Spectral Theory of Graphons

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Abstract

Many new fields in graph theory have develop in the last half century involve and incorporate other branches of mathematics. Utilizing linear algebra, spectral graph theory has been an ever-growing field that studies the eigenvalues of the adjacency matrix and graph Laplacian. These ideas have resulted in the study of a class of highly connected but sparse graphs called expanders. Another recent development in graph theory is the concept of dense graph limits, known as graphons, and how they solve many combinatorial problems. We examine how many core ideas about expander graphs and the adjacency matrix as an operator generalize well to graphons through the lens of functional analysis. In this thesis, we provide proofs for the expander mixing lemma and mixing time for graphons, as well as a few applications of them. In addition, we show strong evidence that an Alon-Boppana type theorem can be proven for graphons.

Contents

1. Introduction

In this paper we will be considering properties of graphs and adjacency matrices, as well as their generalizations in the limiting case. We say that the **adjacency matrix** of a weighted graph $G = (V, E, w)$ is the matrix A_G, where $(A_G)_{ij} = w(i, j)$. For notational purposes, we will refer to $V(G)$ and $E(G)$ as the vertex and edge set of a graph if there are multiple graphs defined, but we will omit G otherwise.

The adjacency matrix has quite a few applications even without analyzing its eigenvalues. For instance, the volume or number of edges from a set $S \subseteq V$ to $T \subseteq V$ is given by $\langle 1_T, A_G 1_S \rangle$, where 1_X is the characteristic vector for a set $X \subseteq V$. Naturally, this can also be viewed as the computing the number of ways to reach T from S using one step in the graph, and so it may be generalized to computing for t steps via $\langle 1_T, A_G^t 1_S \rangle$. We can even redefine other common properties of graphs using the inner product. A graph is connected if and only if for any $S, T \subseteq V$, there exists a $t \geq 0$ such that $\langle 1_T, A_G^t 1_S \rangle \neq 0$. Similarly, the diameter of a graph is minimum Δ such that $t \leq \Delta$ for all $S, T \subseteq V$, or that $\sum_{t=1}^{\Delta} A_G^t$ has all nonzero entries.

Restricting ourselves to undirected graphs, we can view the adjacency matrix as a self-adjoint operator on $\mathbb{R}^{|V|}$, which allows us to use the beloved spectral theorem. From a given adjacency matrix A_G, we have that its multiset of eigenvalues $\{\lambda_i\}_{1 \leq i \leq |V|}$, known as the spectrum, are contained in the reals. In addition, we can create an orthonormal basis of eigenvectors $\{v_i\}_{1\leq i\leq |V|}$ of $\mathbb{R}^|V|$ such that

$$A_G = \sum_{i=1}^{|V|} \lambda_i v_i v_I^T$$

which means that A_G is diagonalizable. When we refer to the eigenvalues of a graph $\lambda_i(G)$, we will be referring to the eigenvalues of its adjacency matrix $\lambda_i(A_G)$. Since the spectral theorem is the foundation of this study, this subfield is called spectral graph theory. Known as one of the fastest growing fields in combinatorics in the last half century, it has provided strong motivations for advancing random matrix theory and creating important algorithms in data science [7, 4].

Now if G was a d-regular graph in that the sum of the edge weights incident to any vertex is d, then it is clear that the row sums of A_G must be d. Thus, we see that d is an eigenvalue of G with eigenvector $\vec{1}$. In fact, it is easy to see that d is the largest eigenvalue of a d-regular graph. If G were a disconnected graph, then d would have geometric multiplicity greater than 1, as $\vec{1}$ could be decomposed into the sum of indicator vectors for the connected components of G. Thus, it is common to refer to d as the trivial eigenvalue of d-regular graphs [4]. Another common notion is the spectral gap, which is the maximal difference between the first eigenvalue and the other eigenvalues of a graph. In short, having a very large spectral gap implies that a graph is highly connected, or that it requires a large number of edges removed to disconnect a graph.

1.1. Expander graphs

This in turn brings us to the notion of graph expanders, which seek to be highly connected yet sparse graphs.

Definition 1.1. *If G is a d-regular graph, we say that G is a* λ-**spectral expander** *if*

$$\lambda \geq \max\{|\lambda_2(G)|, |\lambda_n(G)|\}$$

and that $\lambda(G)$ *is the minimum* λ *such that this is true.*

If we look at the spectrum of the complete graph K_n, we see that it has maximal connectivity as well as spectral gap, since its eigenvalues are $\{n-1, -1, \dots, -1\}$, and so it is a dense 1-spectral expander. Several important results come as a result of this definition, such as the Expander Mixing Lemma. Since their proofs and results are similar in both the finite and limiting case, we will show them later on with graphons. However, one theorem is not quite as direct. Although it was initially derived outside of expander graph research, the Alon-Boppana bound provides a fantastic lower bound of d-regular graphs. In combination with constructions of Ramanujan graphs, we see that there exists an ”optimal” class of d-regular expanders [8].

