, h \in \mathrm{SL}(X)$*, then* $gh \in \mathrm{SL}(X)$.
- *If* $g \in \mathrm{SL}(X)$*, then* $g^{-1} \in \mathrm{SL}(X)$.

Proof. Since the composition of linear transformations (matrix multiplication) is an associative operation of $\mathrm{SL}(X)$, it suffices to check $\mathrm{SL}(X)$ satisfies the group axioms:

- Existence of identity: For any vector representations x of the translation surface X, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = x$. Therefore, the identity exists in $\mathrm{SL}(X)$ which is precisely $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- Closed under composition: Let $g, h \in \mathrm{SL}(X)$, then we have $gX = X$ and $hX = X$. Therefore, we have $(gh)X = g(hX) = gX = X$, which means that $gh \in \mathrm{SL}(X)$.
- Existence of inverse: let $g \in \mathrm{SL}(X)$ where $gX = X$. Since $g \in SL_2(\mathbb{R})$, the inverse $g^{-1} \in SL_2(\mathbb{R})$ satisfies $gg^{-1} = g^{-1}g = id_{SL_2(\mathbb{R})}$. Then,

$$\begin{aligned} gX = X &\implies g^{-1}(gX) = g^{-1}(X) \\ &\implies (g^{-1}g)X = g^{-1}X \\ &\implies X = g^{-1}X \\ &\implies g^{-1} \in \mathrm{SL}(X) \end{aligned}$$

Thus, $\mathrm{SL}(X)$ is a group under linear transformations. □

3.3. $SL_2(\mathbb{Z})$**-orbits.**

Since the origamis are tiled by unit squares, the Veech group admits certain integer structures under linear transformations. In general, we always consider T and S that generate $SL_2(\mathbb{Z})$ to study the behavior of Veech groups and $SL_2(\mathbb{Z})$-orbits of origamis.

Example 3.7. *The action of T and S on Torus are in Figure 14 and 15:*

Proposition 3.8. *The Veech group of a torus is precisely* $SL_2(\mathbb{Z})$.

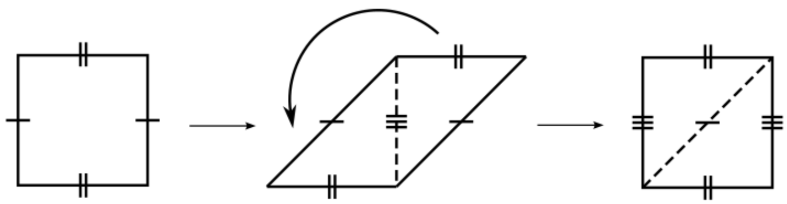

FIGURE 14. Action of T on Torus

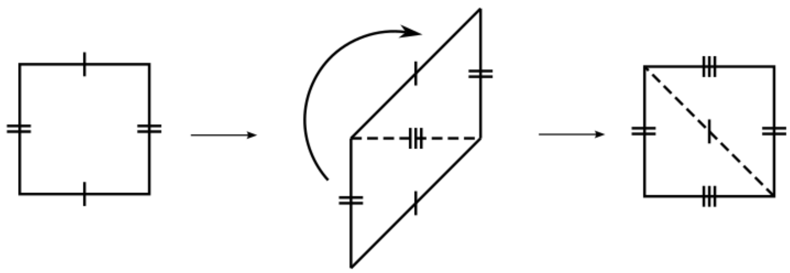

FIGURE 15. Action of S on Torus

Proof. Every element is $SL_2(\mathbb{Z})$ is a product of T and S. Hence, $SL_2(\mathbb{Z}) \subseteq \mathrm{SL}(\mathbb{T})$. In fact, we claim that the other containment is true. Suppose, without loss of generality, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ has an entry not in $\mathbb{Z}$. For example, suppose $a \notin \mathbb{Z}$ $\left(\text{e.g. } \begin{pmatrix} \pi & b \\ c & d \end{pmatrix} \text{ or } \begin{pmatrix} 1/2 & b \\ c & d \end{pmatrix}\right)$. (We work with one of these matrices, and a similar argument works if any other element in g is not in $\mathbb{Z}$.) By equivariance of the $SL_2(\mathbb{R})$-action, the saddle connection $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Lambda_{\mathbb{T}} = \mathrm{prim}(\mathbb{Z}^2)$ gives rise to a saddle connection $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Lambda_{g\mathbb{T}}$ since $\Lambda_{g\mathbb{T}} = g\Lambda_{\mathbb{T}}$. But $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \pi \\ c \end{pmatrix}$ which is clearly not inside of $\Lambda_{\mathbb{T}}$ since it has a non-integer component. We conclude that if g has a non-integer element, then the saddle connections of $\mathbb{T}$ and $g\mathbb{T}$ are different. Hence, $\mathbb{T}$ and $g\mathbb{T}$ are different translation surfaces. Taking the contrapositive tells us that if $\mathbb{T}$ and $g\mathbb{T}$ are the same translation surface, then g must have all components in $\mathbb{Z}$. That is, $g \in SL_2(\mathbb{Z})$. Hence, $\mathrm{SL}(\mathbb{T}) \subseteq SL_2(\mathbb{Z})$. Combining these containments, we showed that the Veech group of the torus $\mathbb{T}$ is exactly $SL_2(\mathbb{Z})$. $\square$

Remark 3.9. *For an arbitrary (reduced) origami $\mathcal{O}$, its Veech group $\mathrm{SL}(\mathcal{O})$ is a subset of $SL_2(\mathbb{Z})$. This was proved in [4].*

Combinatorially, the $SL_2(\mathbb{Z})$-orbits of origami can be realized by two permutations $h, v \in \mathrm{Sym}_n$ in Definition 2.9.

Therefore, on $T(\mathcal{O})$, the right neighbor of a square i is $h(i)$, and the top neighbor of i is $vh^{-1}(i)$. Hence, the action of T on pairs of permutations sends $\mathcal{O} = (h, v)$ to $T(\mathcal{O}) = (h, vh^{-1})$.

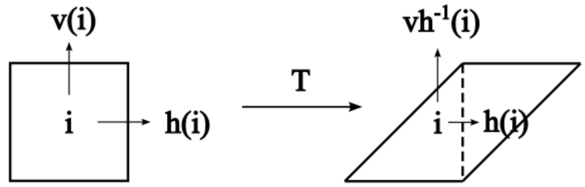

FIGURE 16. Action of T on the pairs of permutations

Similarly, the action of S on pairs of permutations can be obtained through the symmetry of origamis. Hence, S sends $\mathcal{O} = (h, v)$ to $S(\mathcal{O}) = (hv^{-1}, v)$.

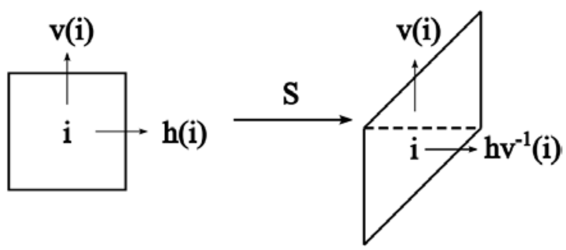

FIGURE 17. Action of S on the pairs of permutations

Remark 3.10. *By the nature of T and S, T preserves all the horizontal waist curves, thus it preserves the horizontal permutation h. Moreover, S preserves the vertical permutation v of origamis.*

Example 3.11. *The $SL_2(\mathbb{Z})$-orbit of the L-shape is in Figure 18. Notice that when T and S apply to the L-shape, even though they "look" similar to the L-shape, but their translation identifications are slightly off. However, applying T and S again will give us the original L-shape back, i.e., T^2(L-shape) = L-shape and S^2(L-shape) = L-shape.*

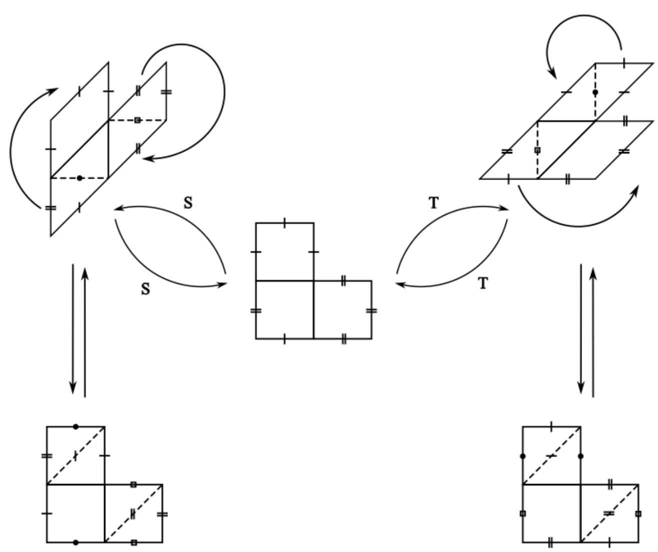

FIGURE 18. $SL_2(\mathbb{Z})$-orbits of L-shape

3.4. **Affine homeomorphisms.** An automorphism of an origami $\mathcal{O} = (X, \omega)$ is a biholomorphism $f : \mathcal{O} \to \mathcal{O}$ with respect to ω. The group of automorphism of an origami $\mathcal{O}$ is denoted by $\mathrm{Aut}(\mathcal{O})$.
An affine homeomorphism $g : (X, \omega) \to (X, \omega)$ of an origami (X, ω) is an orientation preserving homeomorphism with respect to the cone points that is affine in the charts $z \to \int_p^z \omega$. The group of affine homeomorphisms of (X, ω) is denoted by $\mathrm{Aff}(X, \omega)$.
An affine map on $\mathbb{R}^2$ is the action of a linear transformation with a translation. Therefore, from [2], we have the following short exact sequence:

$$1 \to \mathrm{Aut}(\mathcal{O}) \to \mathrm{Aff}(\mathcal{O}) \to \mathrm{SL}(\mathcal{O}) \to 1.$$

In particular, if $\mathrm{Aut}(\mathcal{O}) = 1$, then $\mathrm{Aff}(\mathcal{O}) \cong \mathrm{SL}(\mathcal{O})$.

3.5. **Homology of origamis.** If we want to study how $SL_2(\mathbb{Z})$ acts on an origami, it naturally leads us to study how $SL_2(\mathbb{Z})$ acts on the homology of the origami. Here, the combinatorial definition of origami reveals the essence of the basis of homology and its action of $SL_2(\mathbb{Z})$.
Let $\mathcal{O}$ be an origami with a pair of permutations $h, v \in \mathrm{Sym}_n$ and denote Σ to be the collection of all the corner points in the origami tiled by n squares. For each square $Sq(i), i \in \{1, \dots, n\}$, let σ_i be the horizontally oriented cycle (from left to right) corresponding to the bottom two corner points, and ζ_i to be the vertically oriented cycle (from bottom to top) corresponding to the left two corner points. If we apply h

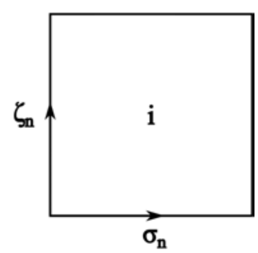

FIGURE 19. Relative homology cycles of an origami

to $Sq(i)$, the bottom side of $h(i)$ is precisely the top side of i, namely $\sigma_{h(i)}$. Similarly, the left side of $v(i)$ is the right side of i, namely $\zeta_{v(i)}$. The relative homology group $H_1(\mathcal{O}, \Sigma, \mathbb{R})$ is the vector space spanned by the cycles σ_i, ζ_i with the relation:

$$\sigma_i + \zeta_{h(i)} = \sigma_{v(i)} + \zeta_n.$$

Essentially, this relation describes the homotopy equivalence of the boundary of $Sq(i)$: the vector space $H_1(\mathcal{O})$ with all cycles in $H_1(\mathcal{O}, \Sigma, \mathbb{R})$ with zero boundaries is the absolute homology of $\mathcal{O}$. Thus, for a genus g origami, $H_1(\mathcal{O})$ is isomorphic to $\mathbb{R}^{2g}$, which has dimension $2g$.
The absolute homology $H_1\mathcal{O}$ naturally induces the structure of symplectic vector space with respect to the intersection form. The intersection form of two oriented curves $\alpha, \beta \in H_1(\mathcal{O})$, denoted as $\langle \alpha, \beta \rangle$, records the number of intersections those two curves have that attribute a sign $\pm$ based on the orientation of two curves:

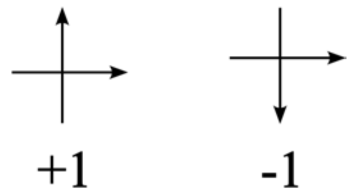

FIGURE 20. Positive or negative sign for corresponding intersections

The tautological plane $H_1^{st}(\mathcal{O})$ is the plane in absolute homology spanned by the cycles

$$\sigma := \sum_{i \in Sq(\mathcal{O})} \sigma_i$$
$$\zeta := \sum_{i \in Sq(\mathcal{O})} \zeta_i.$$

Since σ_i and ζ_i intersects once in the interior of $Sq(i)$, $H_1^{st}(\mathcal{O})$ is a symplectic plane as $\langle \sigma, \zeta \rangle = n = |Sq(\mathcal{O})|$. Therefore, $H_1(\mathcal{O})$ induces a natural decomposition [5]

$$H_1(\mathcal{O}) = H_1^{st}(\mathcal{O}) \oplus H_1^{(0)}(\mathcal{O})$$

where $H_1^{(0)}(\mathcal{O}) = (H_1^{st}(\mathcal{O}))^{\perp} = \{\gamma \in H_1(\mathcal{O}) \mid \langle \gamma, H_1^{st}(\mathcal{O}) = 0 \rangle\} = \{\gamma \in H_1(\mathcal{O}) \mid \int_\gamma \omega = 0\}$. In other words, $H_1^{(0)}(\mathcal{O})$ is the symplectic orthogonal of $H_1^{st}(\mathcal{O})$.

Remark 3.12. *Since our studying object only consists of origamis, the decomposition* $H_1(\mathcal{O}) = H_1^{st}(\mathcal{O}) \oplus H_1^{(0)}(\mathcal{O})$ *is defined over* $\mathbb{Z}$.

4. Actions on Homologies and Kontsevich-Zorich Monodromy of origamis

In this section, the main target is to study the actions on homology and introduce the idea of Kontsevich-Zorich monodromy, which is the one of most critical points of the area.

4.1. **Cylinder decomposition.**

Definition 4.1. *A horizontal cylinder of a translation surface on a saddle connection is a horizontal subset of the translation surface that only consists of a pair of non-horizontal opposite sides identified by translation, that is, a max Euclidean cylinder living on a translation surface with a saddle connection as a boundary that is not contained in a larger horizontal cylinder.*

Definition 4.2. *A vertical cylinder of a translation surface on a saddle connection is a vertical subset of the translation surface that only consists of a pair of non-vertical opposite sides identified by translation, that is, a max Euclidean cylinder living on a translation surface with a saddle connection as a boundary that is not contained in a larger vertical cylinder.*

Definition 4.3. *(General definition of a cylinder) A cylinder is a maximal collection of parallel closed geodesics.*

In general, any origami can decompose as a union of finitely many cylinders of closed geodesics in any arbitrary fixed rational directions. This decomposition is called the cylinder decomposition of origami. For example, the Eierlegende-Wollmilchau (EW) surface can be decomposed into horizontal and vertical cylinders:

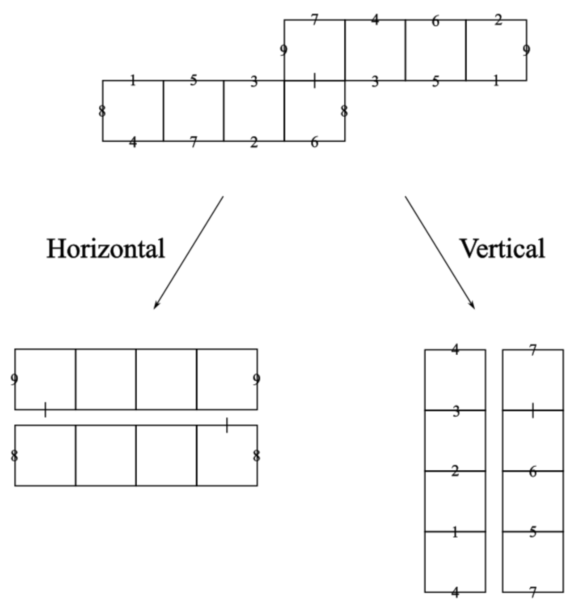

Figure 21. Example of cylinder decomposition of EW surface

4.2. **Dehn twist.** A horizontal cylinder is isometric to a rectangle strip $[l,0]\times[0,h]$ whose vertical identified sides are $\{0\}\times[0,h]$ and $\{l\}\times[0,h]$. The matrix $t=\begin{pmatrix}1 & \frac{l}{h}\\ 0 & 1\end{pmatrix}$ acts on the horizontal cylinder in the way that t shears the horizontal cylinder into a parallelogram that one can cut this parallelogram into two pieces of triangles and glue them back by translation into the original horizontal cylinder.

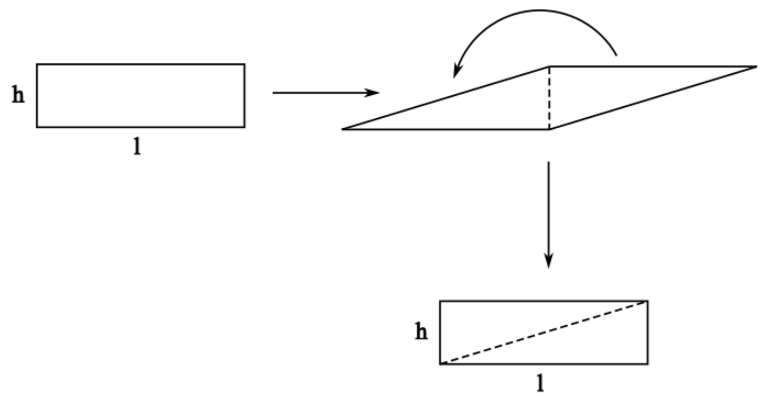

FIGURE 22. Cylinder cutting and pasting

In this process, the horizontal waist curve $\omega=[l,0]\times\{\frac{h}{2}\}$ is preserved since $t\begin{pmatrix}l\\0\end{pmatrix}=\begin{pmatrix}l\\0\end{pmatrix}$. However, the vertical cycles ν connecting the bottom side $[l,0]\times\{0\}$ to the top side $[l,0]\times\{h\}$ acts by t though $t\begin{pmatrix}0\\h\end{pmatrix}=\begin{pmatrix}l\\h\end{pmatrix}$, which tells us that t sends the vertical cycles to

$$t(\nu)=\nu+\omega.$$

The above operation is called the Dehn twist about ω.

4.3. **Kontsevich-Zorich monodromy.** In this part, we will introduce the most critical object that we study in this paper, which is to see how the action on the homology of affine homeomorphisms of origami behaves.

In general, the action on the homology of affine homeomorphisms of an origami $\mathcal{O}$ arises a representation

$$\tilde{\alpha}:\mathrm{Aff}(\mathcal{O})\to\mathrm{Sp}(H_1(\mathcal{O})).$$

The image $\tilde{\alpha}(\mathrm{Aff}(\mathcal{O}))$ respects the decomposition of homology $H_1(\mathcal{O})=H_1^{st}(\mathcal{O})\oplus H_1^{(0)}(\mathcal{O})$ in section 3.5. Moreover, $\tilde{\alpha}|_{H_1^{st}(\mathcal{O})}$ can be realized by the Veech group $\mathrm{SL}(\mathcal{O})$. Therefore, we only have to understand how $\tilde{\alpha}$ behave restricted on the zero holonomy subspace to understand the full behavior of the action of $\mathrm{Aff}(\mathcal{O})$. We define the image restricted on the zero holonomy subspace $\alpha=\tilde{\alpha}|_{H_1^{(0)}}$ as the Kontsevich-Zorich monodromy of $\mathcal{O}$, equivalently saying, an affine homeomorphism A of $\mathcal{O}$ acts on $H_1^{st}(\mathcal{O})$ via linear actions of $SL_2(\mathbb{Z})$ on $\mathbb{Q}^2$, and the subgroup of $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}))$ with a genus

g surface generated by actions on $H_1^{(0)}(\mathcal{O})$ of all affine homeomorphism of $\mathcal{O}$ is the Kontsevich-Zorich monodromy.

In the sense of Sarnak [11], an origami $\mathcal{O}$ has arithmetic/thin Kontsevich-Zorich monodromy if the monodromy is Zariski dense in $\mathrm{Sp}_{2g-2}(\mathbb{R})$ with finite/infinite index in the Zariski closure of the monodromy.

4.4. **Zariski denseness and Arithmeticity of Kontsevich-Zorich monodromy.** In this section, we introduce some interesting and wonderful properties of the Kontsevich-Zorich monodromy.
The Zariski denseness of the Kontsevich-Zorich monodromy Γ of an origami $\mathcal{O}$ of genus g is Zariski dense in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}, \mathbb{R}))$ if the following property holds:

Theorem 4.4. *(Matheus-Möller-Yoccoz [8], Prasad-Rapinchuk [10]) Using the actions on the homology of Dehn twist associated with cylinder decompositions in rational direction, suppose we have*

- *We can combine such Dehn twists to produce a Galois-pinching element* $A \in \mathrm{Sp}(H_1^{(0)}(\mathcal{O}, \mathbb{Z}))$*, i.e., a symplectic matrix whose characteristic polynomial is irreducible over* $\mathbb{Z}$*, splits over* $\mathbb{R}$*, and possesses the largest possible (hyperoctahedral) Galois group among reciprocal polynomials of degree* $2g-2$
- *some Dehn twist induces a non-trivial unipotent element* $B \in \mathrm{Sp}(H_1^{(0)}(\mathcal{O}, \mathbb{R}))$ *such that* $(B - id)(H_1^{(0)}(\mathcal{O}, \mathbb{R}))$ *is not Lagrangian subspace of* $H_1^{(0)}(\mathcal{O}, \mathbb{R})$.

Then, Γ *is Zariski dense in* $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}, \mathbb{R}))$.

Moreover, in Sarnak's sense, the monodromy Γ is arithmetic if it is Zariski dense and has a finite index in $G(\mathbb{Z})$, and Γ is thin if it is Zariski dense and has an infinite index in $G(\mathbb{Z})$ where G is the Zariski closure of its Kontsevich-Zorich monodromy.

4.5. **4-cylinder decomposition and transvections.** To prove the arithmeticity of Γ, we can use the method about cylinder decomposition and transvections to check the finiteness of the index for Γ.