Theorem 1.2 (Alon-Boppana Bound [9])**.** *If G is a d-regular graph on n vertices with diameter* Δ*, then*

$$\lambda_2(G) \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\lfloor \Delta/2 \rfloor}$$

In the rest of this thesis, we will introduce the notion of graphons as limits of graphs, as well as a few of their applications in extremal graph theory. We will then extend our notion of the adjacency matrix to the graphon shift operator, which will allow us to connect ideas from spectral graph theory to graphons. In turn, we will see that many theorems regarding expander graphs have parallel counterparts in graphons, and we will provide evidence that the celebrated Alon-Boppana bound for regular graphs should apply to graphons as well.

2. Graphons

One important area within spectral graph theory is that of the spectrum itself, particularly with random graphs. For certain classes of random graphs, the limiting distribution of the spectrum can be studied as the size of the graph grows. However, as the graph grows larger and larger in vertices, the adjacency matrix grows as well. This has led many to wonder what this matrix would become once the graph grows infinitely large and its edges are dense [3, 2]. In turn, this has resulted in the study of graphons, also known as graph limits.

Definition 2.1. *A* **graphon** *is a symmetric measurable function* $W : [0,1]^2 \to [0,1]$.

The world of graphons has provided many ways to reinterpret topics about finite graphs, such as that of the spectrum of random graphs [11] as well as extremal problems [3, 2]. Although there are other, more general definitions for graphons where the domain is a product measure space Ω^2 and the codomain is unbounded, for the rest of this thesis we will be using $[0,1]^2$ with the Lebesgue measure as our domain, and only considering bounded graphons. To further our understanding, we will redefine some classical properties of graphs in terms of graphons. For instance, we may view $W(x,y)$ as the weight of the edge between x and y. However, we cannot simply count the number of edges between two subsets of $[0,1]$. This is where density, measurability, and integration come into motion.

Definition 2.2. *For graphon W and measurable $S,T \subseteq [0,1]$, the density of edges or volume from S to T is*

$$e(S,T) := \int_{S\times T} W(x,y)dxdy = \int_0^1 \int_0^1 1_{y\in T} W(x,y) 1_{x\in S} dxdy$$

Similarly, W is **connected** *if $e(S,S^C) > 0$ for all $S \subset [0,1]$ with positive measure.*

Similarly, rather than computing the degree of a vertex $x \in [0,1]$ by counting the number of neighbors, we instead find the proportion of $[0,1]$ that are incident to x.

Definition 2.3. *The degree function of a graphon W is defined as*

$$d(x) := \int_0^1 W(x,y)dy$$

We say that W is **d-regular** *if $d(x) = d$ almost everywhere.*

These are parallel to their finite definitions, as we will see later with the graphon shift operator. In the finite definition, computing $e(S,T)$ uses an inner product, just as it will with graphons. The key connection between graphs and graphons is the conversion from growing and scaling infinite sums to integrals.

2.1. Convergence of Graph Limits

For any finite graph G, we can construct a **step graphon** that looks akin to the adjacency matrix A by partitioning $[0,1]$ into $|V(G)|$ evenly-sized blocks, and setting $W(x,y) = A_{i,j}$ for all $(x,y) \in [\frac{i-1}{n}, \frac{i}{n}) \times [\frac{j-1}{n}, \frac{j}{n})$. In turn, we may define convergence of a sequence of graphs through convergence of their step graphons. One norm will be of great importance to us.

Definition 2.4. *The* **cut norm** *for a graphon W is*

$$||W||_{\square} := \sup_{S \subset [0,1]} e(S, S^C)$$

The **cut distance** *for graphons V, W is*

$$\delta_{\square}(V,W) := \inf_{\phi} ||V - W^{\phi}||_{\square}$$

where ϕ is a measure-preserving bijection of $[0,1]$ and $W^{\phi}(x,y) = W(\phi(x), \phi(y))$. If G is a graph, then we say $\delta_{\square}(G,W) := \delta_{\square}(W_G, W)$.