Theorem 4.5. *(Singh-Venkataramana criterion [1], [13]) Suppose that*

- Θ *is a non-degenerate symplectic form on* $\mathbb{Q}^{2n}$ *which is integral on the standard lattice* $\mathbb{Z}^{2n}$
- $\Gamma \subseteq \mathrm{Sp}_{2n}(\mathbb{Z})$ *is a Zariski dense subgroup which contains three transvections* C_1, C_2, C_3 *such that if we write* $\mathbb{Z}w_i = (C - id)(\mathbb{Z}^{2n})$*, then* $\Theta(w_i, w_j) \neq 0$ *for some* i, j.
- $W = \langle w_1, w_2, w_3 \rangle$ *is three dimensional, and the group generated by the restrictions of the* C_i *to* W *contains a non-trivial element of the unipotent radical* $\mathrm{Sp}(W)$

Then the group Γ *has finite index in* $G(\mathbb{Z})$*, which implies* Γ *is arithmetic.*

Then, the cylinder decomposition and Dehn twists help us to form the criterion of checking arithmeticity.
For each direction θ decomposing an origami $\mathcal{O}$ into cylinders with homological dimension 2, we can associate an affine multitwist action on $H_1^{(0)}(\mathcal{O},\mathbb{Q})$ as a transvection. Recall that the moduli of a cylinder Cyl is the quotient of the height(h_{Cyl}) by the circumference(c_{Cyl}). If in some direction the translation flow decomposes $\mathcal{O}$ into a cylinder Cyl_i whose moduli μ_i satisfied $\mu = \frac{n_i}{\lambda}$, i.e. these moduli are commensurable, then there exists a unique affine automorphism C of $\mathcal{O}$ that fixes the boundaries of these cylinders and has derivative $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.
In origami $\mathcal{O}$, every cylinder decomposition represents cylinders of commensurable moduli. Let us denote C to be the affine automorphism of an origami $\mathcal{O}$ associated with the cylinder direction θ. The action of C on $H_1^{st}(\mathcal{O},\mathbb{Q})$ is given by:

$$C_* = id + c_1 f_2 \Theta(\cdot,\gamma_1)\gamma_1 + \gamma_2 f_1 \Theta(\cdot,\gamma_2)\gamma_2$$

where $c_i, f_i \in \mathbb{Q}$ are constants from the geometry of cylinder decomposition.
For every $\beta \in H_1^{(0)}(\mathcal{O},\mathbb{Q})$, $c_1\Theta(\beta,\gamma_1) = -c_2\Theta(\beta,\gamma_2)$. Therefore, if $Z = f_2\gamma_1 - f_1\gamma_2$, the restriction of C_* to $H_1^{(0)}(\mathcal{O},\mathbb{Q})$, which we denote C_Z, is given by the transvection:

$$C_Z - id + c\Theta(\cdot,Z)Z$$

for some $c \in \mathbb{Q}$.
Now, we can describe a metacode from [1] that allows us to apply Theorem 4.4 (The Singh-Venkataramana criterion):

- Find three different 4-cylinder directions $\theta_1,\theta_2,\theta_3$. For each θ_n, we can find an element $X_n \in H_1^{(0)}(\mathcal{O},\mathbb{Q})$ and appropriate affine Dehn multitwist on the origami. When restricted on $\mathrm{Sp}(H_1^{(0)}(\mathcal{O},\mathbb{Q})) \cong \mathrm{Sp}_{2n}(\mathbb{Q})$, this Dehn multitwist defines a transvection C_{X_n}.
- For linearly independent X_1,X_2,X_3, we have $(C_{X_n} - id)(\mathbb{Z}^{2n}) = \mathbb{Z}X_n$ for $n - 1,2,3$ and $\Theta(X_i,X_j) \neq 0, i \neq j$.
- Let W be the $\mathbb{Q}$-vector space generated by $\{X_1,X_2,X_3\}$, then there exists the annihilator element $e \in W$ such that

  $$e = -\frac{\Theta(X_3,X_2)}{\Theta(X_1,X_2)}X_1 - \frac{\Theta(X_3,X_1)}{\Theta(X_2,X_1)}X_2 + X_3.$$

  Moreover, the unipotent radical of $\mathrm{Sp}(W)$ in the ordered base $\{X_i,X_j,e\}$ has the form:

  $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & x & 1 \end{pmatrix}, x,y \in \mathbb{Q}.$$

  Derive the restrictions on C_{X_n} in the basis $\{X_i,X_J,e\}$. Then, it suffices to find an appropriate word in the letters $C^{\pm}_{X_1}, C^{\pm}_{X_2}, C^{\pm}_{X_3}$ that generates the unipotent radical.

5. Basic Properties of a symmetric surface X

In this section, we mainly focus on introducing the symmetric surface X with its basic properties.
Let us consider the surface X in Figure 23 with opposite sides identified by translations of the same symbols and numbers, i.e. the side labeled 3 is identified with another 3 labeled elsewhere, and so on.

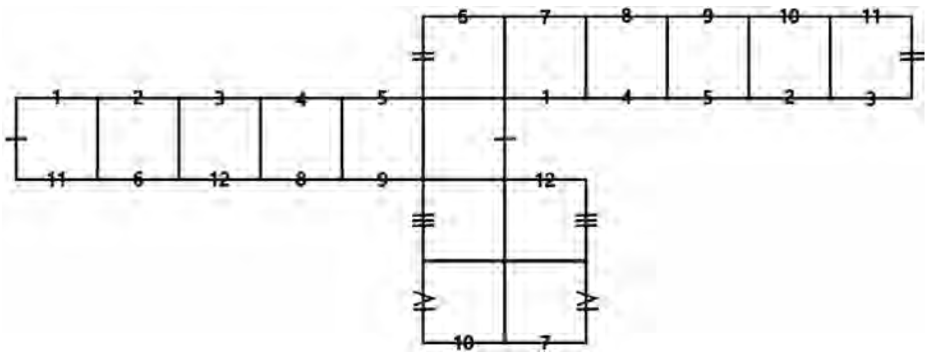

Figure 23. The surface X

We label all the cone points in Figure 24 to facilitate the following propositions.

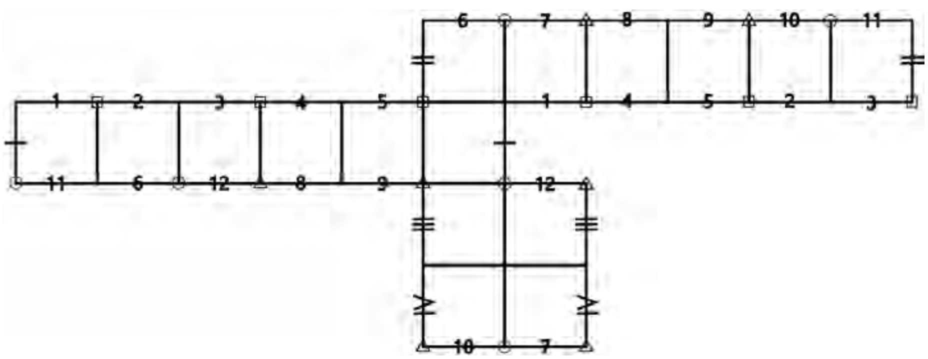

Figure 24. Cone points of X

Proposition 5.1. *The surface X has genus 4.*

Proof. By the translation identities, we have exactly 16 distinct edges with 9 points for our surface. Then, the entire surface has only 1 face since the connected edges inside the surface do not generally contribute to the face count. Therefore, by Euler-Poincaré characteristic, we deduce that

$$\begin{aligned} 2 - 2g &= F - E + V \\ &= 1 - 16 + 9 \\ &\implies g = 4 \end{aligned}$$

□

Moreover, we want to classify what moduli space the surface X live in, which invokes the following proposition.

Proposition 5.2. *The stratum of X is $\mathcal{H}(2,2,2)$*

Proof. Since we have 3 cone points from Figure 24, the cone angles corresponding to each cone point are the following:

- For the hollow circle, it has a total cone angle of $6\pi = (2+1)2\pi$.
- For the hollow rectangle, it has a total cone angle of $6\pi = (2+1)2\pi$.
- For the hollow triangle, it has a total cone angle of $6\pi = (2+1)2\pi$.

The cone angles consists 6 angles of $2\pi = (0+1)2\pi$ and 3 angles of $6\pi = (2+1)2\pi$. Thus, the stratum should be $\mathcal{H}(2,2,2)$. ☐

Remark 5.3. *We can use cone points to double-check the genus, namely, from Proposition 2.20, we have*

$$\begin{aligned} 2g-2 &= \sum_i a_i \\ &= 2+2+2 \\ &\implies g=4 \end{aligned}$$

which verifies Proposition 5.1.

The reason we care about this surface is that the Veech group of X is precisely $SL_2(\mathbb{Z})$, which does not quite happen often since one can only have the weaker statement: the Veech group of an arbitrary origami $\mathcal{O}$ is a subset of $SL_2(\mathbb{Z})$ from Remark 3.9.
Let us assign the permutation on the surface. Consider

$$h = (1\ 2\ 3\ 4\ 5\ 6)(12\ 11\ 10\ 9\ 8\ 7)(13\ 14)(15\ 16)$$

and

$$v = (12\ 2\ 16\ 14\ 10\ 6)(11\ 5\ 15\ 13\ 7\ 1)(3\ 9)(4\ 8)$$

be the permutations that record the horizontal and vertical gluing.

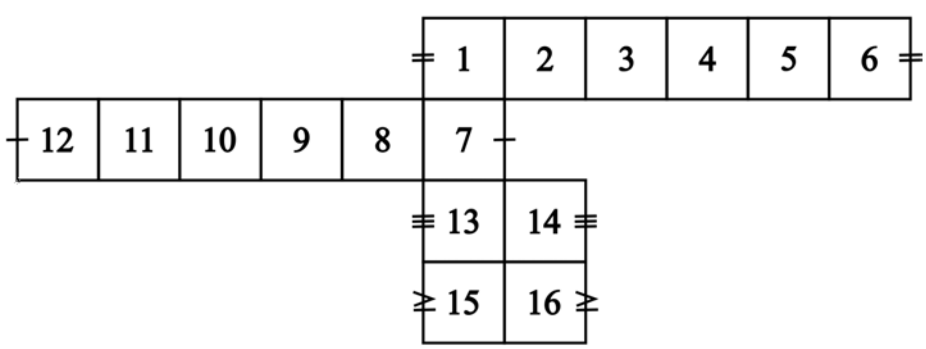

FIGURE 25. Permutations of X

Moreover, let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, we want to prove the following theorem.

Theorem 5.4. *The Veech group* $\mathrm{SL}(X)$ *is precisely* $SL_2(\mathbb{Z})$.

Proof. We need some help from a lemma that essentially provides us with the criterion that determines whether two translation surfaces are the same or not.

Lemma 5.5. *The permutations* $(h, vh^{-1}), (h, v)$ *are simultaneously conjugate via*

$$\psi = (1\ 3\ 5)(2\ 4\ 6)(7\ 10)(8\ 11)(9\ 12)(13)(14)(15\ 16).$$

The permutations $(hv^{-1}, v), (h, v)$ *are simultaneously conjugate via*

$$\phi = (1\ 15)(2\ 14\ 6)(3\ 9)(4)(5\ 7)(8)(10\ 12\ 16)(11\ 13).$$

In other words, these conjugations define the same translation surface up to relabelling the permutation.

Proof. Given h, v in Figure 25, we have that

- $h^{-1} = (6\ 5\ 4\ 3\ 2\ 1)(7\ 8\ 9\ 10\ 11\ 12)(14\ 13)(16\ 15)$
- $v^{-1} = (6\ 10\ 14\ 16\ 2\ 12)(1\ 7\ 13\ 15\ 5\ 11)(9\ 3)(8\ 4)$
- $vh^{-1} = (3\ 16\ 13\ 10\ 5\ 8)(4\ 9\ 6\ 15\ 14\ 7)(1\ 12)(2\ 11)$
- $hv^{-1} = (3\ 8\ 5\ 10\ 13\ 16)(4\ 7\ 14\ 15\ 6\ 9)(1\ 12)(2\ 11)$

Since two cycles are a conjugation of each other if and only if they have the same cycle type, we fix $(13\ 14) \in v$, i.e. we send $13 \mapsto 13, 14 \mapsto 14$ for the pair $(h, vh^{-1}), (h, v)$ and map the permutation of v to vh^{-1} by the fixed labeling, e.g. since we fix 13, then we map 7 (the next label of 13 in v) $\mapsto$ 10 (the next label of 13 in vh^{-1}). Therefore, our corresponding permutation ψ is:

$$\psi = (1\ 3\ 5)(2\ 4\ 6)(7\ 10)(8\ 11)(9\ 12)(13)(14)(15\ 16),$$

which satisfies $\psi(h)\psi^{-1} = h$ and $\psi(vh^{-1})\psi^{-1} = v$. Similarly, for

$$\phi = (1\ 15)(2\ 14\ 6)(3\ 9)(4)(5\ 7)(8)(10\ 12\ 16)(11\ 13),$$

we have $\phi(v)\phi^{-1} = v$ and $\phi(hv^{-1})\phi^{-1} = h$. Hence, the lemma follows from the above choices of $\psi, \phi \in \mathrm{Sym}_n$. ■

By our lemma, since the simultaneous conjugations define the same translation surfaces, we deduce that $T(X) = X$ and $S(X) = X$, which implies T, S generates $\mathrm{SL}(X)$. Because $\langle T, S \rangle = SL_2(\mathbb{Z})$, we have $SL_2(\mathbb{Z}) \subseteq \mathrm{SL}(X)$. On the other hand, since X is a square tiled surface, $\mathrm{SL}(X) \subseteq SL_2(\mathbb{Z})$. Therefore, the Veech group $\mathrm{SL}(X)$ is precisely $SL_2(\mathbb{Z})$. □

Remark 5.6. *We can use the commutator to check Proposition 5.2. The commutator* $[h, v]$ *is:*

$$vhv^{-1}h^{-1} = (1\ 3\ 5)(2)(4)(6)(7\ 15\ 9)(8)(10\ 16\ 12)(11)(13)(14).$$

Since it has 3 non-trivial cycles of length 3, the corresponding total cone angles are all (length of the cycle)$2\pi = 6\pi = (2+1)2\pi$, *which implies* X *lives in* $\mathcal{H}(2,2,2)$.

6. Basis of $H_1(X)$, $H_1^{st}(X)$, and $H_1^{(0)}(X)$

In this section, we explicitly find the basis for $H_1(X), H_1^{st}(X)$, and $H_1^{(0)}$ to facilitate the further propositions about $SL_2(\mathbb{Z})$-orbits.
In Section 3.5, we introduce that the absolute homology $H_1(X)$ has a symplectic vector space with respect to the intersection form. The following proposition serves as a criterion to check the linear independence for the basis of homology.

Proposition 6.1. *If Ω is a real square matrix, then $det(\Omega) \neq 0 \iff$ the vectors that determine Ω are linearly independent.*

Proof. For any given Ω, any linear combination of columns of Ω can be written as Ωx for some $x \in \mathbb{R}^{dim(\Omega)}$. Thus, columns of Ω are linearly dependent $\iff$ there exists some nonzero x such that $\Omega x = 0 \iff 0$ is one of the eigenvalues of $\Omega \iff det(\Omega) = 0$, since the determinant is the product of eigenvalues. Therefore, by taking a contrapositive statement, the desired result follows. □

Let us consider such 8 curves $\{\gamma_1, \ldots, \gamma_8\}$ in the following figures:

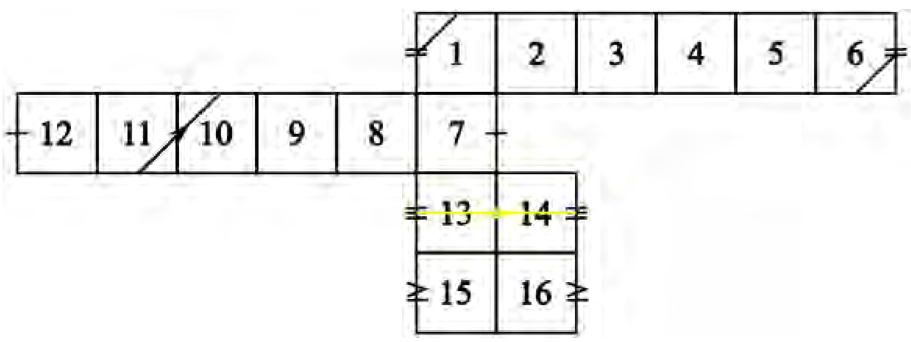

Figure 26. γ_1 and γ_2

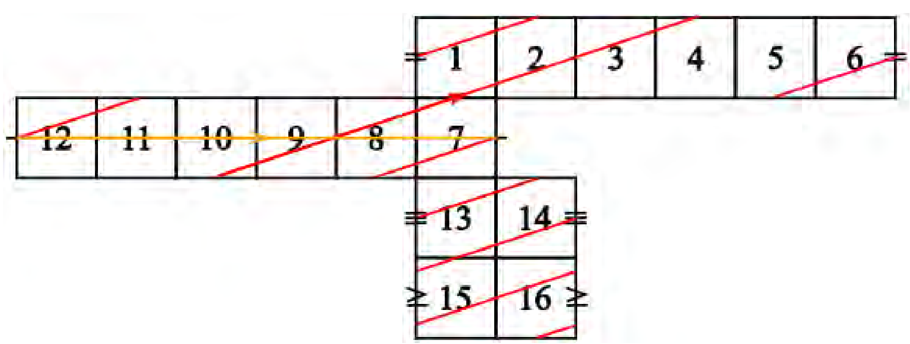

Figure 27. γ_3 and γ_4

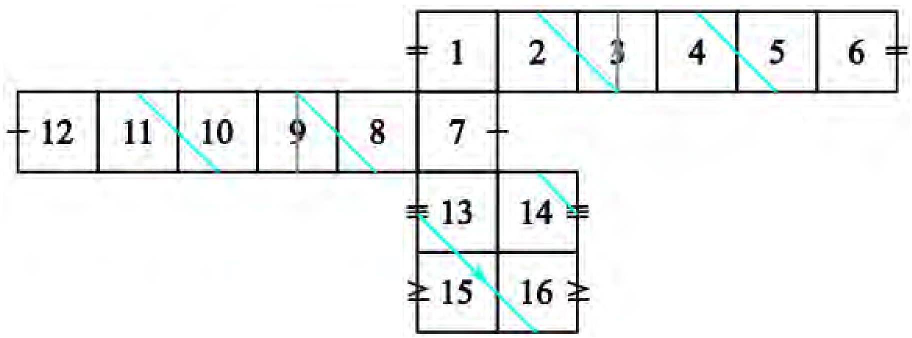

FIGURE 28. γ_5 and γ_6

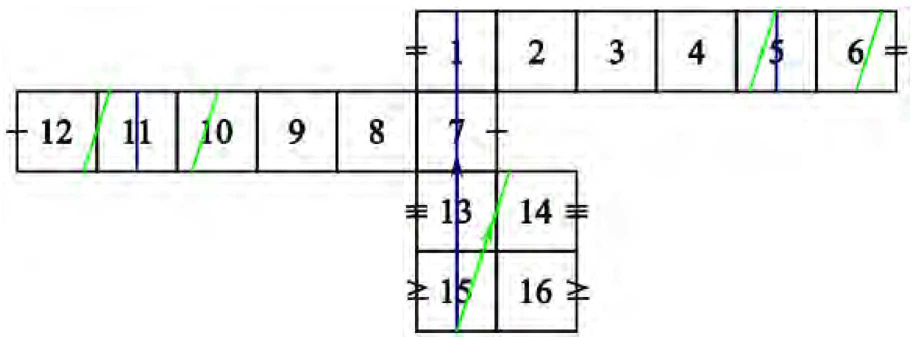

FIGURE 29. γ_7 and γ_8

The color correspondence, directions, and rational slopes of the 8 curves are the following:

- γ_1 : yellow, flow towards the right, slope 0.
- γ_2 : black, flow towards up right, slope 1.
- γ_3 : orange, flow towards the right, slope 0.
- γ_4 : red, flow towards up right, slope $\frac{1}{3}$
- γ_5 grey, flow towards up.
- γ_6 : blue, flow towards the down right, slope -1.
- γ_7 : navy blue, flow towards up.
- γ_8 : green, flow towards up right, slope 3.

Thus, the corresponding intersection matrix of our basis of homology is:

$$\Omega_\gamma = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 & 2 & 2 \\ -1 & 1 & -2 & 0 & 2 & -9 & 7 & 5 \\ 0 & 0 & -1 & -2 & 0 & -1 & 0 & 0 \\ 1 & 1 & 2 & 9 & 1 & 0 & 2 & 3 \\ -1 & -1 & -2 & -7 & 0 & -2 & 0 & -1 \\ -1 & -1 & -2 & -5 & 0 & -3 & 1 & 0 \end{pmatrix} \end{array}$$

Each entry of the matrix, say $\Omega_{\gamma_{i,j}}$, represents the intersection form $\langle \gamma_i, \gamma_j \rangle$, $i, j \in \{1, 2, \ldots, 8\}$.

Proposition 6.2. *These 8 curves determine the basis of $H_1(X)$.*

Proof. Since the determinant $det(\Omega) = 1$, by Proposition 6.1, these 8 curves are linearly independent, which forms the basis of $H_1(X)$. $\square$

Corollary 6.3. *The holonomy vectors for basis of $H_1(X)$ are the following:*
$\mathrm{hol}(\gamma_1) = (2, 0), \mathrm{hol}(\gamma_2) = (2, 2), \mathrm{hol}(\gamma_3) = (6, 0), \mathrm{hol}(\gamma_4) = (18, 6), \mathrm{hol}(\gamma_5) = (0, 2)$, $\mathrm{hol}(\gamma_6) = (6, -6), \mathrm{hol}(\gamma_7) = (0, 6), \mathrm{hol}(\gamma_8) = (2, 6)$.

In general, Section 3.5 gives us a natural decomposition $H_1(X) = H_1^{st}(X) \oplus H_1^{(0)}(X)$. SL$(X)$ acts on $H_1(X) \cong \mathbb{R}^{2g}$ always has a 2-dimensional subspace where we understand this action. This 2-dimensional subspace/plane, the tautological plane $H_1^{st}(X)$, is precisely spanned by $\{\sigma := \sum_{n=1}^{16} \sigma_i, \zeta := \sum_{n=1}^{16} \zeta_i\}$, where for each square i, σ_i, ζ_i represent the horizontal curve and vertical curve correspondingly. It is called "tautological" because for any $g \in \mathrm{SL}(X)$, $\tilde{\alpha}(g) \mid_{H_1^{st}} = g$.

Notice the two waist curves of the horizontal cylinders of length 6 disconnect the surface. Hence, they are homotopic, and their sum is $2\gamma_3$. Similarly, the sum of the two waist curves of the horizontal cylinders of length 2 is given by $2\gamma_1$. The same argument shows that the sum of the vertical waist curves is $2\gamma_5 + 2\gamma_7$. Therefore, we conclude with the following proposition.

Proposition 6.4. *The basis for $H_1^{st}(X)$ is $\{2(\gamma_1 + \gamma_3), 2(\gamma_5 + \gamma_7)\}$.*

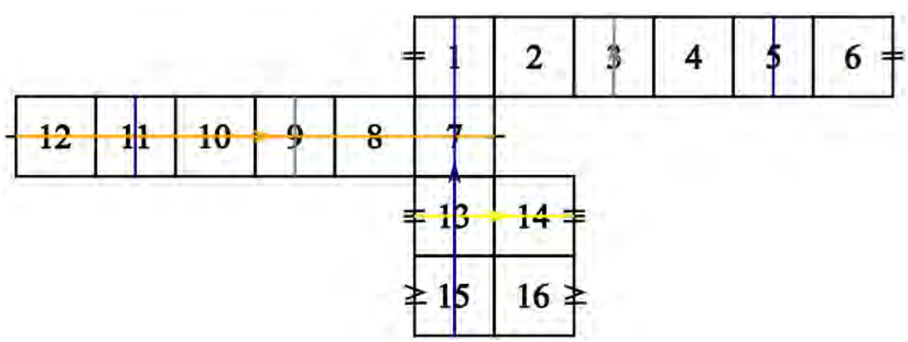

FIGURE 30. Basis for $H_1^{st}(X)$

Since $H_1^{st}(X)$ has dimension 2 and $H_1(X)$ has dimension 8, then

$$H_1^{(0)}(X) = (H_1^{st}(X))^{\perp} = \{\epsilon \in H_1(X) \mid \langle \epsilon, H_1^{st}(X) \rangle = 0\}$$

is naturally a dimension 6 subspace, where we need to find explicitly 6 independent curves as a basis for $H_1^{(0)}(X)$. Through the criterion of the zero holonomy plane

$H_1^{(0)}(X)$, if $\epsilon \in H_1^{(0)}(X)$ and suppose $\epsilon = \sum_{i=1}^{8} b_i\gamma_i, b_i \in \mathbb{R}$, then

$$\langle \epsilon, H_1^{st}(X)\rangle = 0 \implies \begin{cases} \langle \epsilon, 2(\gamma_1+\gamma_3)\rangle = 0 \\ \langle \epsilon, 2(\gamma_5+\gamma_7)\rangle = 0 \end{cases} \implies \begin{cases} 2(\sum_{i=1}^{8} b_i\langle \gamma_i,\gamma_1\rangle) + 2(\sum_{i=1}^{8} b_i\langle \gamma_i,\gamma_3\rangle) = 0 \\ 2(\sum_{i=1}^{8} b_i\langle \gamma_i,\gamma_5\rangle) + 2(\sum_{i=1}^{8} b_i\langle \gamma_i,\gamma_7\rangle) = 0 \end{cases}$$

$$\implies \begin{cases} -2b_2 - 6b_4 - 2b_5 + 6b_6 - 6b_7 - 6b_8 = 0 \\ 2b_1 + 2b_2 + 6b_3 + 18b_4 + 6b_6 + 2b_8 = 0 \end{cases} \tag{1}$$

This system has a degree of freedom of precisely 6. In order to have 6 independent curves, we assign $b_3 = 1, b_4 = 1, b_5 = 1, b_6 = 1, b_7 = 1, b_8 = 1$ since each γ_i is independent of the others.