To gain a more intuitive understanding of the cut distance, we see that the if two graphons are isomorphic to each other in a similar sense to graph isomorphisms, then they should have 0 distance. The cut norm is a measure of how spread out edges are across $[0,1]$. If the majority of differing edges between two graphons are concentrated in a small subset of $[0,1]$, akin to a subclique of a graph, then the cut distance between them is small. If the differing edges are heavily spread out, then the distance will be large.

Now the cut norm is a semi-norm, but we may convert it into a norm if we say a function is equivalent to 0 if it is 0 almost everywhere. Similarly, the cut metric is not a true metric and does not form a metric space with the set of graphons, but it does form a complete metric space over the equivalence classes of graphons that are equal up to a set of measure 0. Now under this cut metric, the set of step graphons generated from finite graphs form a dense subset in the space of graphons, which is the reason why graphons are considered graph limits. However, the cut norm is smaller than the p-norms, and thus convergence in other metrics will imply convergence in cut distance [6].

$$||W||_{\square} \leq ||W||_1 \leq ||W||_2 \leq ||W||_{\infty} \leq 1 \tag{1}$$

To see the result of this limit, we can consider a few sequences of graphs of increasing size. For the weighted complete graph with edge weight p, the limit is the constant $W(x,y) = p$. Similarly, a sequence of Erdős-Renyí random graphs $\{G_{n,p}\}_{n\geq 1}$, where each of the edges appears with constant probability p, will converge under the cut metric to the constant graphon $W(x,y) = p$ as well with high probability. Another interesting example is the balanced stochastic block model, where $V(G)$ is partitioned in k evenly sized blocks. The edges within a block appear with internal probability p, and the edges between blocks appear with external probability q. Under the cut metric, increasing size models converge to a block-diagonal graphon, where the diagonal blocks have value p, and blocks outside of the diagonal have value q. The generalized stochastic block model allows for varying block sizes and different internal or external probabilities, and its graph limit can be similarly expressed in terms of blocks. We now consider an equivalent notion of convergence by first introducing the notion of graph homomorphisms.

Definition 2.5. *A graph homomorphism from graph H to G is a function f : $V(H) \to V(G)$ where $\{i, j\} \in E(H)$ if and only if $\{f(i), f(j)\} \in E(G)$. We denote the set of all homomorphisms from H to G as* $\hom(H,G)$.

Definition 2.6. *Let G and H be finite graphs, and let $n = |V(H)|$. Then the homomorphism density of H in G is*

$$t(H,G) = \frac{|\hom(H,G)|}{|V(G)|^n}$$

For a graphon W, the homomorphism density of H in W is

$$t(H,W) = \int_{[0,1]^n} \prod_{\{i,j\} \in E(H)} W(x_i, x_j) dx_1 \ldots dx_n$$

For instance, we see that $t(K_2,W)$ and $t(K_3,W)$ is the edge and triangle density of a graphon W. This allows us to reframe extremal graph theory problems in terms of graphons. For finite graphs, the majority of Turán problems related edge counts to the appearance of a subgraph G, and so for graphons it is the maximal value of $t(K_2,W)$ until $t(G,W)$ is nonzero. Another important homomorphism density is that of closed walks of length $2k$ in a graphon W, which is precisely $t(C_{2k},W)$ where C_{2k} is the cycle graph. Similar to how the trace of A_G^t is the number of closed walks of length t on G, this will aid us later on in computing the trace of an operator defined by a graphon.

Now, intuition would say that if a graph were to appear at a certain density in a sequence of finite graph, then it should appear equally so in the limit. To capture this, we have the following theorem.

Theorem 2.7 ([3]). *For graphon W and sequence of finite graphs G_n, we have that $G_n \to W$ if and only if $t(H,G_n) \to t(H,W)$ for any finite graph H.*

3. Spectral Theory of Graphons

Similar to how A_G is an operator on $\mathbb{R}^{|V(G)|}$, we introduce an infinite analogue of the adjacency matrix. The following ideas are not unique to graphons, and in fact come from functional analysis, which studies infinite-dimensional vector spaces such as that of $L^2([0,1])$, which our graphons are a subset of.

Definition 3.1. *The graphon shift operator (WSO) for a graphon W on $f \in L^2([0,1])$ is*

$$(T_W f)(y) = \int_0^1 W(x,y) f(x) dx$$

By definition, we see that if a graphon is d-regular, then the graphon shift operator of a constant function c is dc. This resembles how, for d-regular graph G with adjacency matrix A_G, we see that $A_G\vec{1} = d\vec{1}$, where $\vec{1}$ is an eigenvector. This brings us to the notion of an eigenfunction.