Proposition 6.5. *The $H_1^{(0)}(X)$ basis is $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6\}$*

- $\epsilon_1 = -3\gamma_1 + \gamma_3$
- $\epsilon_2 = -6\gamma_1 - 3\gamma_2 + \gamma_4$
- $\epsilon_3 = \gamma_1 - \gamma_2 + \gamma_5$
- $\epsilon_4 = -6\gamma_1 + 3\gamma_2 + \gamma_6$
- $\epsilon_5 = 3\gamma_1 - 3\gamma_2 + \gamma_7$
- $\epsilon_6 = 2\gamma_1 - 3\gamma_2 + \gamma_8$

Proof. By solving the system in (1), the result follows. □

Remark 6.6. *Let us use some sanity check to verify all the ϵ_i's are in $H_1^{(0)}(X)$, i.e. they all have zero holonomies.*

- $\mathrm{hol}(\epsilon_1) = -3\mathrm{hol}(\gamma_1) + \mathrm{hol}(\gamma_3) = -3(2,0) + (6,0) = (0,0)$
- $\mathrm{hol}(\epsilon_2) = -6\mathrm{hol}(\gamma_1) - 3\mathrm{hol}(\gamma_2) + \mathrm{hol}(\gamma_4) = -6(2,0) - 3(2,2) + (18,6) = (0,0)$
- $\mathrm{hol}(\epsilon_3) = \mathrm{hol}(\gamma_1) - \mathrm{hol}(\gamma_2) + \mathrm{hol}(\gamma_5) = (2,0) - (2,2) + (0,2) = (0,0)$
- $\mathrm{hol}(\epsilon_4) = -6\mathrm{hol}(\gamma_1) + 3\mathrm{hol}(\gamma_2) + \mathrm{hol}(\gamma_6) = -6(2,0) + 3(2,2) + (6,-6) = (0,0)$
- $\mathrm{hol}(\epsilon_5) = 3\mathrm{hol}(\gamma_1) - 3\mathrm{hol}(\gamma_2) + \mathrm{hol}(\gamma_7) = 3(2,0) - 3(2,2) + (0,6) = (0,0)$
- $\mathrm{hol}(\epsilon_6) = 2\mathrm{hol}(\gamma_1) - 3\mathrm{hol}(\gamma_2) + \mathrm{hol}(\gamma_8) = 2(2,0) - 3(2,2) + (2,6) = (0,0)$

With all the above, let us summarize an important theorem

Theorem 6.7. *The basis for $H_1(X)$ is $\{\gamma_1, \ldots, \gamma_8\}$ in Proposition 6.2. The basis for the tautological plane $H_1^{st}(X)$ is $\sigma = 2(\gamma_1+\gamma_3)$ and $\zeta = 2(\gamma_5+\gamma_7)$ in Proposition 6.5. The basis for zero holonomy space $H_1^{(0)}(X)$ is $\epsilon_1, \ldots, \epsilon_6$ in Proposition 6.5.*

7. Action of the affine group and Identification of the Monodromy group

In this section, we want to understand how $SL_2(\mathbb{Z})$ acts on X and identify the Kontsevich-Zorich monodromy group. In particular, it is significant to understand the action of $SL_2(\mathbb{Z})$ on the basis of $H_1(X)$. Since $\mathrm{SL}(X) = SL_2(\mathbb{Z})$, it suffice to understand how T and S acts on the basis of $H_1(X)$.
Let us take γ_2 as an example.
The horizontal cylinder decomposition with permutation of X with γ_2 labeled is:

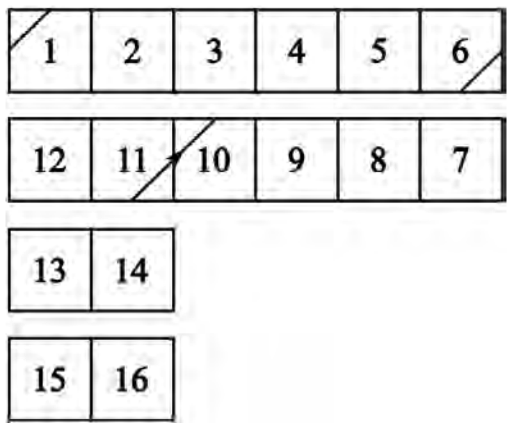

Figure 31. γ_2 in X horizontal cylinder decomposition

Then, applying T, γ_2 will sheer slightly by preserving the bottom of the curve and shifting 1 unit right on the top:

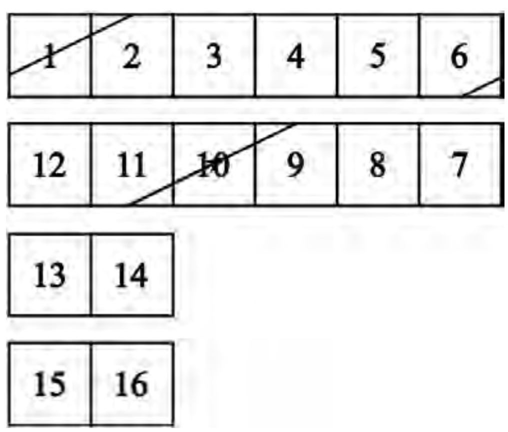

Figure 32. Apply T to γ_2

By Lemma 5.5, using ψ to adjust the surface and the position of γ_2:

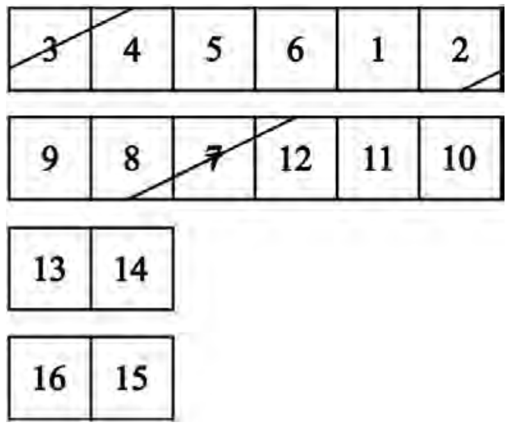

FIGURE 33. Permutating X by ψ

After permuting, we obtain $\tilde{\gamma_2}$ which is the image of γ_2 under the action of T:

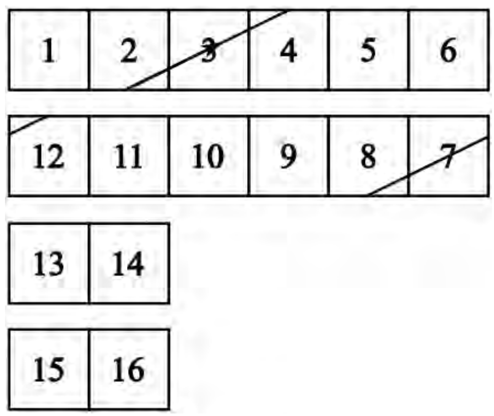

FIGURE 34. $\tilde{\gamma_2}$ under T in X

Since $\tilde{\gamma_2}$ lives in X, we can express $\tilde{\gamma_2}$ through linear combinations of the basis of $H_1(X)$.
Intersecting with each basis of $H_1(X)$, the intersection vector of $\tilde{\gamma_2}$ with $\{\gamma_1, \ldots, \gamma_8\}$ is

$$t = \begin{array}{cc} & \begin{array}{cccccccc} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \end{array} \\ \tilde{\gamma_2} & \begin{pmatrix} 0 & 0 & -1 & -1 & 1 & -2 & 1 & 0 \end{pmatrix} \end{array}$$

Now, suppose $\tilde{\gamma_2} = \sum_{i=1}^{8} \tilde{c}_i \gamma_i, \tilde{c}_i \in \mathbb{R}$, the intersection form of $\tilde{\gamma_2}$ with each the basis of $H_1(X)$ has the expression, for $i, j \in \{1, \ldots, 8\}$,

$$t_j = \langle \tilde{\gamma_2}, \gamma_j \rangle = \left\langle \sum_{i=1}^{8} \tilde{c}_i \gamma_i, \gamma_j \right\rangle = \sum_{i=1}^{8} \tilde{c}_i \langle \gamma_i, \gamma_j \rangle = \sum_{i=1}^{8} \tilde{c}_i \Omega_{i,j}.$$

where t_j is the j^{th} entry of t.

Therefore, the augmented matrix for intersection matrix with respect to $\tilde{\gamma}_2$ coefficients is:

$$\left(\begin{array}{cccccccc|c}
0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & -1 & -1 & 0 \\
0 & -1 & 0 & -2 & -1 & 2 & -2 & -2 & -1 \\
1 & -1 & 2 & 0 & -2 & 9 & -7 & -5 & -1 \\
0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 \\
-1 & -1 & -2 & -9 & -1 & 0 & -2 & -3 & -2 \\
1 & 1 & 2 & 7 & 0 & 2 & 0 & 1 & 1 \\
1 & 1 & 2 & 5 & 0 & 3 & -1 & 0 & 0
\end{array}\right)$$

Solving this matrix, we deduce that $\tilde{\gamma}_2 = \tilde{\alpha}(T)(\gamma_2) = -4\gamma_1 - 2\gamma_3 + \gamma_4 + \gamma_5 + \gamma_6$.
A similar procedure applies to all basis of $H_1(X)$ and notices that T preserves any horizontally closed geodesics, we have

- $\gamma_1 \xrightarrow{\tilde{\alpha}(T)} \tilde{\gamma}_1 = \gamma_1$
- $\gamma_2 \xrightarrow{\tilde{\alpha}(T)} \tilde{\gamma}_2 = -4\gamma_1 - 2\gamma_3 + \gamma_4 + \gamma_5 + \gamma_6$
- $\gamma_3 \xrightarrow{\tilde{\alpha}(T)} \tilde{\gamma}_3 = \gamma_3$
- $\gamma_4 \xrightarrow{\tilde{\alpha}(T)} \tilde{\gamma}_4 = 7\gamma_1 - \gamma_2 + 5\gamma_3 - \gamma_4 + \gamma_5 + 2\gamma_7$
- $\gamma_5 \xrightarrow{\tilde{\alpha}(T)} \tilde{\gamma}_5 = \gamma_2$
- $\gamma_6 \xrightarrow{\tilde{\alpha}(T)} \tilde{\gamma}_6 = -\gamma_7$
- $\gamma_7 \xrightarrow{\tilde{\alpha}(T)} \tilde{\gamma}_7 = \gamma_1 - \gamma_2 + \gamma_3 + \gamma_5 + \gamma_7$
- $\gamma_8 \xrightarrow{\tilde{\alpha}(T)} \tilde{\gamma}_8 = -\gamma_1 - \gamma_3 + \gamma_4 + \gamma_7 - \gamma_8$

Therefore, for $\tilde{\alpha}(T)$, the matrix of group action $\mathrm{SL}(X) \curvearrowright H_1(X)$ under T with respect to the basis $\{\gamma_1, \ldots, \gamma_8\}$ is:

$$\tilde{\alpha}(T) = \begin{pmatrix}
1 & -4 & 0 & 7 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\
0 & -2 & 1 & 5 & 0 & 0 & 1 & -1 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

A similar logic applies to the action of S on the surface X with a slightly different cylinder decomposition that we use for the action of T.

For S, decomposing X into vertical cylinders provides permutation preserving property, again, let us how γ_2 behaves. The vertical cylinder decomposition with permutation of X with γ_2 labeled is:

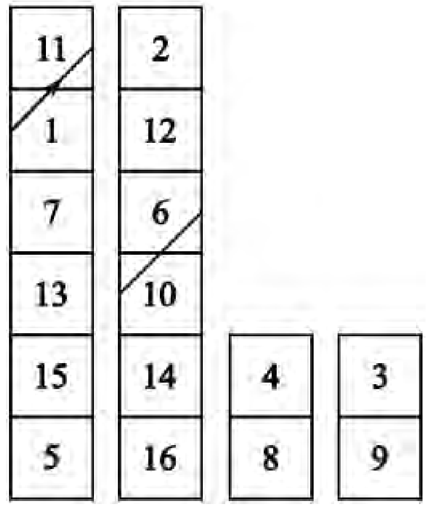

FIGURE 35. γ_2 in X vertical cylinder decomposition

Then, applying S, γ_2 will sheer slightly by preserving the left of the curve and shifting 1 unit up on the right:

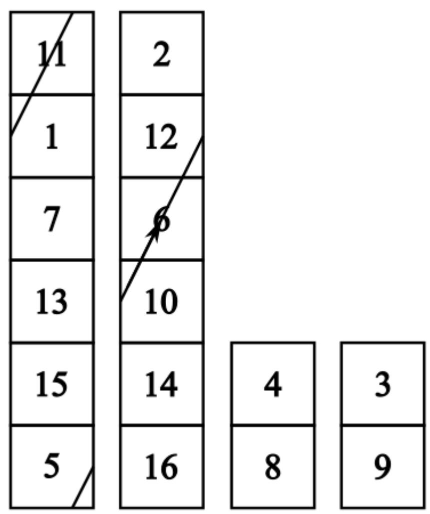

FIGURE 36. Apply S to γ_2

By Lemma 5.5, using ϕ to adjust the surface and the position of γ_2:

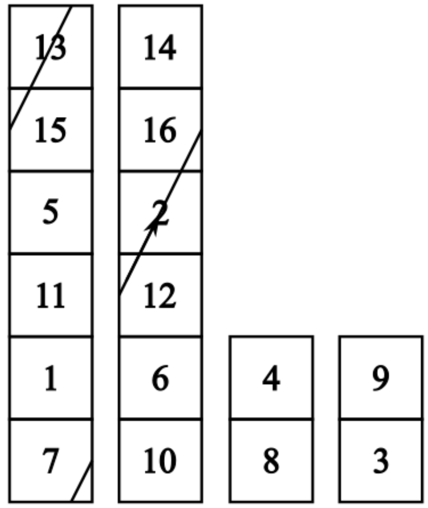

FIGURE 37. Permutating X by ϕ

After permuting, we obtain $\bar{\gamma}_2$ which is the image of γ_2 under the action of S:

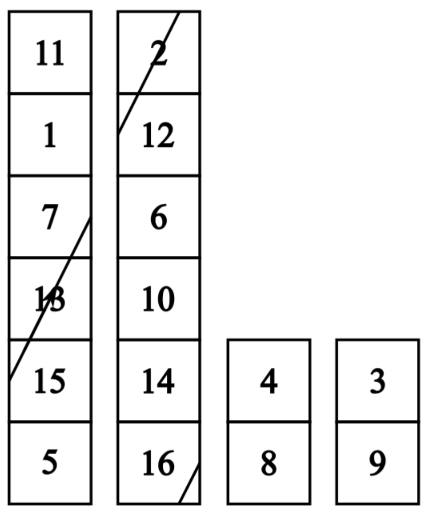

FIGURE 38. $\tilde{\gamma}_2$ under S in X

Again, the intersection vector of $\bar{\gamma}_2$ with $\{\gamma_1, \ldots, \gamma_8\}$ is

$$s = \begin{array}{c} \\ \bar{\gamma}_2 \end{array} \begin{array}{c} \begin{array}{cccccccc} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \end{array} \\ \begin{pmatrix} -1 & 0 & -1 & -5 & 0 & -2 & 1 & 0 \end{pmatrix} \end{array}$$

Suppose $\bar{\gamma}_2 = \sum_{i=1}^{8} \bar{c}_i \gamma_i, \bar{c}_i \in \mathbb{R}$, the intersection form of $\bar{\gamma}_2$ with each the basis of $H_1(X)$ has the expression, for $i, j \in \{1, \ldots, 8\}$,

$$s_j = \langle \bar{\gamma}_2, \gamma_j \rangle = \left\langle \sum_{i=1}^{8} \bar{c}_i \gamma_i, \gamma_j \right\rangle = \sum_{i=1}^{8} \bar{c}_i \langle \gamma_i, \gamma_j \rangle = \sum_{i=1}^{8} \bar{c}_i \Omega_{i,j}.$$

where s_j is the j^{th} entry of s.
The augmented coefficient matrix is:

$$\left(\begin{array}{cccccccc|c}
0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & -1 \\
0 & 0 & 1 & 1 & 0 & 1 & -1 & -1 & 0 \\
0 & -1 & 0 & -2 & -1 & 2 & -2 & -2 & -1 \\
1 & -1 & 2 & 0 & -2 & 9 & -7 & -5 & -5 \\
0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & -2 & -9 & -1 & 0 & -2 & -3 & -2 \\
1 & 1 & 2 & 7 & 0 & 2 & 0 & 1 & 1 \\
1 & 1 & 2 & 5 & 0 & 3 & -1 & 0 & 0
\end{array}\right)$$

Therefore, $\bar{\gamma_2} = \tilde{\alpha}(S)(\gamma_2) = 2\gamma_1 + \gamma_3 - \gamma_5 - \gamma_6 + \gamma_7 - \gamma_8$.
S sends γ_i to its image $\bar{\gamma_i}_S$. Since S preserves any vertically closed geodesics, we have

- $\gamma_1 \xrightarrow{\tilde{\alpha}(S)} \bar{\gamma_1} = \gamma_2$
- $\gamma_2 \xrightarrow{\tilde{\alpha}(S)} \bar{\gamma_2} = 2\gamma_1 + \gamma_3 - \gamma_5 - \gamma_6 + \gamma_7 - \gamma_8$
- $\gamma_3 \xrightarrow{\tilde{\alpha}(S)} \bar{\gamma_3} = \gamma_1 - \gamma_2 + \gamma_3 + \gamma_5 + \gamma_7$
- $\gamma_4 \xrightarrow{\tilde{\alpha}(S)} \bar{\gamma_4} = 4\gamma_1 + 4\gamma_3 - \gamma_4 + 3\gamma_5 + 2\gamma_7 + 2\gamma_8$
- $\gamma_5 \xrightarrow{\tilde{\alpha}(S)} \bar{\gamma_5} = \gamma_5$
- $\gamma_6 \xrightarrow{\tilde{\alpha}(S)} \bar{\gamma_6} = \gamma_3$
- $\gamma_7 \xrightarrow{\tilde{\alpha}(S)} \bar{\gamma_7} = \gamma_7$
- $\gamma_8 \xrightarrow{\tilde{\alpha}(S)} \bar{\gamma_8} = \gamma_1 + \gamma_2 + 2\gamma_7 - \gamma_8$

Hence, for $\tilde{\alpha}(S)$, the matrix of group action $\mathrm{SL}(X) \curvearrowright H_1(X)$ under S with respect to the basis $\{\gamma_1, \ldots, \gamma_8\}$ is:

$$\tilde{\alpha}(S) = \begin{pmatrix}
0 & 2 & 1 & 4 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 4 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 3 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 \\
0 & -1 & 0 & 2 & 0 & 0 & 0 & -1
\end{pmatrix}$$

Given the above clarification of how $SL_2(\mathbb{Z})$ acts on the basis of homology and recall Kontsevich-Zorich monodromy $\alpha = \tilde{\alpha} \mid_{H_1^{(0)}(X)}$ in Section 4.3, the main result in this section is the following:

Theorem 7.1. *The Kontsevich-Zorich monodromy group of X is generated by the following two matrices:*

$$\alpha(T) = \begin{pmatrix} 1 & 11 & 2 & -6 & 7 & 5 \\ 0 & -4 & -1 & 3 & -3 & -2 \\ 0 & -2 & -1 & 3 & -2 & -3 \\ 0 & -3 & -1 & 3 & -3 & -3 \\ 0 & 2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \alpha(S) = \begin{pmatrix} 1 & 1 & -1 & 4 & -3 & -3 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 2 & -3 & 3 & 3 \\ 0 & 3 & 1 & -3 & 3 & 3 \\ 1 & -1 & -1 & 3 & -2 & -1 \\ 0 & 5 & 1 & -3 & 3 & 2 \end{pmatrix}$$

Proof. Recall from Section 4.3, $\tilde{\alpha} : \text{Aff}(X) \to \text{Sp}_8(\mathbb{R})$ denotes the representation arising from the action of the affine diffeomorphisms on X. In the following, we precisely compute the action of the Veech group, and all the calculations and matrices only make sense up to the action of $\text{Aut}(X)$. Moreover, by the nature of surface X, $\text{Aut}(X) = 1 \implies \text{Aff}(X) = \text{SL}(X)$ from Section 3.4. Essentially, we compute the action on $\text{SL}(X)$. Let us borrow the definition from Section 4.3 and define $\alpha = \tilde{\alpha} \mid_{H_1^{(0)}(X)}$ to be the action on the zero holonomy subspace. Since $\tilde{\alpha}$ respects the decomposition of homology $H_1(X) = H_1^{st}(X) \oplus H_1^{(0)}(X)$, the restriction on the zero holonomy subspace makes sense and has the following relation about $\{\epsilon_1, \dots, \epsilon_6\}$ and $\{\gamma_1, \dots, \gamma_8\}$

$$\alpha(\epsilon_i) = \alpha\left(\sum_{j=1}^{8} a_j \gamma_j\right) = \sum_{j=1}^{8} a_j \tilde{\alpha}(\gamma_j), a_j \in \mathbb{R}. \tag{2}$$

Since we restrict to $H_1^{(0)}(X)$, we need to express $\alpha(\epsilon_i)$ in terms of the basis of $H_1^{(0)}(X)$, i.e. the linear combination of $\{\epsilon_1, \dots, \epsilon_6\}$. We need some help with the following lemma.

Lemma 7.2. *Suppose for each $i \in \{1, \dots, 6\}$, $\alpha(\epsilon_i)$ has two linear representations by different basis, i.e. $\alpha(\epsilon_i) = \sum_{j=1}^{8} a_j \gamma_j, a_j \in \mathbb{R}$ and $\alpha(\epsilon_i) = \sum_{k=1}^{6} d_k \epsilon_k, d_k \in \mathbb{R}$, then $d_k = a_{k+2}$.*

Proof. We compute

$$\begin{aligned} \sum_{k=1}^{6} d_k \epsilon_k &= d_1(-3\gamma_1 + \gamma_3) + d_2(-6\gamma_1 - 3\gamma_2 + \gamma_4) + d_3(\gamma_1 - \gamma_2 + \gamma_5) \\ &\quad + d_4(-6\gamma_1 + 3\gamma_2 + \gamma_6) + d_5(3\gamma_1 - 3\gamma_2 + \gamma_7) + d_6(2\gamma_1 - 3\gamma_2 + \gamma_8) \\ &= (-3d_1 - 6d_2 + d_3 - 6d_4 + 3d_5 + 2d_6)\gamma_1 + (-3d_2 - d_3 + 3d_4 - 3d_5 - 3d_6)\gamma_2 \\ &\quad + d_1\gamma_3 + d_2\gamma_4 + d_3\gamma_5 + d_4\gamma_6 + d_5\gamma_7 + d_6\gamma_8. \end{aligned}$$

Also, $\alpha(\epsilon_i) = a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 + a_4\gamma_4 + a_5\gamma_5 + a_6\gamma_6$. By corresponding the coefficients, we have $d_1 = a_3, d_2 = a_4, d_3 = a_5, d_4 = a_6, d_5 = a_7, d_6 = a_8$, which is precisely what we want. ∎

Since $\mathrm{SL}(X) = SL_2(\mathbb{Z})$, we only care about T and S, the only 2 generators of the Veech group. The strategy is that we find the action of the generators on each of the basis of the $H_1^{(0)}(X)$ to encode the data of the action on homology restricted on the Kontsevich-Zorich monodromy. Using (2), Lemma 7.2, $\tilde{\alpha}(T)$, and $\tilde{\alpha}(S)$, we have the following matrix representations.