Definition 3.2. *An* **eigenvalue** *and* **eigenfunction** *are a pair (λ, f) where $f \in L^2([0,1])$ and $(T_W f)(x) = \lambda f(x)$.*

Similar to how trace can be defined for matrices using the sum of the diagonal as well as the sum of its eigenvalues, we may extend this to general operators on Hilbert spaces. Since T_W is a self-adjoint integral operator over a compact Hilbert space, we have that T_W is a trace-class operator where a trace functional can be defined [1]. With $\{f_i\}_{i\geq 1}$ as an orthonormal basis of $L^2([0,1])$, we say that the trace is

$$\mathrm{Tr}(T_W) := \sum_{i=1}^{\infty} \langle T_W f_i, f_i \rangle \tag{2}$$

Furthermore, Lidskii's theorem states that this trace is equal to the sum of the eigenvalues of T_W. However, since T_W is self-adjoint and compact, this means the beloved spectral theorem still applies, which provides us with an orthonormal eigenfunction basis for $L^2([0,1])$ for a given graphon W [1, 3]. If $\{(\lambda, f_i)\}_{i\in\mathbb{Z}}$ are our eigenpairs, then

$$W(x,y) = \sum_{i\in\mathbb{Z}} \lambda_i f_i(x) f_i(y)$$

This will be quite useful in our later proofs. In fact, this also implies that the set of nonzero eigenvalues is countable [3]. However, as graphons are the limits of graphs, this begs the question of how the eigenvalues of graphs relate to the eigenvalues of the limit.

Theorem 3.3 ([3]). *Let the sequence graphs $\{G_n\}_{n\geq 1}$ converge to graphon W and let $v_n = |V(G_n)|$. Define $\lambda_i(G_n)$ and $\lambda_i(W)$ to be the ith largest positive eigenvalues of G_n and W respectively for $i \geq 1$, and define $\lambda_i'(G_n)$ and $\lambda_i'(W)$ to be the ith largest negative eigenvalues accordingly. Then* $\lim_{n\to\infty} \lambda_i(G_n)/v_n = \lambda_i(W)$

and $\lim_{n\to\infty} \lambda_i'(G_n)/v_n = \lambda_i'(W)$ *for all* $i \neq 0$.

The intuition behind this statement is that we may compute the eigenvalues of A_G using a finite sum over the entries of an eigenvector. As the number of entries grows towards infinity however, this sum can be viewed as a Riemann sum. Adding a normalizing factor and taking the limit hence changes this process into an integral, akin to the graphon shift operator. As most of the eigenvalues of a graph are $O(\sqrt{|V(G)|})$, we see that they vanish in the limit, which explains why the eigenvalues of graphons accumulate at 0 [10].

3.1. Expander Graphons

For the rest of this thesis, we will only be concerned with d-regular graphons. Similar to how the largest eigenvalue of a k-regular graph is k, this assumption gives us that the first eigenvalue $\lambda_1(W)$ is d, which has eigenfunction 1. In turn, we regard this as the trivial eigenvalue. For the same reason as finite graphs, d will have multiplicity 1 if and only if W is connected.

Definition 3.4. *We say that a graphon* W *is a* **λ-spectral expander** *if*

$$\lambda \geq \max\{|\lambda_2|, |\lambda_{-1}|\}$$

and that $\lambda(W)$ *is the minimum* λ *such that this is true.*

This definition allows use to refer to expanders as classes of graphons where the above properties hold. We now present our first result, which generalizes the Expander Mixing Lemma to graphons.

Lemma 3.5 (Expander Mixing Lemma for Graphons)**.** *If W is a d-regular λ-expander graphon, then for measurable $S,T \subset [0,1]$*

$$|e(S,T) - d|S||T|| \leq \lambda(W)\sqrt{|S||T|(1-|S|)(1-|T|)}$$

Proof. Note that

$$\begin{aligned} e(S,T) &= \int_0^1 1_{y\in T} \int_0^1 W(x,y) 1_{x\in S} dxdy \\ &= \int_0^1 [(1_{y\in T} - |T|) + |T|] \int_0^1 W(x,y)[(1_{x\in S} - |S|) + |S|] dxdy \end{aligned} \tag{3}$$

Since $1_{y\in T} - |T|$ is orthogonal to any constant function and W is d-regular, we have that (3) is equivalent to

$$|T||S| \int_0^1 \int_0^1 W(x,y) dxdy + \int_0^1 (1_{y\in T} - |T|) \int_0^1 W(x,y)(1_{x\in S} - |S|) dxdy$$

As W is d-regular, we have that the first term above is $d|S||T|$. Thus,

$$e(S,T) - d|S||T| = \int_0^1 (1_{y\in T} - |T|) \int_0^1 W(x,y)(1_{x\in S} - |S|) dxdy$$