For $\alpha(T)$,

$$\begin{aligned}
\alpha(T)(\epsilon_1) &= \alpha(T)(-3\gamma_1 + \gamma_3) \\
&= -3\tilde{\alpha}(T)(\gamma_1) + \tilde{\alpha}(T)(\gamma_3) \\
&= -3\gamma_1 + \gamma_3 \\
&= \epsilon_1 \\
\alpha(T)(\epsilon_2) &= \alpha(T)(-6\gamma_1 - 3\gamma_2 + \gamma_4) \\
&= -6\tilde{\alpha}(T)(\gamma_1) - 3\tilde{\alpha}(T)(\gamma_2) + \tilde{\alpha}(T)(\gamma_4) \\
&= 13\gamma_1 - \gamma_2 + 11\gamma_3 - 4\gamma_4 - 2\gamma_5 - 3\gamma_6 + 2\gamma_7 \\
&= 11\epsilon_1 - 4\epsilon_2 - 2\epsilon_3 - 3\epsilon_4 + 2\epsilon_5 \\
\alpha(T)(\epsilon_3) &= \alpha(T)(\gamma_1 - \gamma_2 + \gamma_5) \\
&= \tilde{\alpha}(T)(\gamma_1) - \tilde{\alpha}(T)(\gamma_2) + \tilde{\alpha}(T)(\gamma_5) \\
&= 5\gamma_1 + \gamma_2 + 2\gamma_3 - \gamma_4 - \gamma_5 - \gamma_6 \\
&= 2\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 \\
\alpha(T)(\epsilon_4) &= \alpha(T)(-6\gamma_1 + 3\gamma_2 + \gamma_6) \\
&= -6\tilde{\alpha}(T)(\gamma_1) + 3\tilde{\alpha}(T)(\gamma_2) + \tilde{\alpha}(T)(\gamma_6) \\
&= -18\gamma_1 - 6\gamma_3 + 3\gamma_4 + 3\gamma_5 + 3\gamma_6 \\
&= -6\epsilon_1 + 3\epsilon_2 + 3\epsilon_3 + 3\epsilon_4 - \epsilon_5 \\
\alpha(T)(\epsilon_5) &= \alpha(T)(3\gamma_1 - 3\gamma_2 + \gamma_7) \\
&= 3\tilde{\alpha}(T)(\gamma_1) - 3\tilde{\alpha}(T)(\gamma_2) + \tilde{\alpha}(T)(\gamma_7) \\
&= 16\gamma_1 - \gamma_2 + 7\gamma_3 - 3\gamma_4 - 2\gamma_5 - 3\gamma_6 + \gamma_7 \\
&= 7\epsilon_1 - 3\epsilon_2 - 2\epsilon_3 - 3\epsilon_4 + \epsilon_5 \\
\alpha(T)(\epsilon_6) &= \alpha(T)(2\gamma_1 - 3\gamma_2 + \gamma_8) \\
&= 2\tilde{\alpha}(T)(\gamma_1) - 3\tilde{\alpha}(T)(\gamma_2) + \tilde{\alpha}(T)(\gamma_8) \\
&= 13\gamma_1 + 5\gamma_3 - 2\gamma_4 - 3\gamma_5 - 3\gamma_6 + \gamma_7 - \gamma_8 \\
&= 5\epsilon_1 - 2\epsilon_2 - 3\epsilon_3 - 3\epsilon_4 + \epsilon_5 - \epsilon_6
\end{aligned}$$

Hence, $\alpha(T) = \begin{pmatrix} 1 & 11 & 2 & -6 & 7 & 5 \\ 0 & -4 & -1 & 3 & -3 & -2 \\ 0 & -2 & -1 & 3 & -2 & -3 \\ 0 & -3 & -1 & 3 & -3 & -3 \\ 0 & 2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$.

For $\alpha(S)$,

$$\begin{aligned}
\alpha(S)(\epsilon_1) &= \alpha(S)(-3\gamma_1 + \gamma_3) \\
&= -3\tilde{\alpha}(S)(\gamma_1) + \tilde{\alpha}(S)(\gamma_3) \\
&= \gamma_1 + -4\gamma_2 + \gamma_3 + \gamma_5 + \gamma_7 \\
&= \epsilon_1 + \epsilon_3 + \epsilon_5 \\
\alpha(S)(\epsilon_2) &= \alpha(S)(-6\gamma_1 - 3\gamma_2 + \gamma_4) \\
&= -6\tilde{\alpha}(S)(\gamma_1) - 3\tilde{\alpha}(S)(\gamma_2) + \tilde{\alpha}(S)(\gamma_4) \\
&= -2\gamma_1 - 6\gamma_2 + \gamma_3 - \gamma_4 + 6\gamma_5 + 3\gamma_6 - \gamma_7 + 5\gamma_8 \\
&= \epsilon_1 - \epsilon_2 + 6\epsilon_3 + 3\epsilon_4 - \epsilon_5 + 5\epsilon_6 \\
\alpha(S)(\epsilon_3) &= \alpha(S)(\gamma_1 - \gamma_2 + \gamma_5) \\
&= \tilde{\alpha}(S)(\gamma_1) - \tilde{\alpha}(S)(\gamma_2) + \tilde{\alpha}(S)(\gamma_5) \\
&= -2\gamma_1 + \gamma_2 - \gamma_3 + 2\gamma_5 + \gamma_6 - \gamma_7 + \gamma_8 \\
&= -\epsilon_1 + 2\epsilon_3 + \epsilon_4 - \epsilon_5 + \epsilon_6 \\
\alpha(S)(\epsilon_4) &= \alpha(S)(-6\gamma_1 + 3\gamma_2 + \gamma_6) \\
&= -6\tilde{\alpha}(S)(\gamma_1) + 3\tilde{\alpha}(S)(\gamma_2) + \tilde{\alpha}(S)(\gamma_6) \\
&= 6\gamma_1 - 6\gamma_2 + 4\gamma_3 - 3\gamma_5 - 3\gamma_6 + 3\gamma_7 - 3\gamma_8 \\
&= 4\epsilon_1 - 3\epsilon_3 - 3\epsilon_4 + 3\epsilon_5 - 3\epsilon_6 \\
\alpha(S)(\epsilon_5) &= \alpha(S)(3\gamma_1 - 3\gamma_2 + \gamma_7) \\
&= 3\tilde{\alpha}(S)(\gamma_1) - 3\tilde{\alpha}(S)(\gamma_2) + \tilde{\alpha}(S)(\gamma_7) \\
&= -6\gamma_1 + 3\gamma_2 - 3\gamma_3 + 3\gamma_5 + 3\gamma_6 - 2\gamma_7 + 3\gamma_8 \\
&= -3\epsilon_1 + 3\epsilon_3 + 3\epsilon_4 - 2\epsilon_5 + 3\epsilon_6 \\
\alpha(S)(\epsilon_6) &= \alpha(S)(2\gamma_1 - 3\gamma_2 + \gamma_8) \\
&= 2\tilde{\alpha}(S)(\gamma_1) - 3\tilde{\alpha}(S)(\gamma_2) + \tilde{\alpha}(S)(\gamma_8) \\
&= -5\gamma_1 + 3\gamma_2 - 3\gamma_3 + 3\gamma_5 + 3\gamma_6 - \gamma_7 + 2\gamma_8 \\
&= -3\epsilon_1 + 3\epsilon_3 + 3\epsilon_4 - \epsilon_5 + 2\epsilon_6
\end{aligned}$$

Hence, $\alpha(S) = \begin{pmatrix} 1 & 1 & -1 & 4 & -3 & -3 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 2 & -3 & 3 & 3 \\ 0 & 3 & 1 & -3 & 3 & 3 \\ 1 & -1 & -1 & 3 & -2 & -1 \\ 0 & 5 & 1 & -3 & 3 & 2 \end{pmatrix}$.

The Kontsevich-Zorich monodromy group is $\langle \alpha(T), \alpha(S) \rangle$. $\square$

8. Zariski denseness and Arithmeticity

In this section, We further explore the properties of this monodromy group, denoted as $\Gamma = \langle \alpha(T), \alpha(S) \rangle$, in the sense of Sarnak.

Theorem 8.1. *The monodromy group* Γ *is Zariski dense in* $\mathrm{Sp}_6(\mathbb{R})$*.*

Proof. Recall Theorem 4.4 in Section 4.4, we need to find two elements A, B in $\mathrm{Sp}_6(H_1^{(0)}(X))$ that satisfies:

(1) A is a Galois-pinching element splits over $\mathbb{R}$, and possesses the largest possible Galois group among reciprocal polynomials of degree 6 (namely, isomorphic to the hyperoctahedral group of order $2^3 \cdot 3! = 48$ viewed as the centralizer of the involution $x \to x^{-1}$ on the set of roots).
(2) some Dehn twist induces a non-trivial unipotent element B that $(B - id)(H_1^{(0)}(X))$ is not a Lagrangian subspace of $H_1^{(0)}(X)$.

Let us consider $A = \alpha(T)^2\alpha(S)^2\alpha(T)\alpha(S)^3$, then the characteristic polynomial of A is $f_A(x) = 1 - x - 83x^2 + 200x^3 - 83x^4 - x^5 + x^6$, which implies, by symmetry and involution $x \to x^{-1}$, the reciprocal polynomial $x^6 f_A(x^{-1}) = f_A(x)$. Since the roots of $f_A(x)$ are all real values, it splits over $\mathbb{R}$. Since the permutation group acts on a set of cardinality 6, the order of the Galois group is precisely $2^4 \cdot 3 = 48$ which matches with the order of the hyperoctahedral group. Giving the thanks to Matheus-Möller-Yoccoz's criterion, the Galois group of $f_A(x)$ is isomorphic to the hyperoctahedral group that satisfies (1).
Moreover, let $B = \alpha(T)^6$. The dimension of $(B - id)(H_1^{(0)}(X))$ is precisely the dimension of the nullspace for its eigenspaces. The nullspace consists of the following basis:

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which has dimension 5. Yet, if $(B - id)(H_1^{(0)}(X))$ is a Lagrangian subspace, then $\dim((B - id)(H_1^{(0)}(X))) = \frac{\dim(H_1^{(0)}(X))}{2} = \frac{6}{2} = 3 \neq 5$. Therefore, the dimension of such subspace $(B - id)(H_1^{(0)}(X))$ with given element B is not Lagrangian, and it satisfies (2).
Together with the criterion, the monodromy Γ is Zariski dense in $\mathrm{Sp}_6(\mathbb{R})$. $\square$

With the Zariski denseness, it is possible that we can prove arithmeticity using Theorem 4.5 in Section 4.5.

Theorem 8.2. *The monodromy group* Γ *is arithmetic.*

Proof. Recall Theorem 4.5, the non-degenerate symplectic form Θ is the intersection form we use throughout the process. Then, it suffices to find 3 transvections with respect to 3 rational directions of cylinder decomposition.
Using Proposition 6.5, a basis for $H_1^{(0)}(X)$ is given by the elements:

- $\epsilon_1 = -3\gamma_1 + \gamma_3$
- $\epsilon_2 = -6\gamma_1 - 3\gamma_2 + \gamma_4$
- $\epsilon_3 = \gamma_1 - \gamma_2 + \gamma_5$
- $\epsilon_4 = -6\gamma_1 + 3\gamma_2 + \gamma_6$
- $\epsilon_5 = 3\gamma_1 - 3\gamma_2 + \gamma_7$
- $\epsilon_6 = 2\gamma_1 - 3\gamma_2 + \gamma_8$

Then we consider three 4-cylinder decompositions with rational directions $\theta_1 = (1,0), \theta_2 = (0,1), \theta_3 = (1,2)$.
For θ_1, the corresponding zero holonomy element for transvection is $w_1 = 2\gamma_3 - 6\gamma_1 \in H_1^{(0)}(X)$.

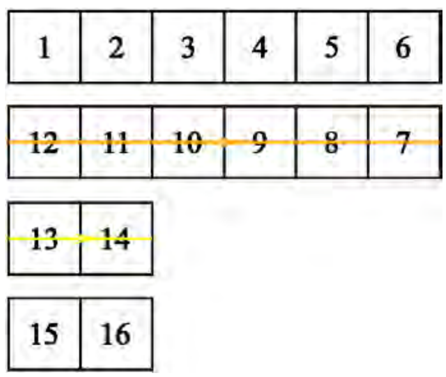

FIGURE 39. θ_1 4-cylinder decomposition with γ_1 and γ_3

For θ_2, $w_2 = 2\gamma_7 - 6\gamma_5 \in H_1^{(0)}(X)$.

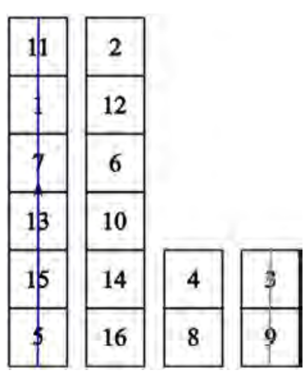

FIGURE 40. θ_2 4-cylinder decomposition with γ_5 and γ_7

For θ_3, let α (solid line) be the waist curves of the "longer" cylinder (light purple) and β (dotted line) be the waist curves of the "shorter" cylinder (light yellow). Then, $w_3 = 2\alpha - 6\beta \in H_1^{(0)}(X)$.