The right hand of this equation contains an inner integral which is notably the graphon shift operator for W on $1_{x\in S} - |S|$. Now let $f_1, f_2, \ldots$ be an orthonormal basis of eigenfunctions of W for $L^2([0,1])$ and order the eigenfunctions and eigenvalues such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0$. Thus let $1_{x\in S} = \sum_i s_i f_i$ be the eigenfunction decomposition of the indicator function, and note that since $1_{x\in S} - |S|$ is orthogonal to $f_1 = 1$, we have that $s_1 = 0$. Hence,

$$\begin{aligned}
|e(S,T) - d|S||T|| &= \left| \int_0^1 (1_{y\in T} - |T|) \int_0^1 W(x,y)(1_{x\in S} - |S|)dxdy \right| \\
&= \left| \int_0^1 (1_{y\in T} - |T|) \sum_{i=2}^{\infty} \lambda_i s_i f_i(y) dy \right| \\
&\leq \lambda(W) \left| \sum_{i=2}^{\infty} \int_0^1 (1_{y\in T} - |T|) s_i f_i(y) dy \right|
\end{aligned}$$

We then apply Cauchy-Schwarz to see that

$$\begin{aligned}
|e(S,T) - d|S||T|| &\leq \lambda(W) \sqrt{\int_0^1 (1_{y\in T} - |T|)^2 dy} \sqrt{\int_0^1 \left(\sum_{i=2}^{\infty} s_i f_i(y) \right)^2 dy} \\
&\leq \lambda(W) \sqrt{\int_0^1 (1_{y\in T} - |T|)^2 dy} \sqrt{\sum_{i,j=2}^{\infty} \int_0^1 s_i s_j f_i(y) f_j(y) dy} \\
&\leq \lambda(W) \sqrt{|T|(1-|T|)} \sqrt{\sum_{i=2}^{\infty} s_i^2 \int_0^1 f_i^2(y) dy} \\
&\leq \lambda(W) \sqrt{|S|(1-|S|)|T|(1-|T|)} \qquad \square
\end{aligned}$$

One way to interpret the Expander Mixing lemma is that if $\lambda(W)$ is small, then W is very close to the constant graphon $W'(x,y) = d$ on any measurable subset of $[0,1]^2$. This provides another reason as to why if graphon W is d-regular and a λ-spectral expander where $\lambda < d$, then W is connected. Assuming that there exists an $S \subset [0,1]$ with positive measure such that $e(S,[0,1] \setminus S) = 0$, we may apply

Lemma 3.5, to see that

$$d|S|(1-|S|) \leq \lambda(W)|S|(1-|S|) < d|S|(1-|S|)$$

which is a contradiction. The Expander Mixing Lemma also gives a bound on the diameter of a graphon as well, given below.

Definition 3.6. *The* **diameter** *of a graphon W is the minimal value Δ such that for any $A, B \subset [0,1]$ with positive measure, we have that for some $0 \leq t \leq \Delta$,*

$$\langle 1_{x\in B}, (T_W^t 1_{y\in A})\rangle > 0$$

If none exists, we say that the diameter is infinite.

If $W(x,y) > 0$ almost everywhere, then W clearly has diameter 1. Now if W is a d-regular graphon where $d > 1/2$, then a short argument shows that the diameter of W must be at most 2. For other values of d, it is unclear if W has finite diameter or not. However, the Expander Mixing lemma is a strong tool to gain some insight into this.

Corollary 3.7. *If $\lambda(W) < d/2$, we have that $\Delta(W) = O(1)$.*

Proof. It suffices to show that for any $A \subset [0,1]$ where $d \leq |A| \leq 1/2$, there is an $r = O(1)$ where

$$N_r(A) := \left|\bigcup_{t=0}^{r} \operatorname{supp}(T_W^t 1_{x\in A})\right| > \frac{1}{2}$$

We only need to consider the case when $|A| \geq d$, as because W is d-regular, any application of T_W on $1_{x\in A}$ will have support of measure at least d, and so the diameter changes by at most 2. In essence, $N_r(A)$ is the measure of the set of all points reachable from A using T_W in at most r steps. By the Expander Mixing lemma, we have that

$$e(A,A) < d|A|^2 + \lambda(W)|A| < \left(\frac{d}{2} + \lambda\right)|A|$$

Since $e(A,A) + e(A,A^C) = d|A|$, it follows that $\varepsilon|A| = e(A,A^C) = (d/2 - \lambda)|A|$ is positive. Since W is d-regular, we thus have that $N_{r+1}(A) \geq N_r(A)(1+\varepsilon)$. Hence $r = \log_{1+\varepsilon}(1/(2|A|)) = O(1)$. For $A, B \subseteq [0,1]$ with positive measure, let r_A, r_B be found using the above method. If $R = \max\{r_A, r_B\}$, then this implies that $|N_R(A) \cap N_R(B)| > 0$ and thus $|N_{2R}(A) \cap B| > 0$, as desired. In other words, the above implies that $\langle T_W^R 1_{x\in A}, T_W^R 1_{x\in B}\rangle \neq 0$, and as the shift operator is self-adjoint, it is equivalent to $\langle T_W^{2R} 1_{x\in A}, 1_{x\in B}\rangle \neq 0$. □

In addition, we may also relate eigenvalues to some combinatorial properties. For example, in graphs, a valid color is a mapping where all vertices of a given color must have no edges between them. With graphons, we say that a k**-coloring** of a graphon W is a measurable function $c : [0,1] \to \{1, \ldots, k\}$ where W is 0 almost everywhere on $(c^{-1}(i))^2$ for all $1 \leq i \leq k$. The $\{chromaticnumber\}$ of a graphon W is the minimum k such that a valid k-coloring of W exists. With the Expander Mixing Lemma, we can relate the chromatic number of a graphon to its eigenvalues.

Corollary 3.8. *If W is a d-regular λ-spectral expander, then its chromatic number* *is greater than* $d/\lambda(W)$.

Proof. If $c : [0,1] \to \{1,\ldots,k\}$ is a valid k-coloring of W, then we have that W is 0 almost everywhere on $(c^{-1}(i))^2$, and thus that $e(c^{-1}(i), c^{-1}(i)) = 0$. It follows that if $S_i = c^{-1}(i)$, then by the Expander Mixing Lemma,

$$d|S_i|^2 \leq \lambda(W)|S_i|(1-|S_i|) < \lambda(W)|S_i|$$

and so $|S_i| < \lambda(W)/d$. Since this applies for all $i \in \{1,\ldots,k\}$ and $\sum_{i=1}^k |S_i| = 1$, we have that $k > d/\lambda(W)$. □

We now introduce a concept similar to the Expander Mixing lemma, but with regard to how easily one can get ”lost” in the graphon from any initial position.

Definition 3.9. *For graphon* W *and* $\varepsilon > 0$*, the mixing time* $M(W,\varepsilon)$ *of* W *is the minimal* t *such that for all* $f \in L^2([0,1])$*,*

$$||T_W^t f - 1||_1 \leq \varepsilon$$

This is equivalent to saying that for any initial distribution f, we are ε close to the uniform distribution/constant function 1 after t applications of the graphon shift operator.

Theorem 3.10. *Let* W *be a* d*-regular* λ*-expander graphon. The mixing time of* W *is at most*

$$M(W,\varepsilon) \leq \left\lceil \frac{\log \varepsilon}{\log \lambda} \right\rceil$$

Proof. Note that for $f \in L^2([0,1])$ where f is a distribution over $[0,1]$, we have that $f - ||f||_2 = f - 1$ is orthogonal to 1. Decomposing f into an orthonormal eigenfunction basis of $L^2([0,1])$, we see that

$$\begin{aligned}\int_0^1 ((T_W f)(y) - 1)^2 dy &= \int_0^1 \left(\int_0^1 W(x,y)(f(x) - 1)dx \right)^2 dy \\ &\leq \lambda(W)^2 \int_0^1 (f(y) - 1)^2 dy\end{aligned}$$

It follows that $||T_W^t f - 1||_2 \leq \lambda(W)^t ||f - 1||_2 \leq \lambda(W)^t$ by repeated application of the operator, and thus by Cauchy-Schwarz,

$$||T_W^t f - 1||_1 \leq ||1||_2 ||T_W^t f - 1||_2 \leq \lambda(W)^t$$

Solving for t such that $\lambda(W)^t$ is at most ε gives our desired result. $\square$

Corollary 3.11. *If W is d-regular and connected, then for $\varepsilon < d$, we have that*

$$\Delta(W) \leq M(W, \varepsilon) + 1 \leq \left\lceil \frac{\log \varepsilon}{\log \lambda} \right\rceil + 1$$