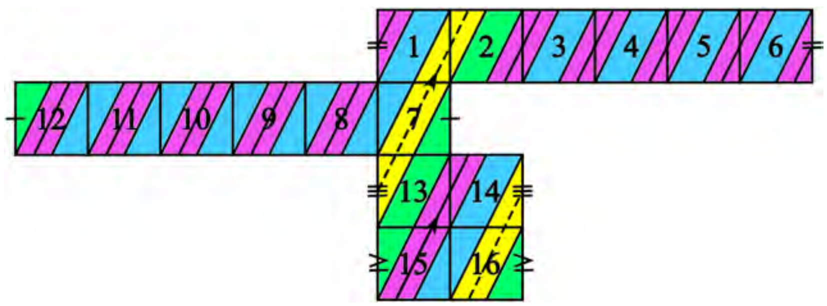

FIGURE 41. θ_3 4-cylinder decomposition with α and β

Notice that $\gamma_1, \gamma_3, \gamma_5, \gamma_7, \alpha, \beta$ are homotopically equivalent to other cylinders of the same length in their corresponding cylinder decompositions. Therefore, they all have the homological dimension of 2 which satisfies the condition for the Singh-Venkataramana criterion in Section 4.5.
The intersection matrix between the basis of $H_1(X)$ and the corresponding waist curves of the directions is:

$$\Omega_w = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \end{array} \begin{array}{c} \begin{array}{cccccc} \gamma_1 & \gamma_3 & \gamma_5 & \gamma_7 & \alpha & \beta \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 & 1 \\ -1 & -2 & 2 & 7 & 10 & 5 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 2 & 1 & 2 & 7 & 2 \\ -1 & -2 & 0 & 0 & -2 & -1 \\ -1 & -2 & 0 & 1 & -1 & 0 \end{pmatrix} \end{array}$$

The transvections act on ϵ_i as follows:

- $C_{w_1}(\epsilon_1) = \epsilon_1 + 2\Theta(\epsilon_1, \gamma_3)\gamma_3 + 6\Theta(\epsilon_1, \gamma_1)\gamma_1 = \epsilon_1$
  $C_{w_1}(\epsilon_2) = \epsilon_2 + 2\gamma_3 - 6\gamma_1 = \epsilon_2 + w_1$
  $C_{w_1}(\epsilon_3) = \epsilon_3$
  $C_{w_1}(\epsilon_4) = \epsilon_4 - 2\gamma_3 + 6\gamma_1 = \epsilon_4 - w_1$
  $C_{w_1}(\epsilon_5) = \epsilon_5 + 2\gamma_3 - 6\gamma_1 = \epsilon_5 + \gamma_1$
  $C_{w_1}(\epsilon_6) = \epsilon_6$
- $C_{w_2}(\epsilon_1) = \epsilon_1 - 2\gamma_7 + 6\gamma_5 = \epsilon_1 - w_2$
  $C_{w_2}(\epsilon_2) = \epsilon_2 - 4\gamma_7 + 12\gamma_5 = \epsilon_2 - 2w_2$
  $C_{w_2}(\epsilon_3) = \epsilon_3$
  $C_{w_2}(\epsilon_4) = \epsilon_4 - 2\gamma_7 + 6\gamma_5 = \epsilon_4 - w_2$
  $C_{w_2}(\epsilon_5) = \epsilon_5$
  $C_{w_2}(\epsilon_6) = \epsilon_6$

- $C_{w_3}(\epsilon_1) = \epsilon_1 + 4\alpha - 12\beta = \epsilon_1 + 2w_3$
  $C_{w_3}(\epsilon_2) = \epsilon_2 + 2\alpha - 6\beta = \epsilon_2 + w_3$
  $C_{w_3}(\epsilon_3) = \epsilon_3 - 2\alpha + 6\beta = \epsilon_3 - w_3$
  $C_{w_3}(\epsilon_4) = \epsilon_4 + 8\alpha - 24\beta = \epsilon_4 + 4w_3$
  $C_{w_3}(\epsilon_5) = \epsilon_5 - 4\alpha + 12\beta$
  $C_{w_3}(\epsilon_6) = \epsilon_6 - 4\alpha + 12\beta = \epsilon_6 - 2w_3$

Then, in terms of the basis of $H_1^{(0)}(X)$, $w_1 = 2\epsilon_1, w_2 = 2\epsilon_5 - 6\epsilon_3$ directly from the construction. Yet, we have to find $w_3 = 2\alpha - 6\beta$ in terms of the linear combination of the basis. Suppose $w_3 = \sum_{i=1}^{6} g_i\epsilon_i, g_i \in \mathbb{R}$, the intersection matrix of the basis of $H_1^{(0)}(X)$ is:

$$\Omega_\epsilon = \begin{array}{c} \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{array} \begin{array}{c} \begin{array}{cccccc} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \end{array} \\ \begin{pmatrix} 0 & -4 & 0 & 4 & -4 & -4 \\ 4 & 0 & 0 & 9 & -8 & -9 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -4 & -9 & 1 & 0 & -1 & -1 \\ 4 & 8 & 0 & 1 & 0 & 0 \\ 4 & 9 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Using Ω_w and $w_3 = 2\alpha - 6\beta$, we have $\begin{cases} \Theta(w_3, \epsilon_1) = -16 \\ \Theta(w_3, \epsilon_2) = -8 \\ \Theta(w_3, \epsilon_3) = 8 \\ \Theta(w_3, \epsilon_4) = -32 \\ \Theta(w_3, \epsilon_5) = 16 \\ \Theta(w_3, \epsilon_6) = 16 \end{cases}$,

whereas using Ω_ϵ and $w_3 = \sum_{i=1}^{6} g_i\epsilon_i$, we have $\begin{cases} \Theta(w_3, \epsilon_1) = 4g_2 - 4g_4 + 4g_5 + 4g_6 \\ \Theta(w_3, \epsilon_2) = -4g_1 - 9g_4 + 8g_5 + 9g_6 \\ \Theta(w_3, \epsilon_3) = g_4 \\ \Theta(w_3, \epsilon_4) = 4g_1 + 9g_2 - g_3 + g_5 + g_6 \\ \Theta(w_3, \epsilon_5) = -4g_1 - 8g_2 - g_4 \\ \Theta(w_3, \epsilon_6) = -4g_1 - 9g_2 - g_4 \end{cases}$.

Therefore, $w_3 = -6\epsilon_1 + 12\epsilon_3 + 8\epsilon_4 - 4\epsilon_5 + 8\epsilon_6$.
In the metacode, the annihilator, $C_w(e) = 3$, the formula give us:

$$e = -\frac{\Theta(w_3, w_2)}{\Theta(w_1, w_2)}w_1 - \frac{\Theta(w_3, w_1)}{\Theta(w_2, w_1)}w_2 + w_3 = -w_1 + 2w_2 + w_3.$$

We choose the basis $\{w_1, w_3, e\}$, and notice that $e - w_3 + w_1 = 4\epsilon_5 - 12\epsilon_3 = 2w_2$. The action of the transvection on the rest of this basis is given by:

- $C_{w_1}(w_3) = C_{w_1}(-6\epsilon_1 + 12\epsilon_3 + 8\epsilon_4 - 4\epsilon_5 + 8\epsilon_6) = w_3 - 12w_1$
- $C_{w_3}(w_1) = w_1 + 4w_3$
- $C_{w_2}(w_1) = w_1 - 2w_2 = w_1 - e + w_3 - w_1 = w_3 - e$
  $C_{w_2}(w_3) = w_3 - 2w_2 = w_3 - e + w_3 - w_1 = 2w_3 - e - w_1$

So, the transvections with respect to the basis $\{w_1, w_3, e\}$ are:

$$C_{w_1} = \begin{pmatrix} 1 & -12 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_{w_2} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}, C_{w_3} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The non-trivial unipotent radical word generated by the letters $C_{w_1}^{\pm}, C_{w_2}^{\pm}, C_{w_3}^{\pm}$ is:

$$C_{w_1} C_{w_2} C_{w_1}^{-1} C_{w_2}^{-1} C_{w_3}^{3} C_{w_1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 12 & 144 & 1 \end{pmatrix},$$

which tells us, by the metacode, Γ is arithmetic.

$\square$

Appendix A. Measure Theory and Poincaré Recurrence

We develop some theories needed to prove Masur's result on the density of saddle connection directions on the circle.

Definition A.1. *A measure μ on a measure space $(X, \mathcal{A})$ is a function*

$$\mu : A \to [0, \infty]$$

such that

1. *$\mu(\varnothing) = 0$.*
2. *If $(A_i)_{i \in \mathbb{N}}$ is a countable disjoint collection of sets in $\mathcal{A}$, then*

$$\mu\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i).$$

Further, we say $(X, \mathcal{A}, \mu)$ is a measure space.

Definition A.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space. A transformation $T : X \to X$ is measurable if for any measurable set $A \in \mathcal{B}$, the preimage is measurable, i.e. $T^{-1}(A) \in \mathcal{B}$.*

Definition A.3. *Let $(X, \mathcal{B}, \mu)$ be a measure space. A transformation $T : X \to X$ is measure-preserving if it is measurable and for all measurable sets $A \in \mathcal{B}$,*

$$\mu(T^{-1}(A)) = \mu(A)$$

In addition, if T^{-1} exists almost everywhere and is measurable, then we say T is an invertible measure-preserving map. If $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is measure-preserving, then the measure μ is T-invariant, $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system, and T is a measure-preserving transformation.

Theorem A.4 (Poincaré Recurrence Theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space, T be a transformation preserves μ, and μ be a finite measure. Then, for any $E \in \mathcal{B}$ with $\mu(E) > 0$, μ-almost every point $x \in E$ is infinitely recurrent to E, i.e., the set of points $x \in E$ such that x returns to E infinitely many times has measure equal to $\mu(E)$.*

Proof. For $N \geq 0$, consider the union

$$E_N = \bigcup_{n=N}^{\infty} T^{-n} E.$$

Then, $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ and $E \subseteq E_0$. For all N,

$$E_{N+1} = T^{-1} E_N.$$

As T is measure-preserving,

$$\mu(E_N) = \mu(T^{-1} E_N) = \mu(E_{N+1})$$

for all N. Then, by induction, $\mu(E_N) = \mu(E_0)$, and

$$\mu\left(\bigcap_{N=0}^{\infty} E_N\right) = \lim_{N\to\infty} \mu(E_N) = \lim_{N\to\infty} \mu(E_0) = \mu(E_0).$$

Then, let

$$x \in \bigcap_{N=0}^{\infty} E_N = \bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}E.$$

For all $N \geq 0$, there exists $n \geq N$ such that $T^n x \in E$. Therefore, the set $\bigcap\limits_{N=0}^{\infty} E_N$ is the set of points x that enter E infinitely many times.

Claim: $\mu\left(E_0 \setminus \left(\bigcap\limits_{N=0}^{\infty} E_N\right)\right) = 0$.

Since $\mu\left(\bigcap\limits_{N=0}^{\infty} E_N\right) = \mu(E_0)$ and $\bigcap\limits_{N=0}^{\infty} E_N \subseteq E_0$, we have

$$\mu\left(E_0 \setminus \bigcap_{N=0}^{\infty} E_N\right) = \mu(E_0) - \mu\left(\bigcap_{N=0}^{\infty} E_N\right) = 0.$$

Proof. Let

$$F = E \cap \left(\bigcap_{N=0}^{\infty} E_N\right).$$

We have

$$\begin{aligned}
\mu(F) &= \mu\left(E \cap \left(\bigcap_{N=0}^{\infty} E_N\right)\right) \\
&= \mu\left(E \cap \left(E_0 \setminus \left(E_0 \setminus \left(\bigcap_{N=0}^{\infty} E_N\right)\right)\right)\right) \\
&= \mu(E \cap E_0) - \mu\left(E \cap \left(E_0 \setminus \left(\bigcap_{N=0}^{\infty} E_N\right)\right)\right) \\
&= \mu(E \cap E_0) \\
&= \mu(E).
\end{aligned}$$

■

Thus, our desired result follows from above. □

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