Proof. For any $A \subseteq [0,1]$, let $Z = \text{supp}(T_W^t 1_{x \in A})^C = \{y \in [0,1] : (T_W^t 1_{x \in A})(y) = 0\}$ be the set of points that are unreachable from A after t applications of the graphon shift operator. We have by Theorem 3.10 that $||T_W^t 1_{x \in A} - 1||_1 \leq \varepsilon < d$, which implies that $|Z| < d$. However, since W is d-regular, we see that for $x \in Z$, that $d(x) = d > |Z|$, and so each $x \in Z$ has a set of neighbors in the support of $T_W^t 1_{x \in A}$

of measure at least $|Z| - d$. This is equivalent to saying that $\langle T_W^t 1_{x \in A}, T_W 1_{x \in Z} \rangle > 0$. Hence, we see that $\Delta(W) \leq M(W, \varepsilon) + 1$. □

Notably, since $\lambda < d$ by assumption that W is connected, we may find an $\varepsilon \in (\lambda, d)$ such that this bound is better than the one given by the Expander Mixing lemma. Critically, this shows that d-regular connected graphons always have finite diameter, which a core part of the Alon-Boppana bound that we will see in the next section. In fact, for most graphons, this implies that their diameter is at most 2. This is actually consistent in the finite graph case, as graphons represent graphs with $O(n)$ edge density, which implies a very small diameter [10]. However, this d-regularity assumption is necessary. If there was no lower bound greater than 0 for the degrees of the graphon, then it is possible for a connected graphon to have infinite diameter [5].

One major application of graphons is that they are very ”efficient” graphs in terms of sampling vertices via random walks without many edges. If k is a prime power, then constructions of k-regular expander graphs exist. However, we cannot sample a regular expander graph on n vertices from a d-regular graphon, as the density of edges in the graph will always be $O(dn)$. In turn, the shift from sparse to dense expander graphs is not particularly useful, but we still do not know if expander graphons hold particular properties that sparse expander graphs cannot.

3.2. Evidence for Alon-Boppana for Graphons

Since $\Delta(W)$ is finite by Corollary 3.11 for small enough ε, we have a strong motivation for finding a bound on $\lambda(W)$ similar to that of Theorem 1.2. Thus, we now provide scaffolding towards a proof of the Alon-Boppana for graphons, in which for d-regular graphon W, we have that $\lambda(G) \geq O(\sqrt{d}) - o(1)$. We first begin by finding finite regular weighted graphs to compare W to. This allows us to create regular graphons that are increasingly close to W which must be the sum of step functions.

Lemma 3.12. *For d-regular graphon W, there exists a sequence of weighted graphs $\{G_n\}_{n\geq 1}$ where $G_n \to W$ and $|V(G_n)| = n$ while the sum of edge weights incident to any vertex in G_n is dn.*

Proof. We partition $[0,1]$ into n uniform intervals $B_i = [\frac{i-1}{n}, \frac{i}{n})$ for $1 \leq i < n$ and $B_n = [\frac{n-1}{n}, 1]$, and create a step graphon W_n as follows: for $(x,y) \in B_i \times B_j$,

$$W_n(x,y) = n^2 \int_{B_i \times B_j} W(x,y)dxdy$$

From there, we create G_n on vertices $\{1, 2, \ldots, n\}$ by letting the edge weight between vertices $a, b \in V(G_n)$ be the value of W_n on $B_a \times B_b$. Since simple functions are dense in $L^2([0,1]^2)$, it is clear that $||W_n - W||_1$ converges to 0, and so by equation (1) the graphs G_n converge to W in cut distance. □

In order to lower bound the number of closed walks on a graphon, we apply d-regularity in order to count the homomorphism density of trees in the graphon.

Lemma 3.13. *For d-regular graphon W, if T is a finite tree where $k = |E(T)|$, then $t(T, W) = d^k$.*

Proof. We will induct on the number of vertices in the tree. For $k = 1$, we see that $t(T,W)$ is the degree function, which is always d. Now assume the statement holds for $k-1$ edges. For any tree T with k edges, we may order the vertices using breadth-first search with labels $\{1,\ldots,k+1\}$. Without loss of generality, let $\{k,k+1\}$ be the last edge visited. Then

$$\begin{aligned} t(T,W) &= \int_{[0,1]^{k+1}} \prod_{\{i,j\}\in E(T)} W(x_i,x_j)dx_1\ldots dx_{k+1} \\ &= \int_{[0,1]^k} \left(\int_{[0,1]} W(x_k,x_{k+1})dx_{k+1} \right) \prod_{\{i,j\}\in E(T)\setminus\{\{k,k+1\}\}} W(x_i,x_j)dx_1\ldots dx_k \\ &= d\int_{[0,1]^k} \prod_{\{i,j\}\in E(T)\setminus\{\{k,k+1\}\}} W(x_i,x_j)dx_1\ldots dx_k \end{aligned}$$

Removing $\{k,k+1\}$ and vertex $k+1$ from T results in a tree T' with $k-1$ edges, and thus

$$t(T,W) = dt(T',W) = d^k$$

by hypothesis, completing the induction. □

If we let $\{G_n\}_{n\geq 1}$ be the sequence of dn-regular graphs from Lemma 3.12 where $G_n \to W$ and $n = |V(G)|$, then Theorem 3.3 implies that the eigenvalues of the step graphons W_{G_n} created from G_n converge to those of W. Hence, any lower bound of $\lambda(W_{G_n})$ will converge in the limit supremum to a lower bound for $\lambda(W)$. As W_n is d-regular, its first eigenvalue is d. However, as G_n has n eigenvalues, it follows that W_n has n nonzero eigenvalues, and so

$$\mathrm{Tr}(T_{W_n}^{2k}) \leq d^{2k} + (n-1)\lambda(W)^{2k} \tag{4}$$

Note that $\mathrm{Tr}(T_{W_n}^{2k}) = t(C_{2k}, W_n)$ is the density of closed walks on W_n, where C_{2k} is the cycle graph. If there existed a lower bound of $\mathrm{Tr}(T_{W_n}^{2k})$ using Lemma 3.13 such that $\mathrm{Tr}(T_{W_n}^{2k}) \geq O(nd^k)$, then we may let k grow sublogarithmically with n, and this would thus grant our desired result.

However, a few issues arise in attempting to find a similar result to Theorem 1.2. In Alon's original proof, the vector constructed to utilize variational characterization of eigenvalues is heavily altered by vertices at the end and center of maximal paths. As we have seen, most graphons only have a diameter of 2, and so all subsets of $[0,1]$ in a maximal path will have a large contribution of mass to any function created. Thus, adapting this proof is not an effective method. In addition, the more combinatorial proof techniques used for Theorem 1.2 take advantage of trace and the infinite d-regular tree. However, this method does not generalize well to weighted graphs, which we are concerned with. In turn, a proof of this type may be viable for d-regular $\{0,1\}$-valued graphons, but not for general regular graphons.

4. Conclusion

In this thesis, we discussed spectral graph theory and graphons, and extended several results from expander graphs to d-regular graphons. This latter portion develops new spectral results for graphons and analyzes some of its implications.

Some results such as the Expander Mixing Lemma generalized very well to graphons, as high density in edges only helps these results. In turn, we have seen unintuitive properties of graphons, such as the finite diameter of connected ones. In contrast however, a high density of edges often implies high connectivity, which is equivalent to having $\lambda(G)$ be very small. Thus, the edge density of graphons implies that a much greater hurdle lies in proving an Alon-Boppana-type result for general graphons, just as it does for general graphs. This reminds us that even in the limiting case, a paradigm shift occurs with edge density. Although some wonderful ideas may appear to generalize, the failure to acknowledge this paradigm shift may lead them astray.

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References

[1] Introductory Functional Analysis with Applications, Wiley classics library, Wiley India Pvt. Limited, 2007.

[2] C. Borgs, J. Chayes, L. Lovasz, V. T. Sos, K. Vesztergombi, Convergent Sequences of Dense Graphs II: Multiway Cuts and Statistical Physics, Annals of Mathematics (2012) 151–219.

[3] C. Borgs, J. Chayes, L. Lovász, V. Sós, K. Vesztergombi, Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, Advances in Mathematics 219 (2008) 1801–1851.

[4] F. Chung, Spectral Graph Theory, number no. 92 in CBMS Regional Conference Series, Conference Board of the Mathematical Sciences, 1997.

[5] A. Khetan, M. Mj, Cheeger inequalities for graph limits, 2018.

[6] L. Lovász, Large Networks and Graph Limits., volume 60 of *Colloquium Publications*, American Mathematical Society, 2012.

[7] Y. Ma, Y. Fu, Manifold Learning Theory and Applications, CRC Press, 2012.

[8] A. Marcus, D. A. Spielman, N. Srivastava, Interlacing Families I: Bipartite Ramanujan Graphs of All Degrees, 2014.

[9] A. Nilli, On the second eigenvalue of a graph, Discrete Mathematics 91 (1991) 207–210.

[10] K. Tikhomirov, P. Youssef, The spectral gap of dense random regular graphs, The Annals of Probability 47 (2019) 362 – 419.

[11] Y. Zhu, A graphon approach to limiting spectral distributions of Wigner-type matrices, Random Structures Algorithms 56 (2019) 251–279.