University of California San Diego
DEPARTMENT OF MATHEMATICS

Kleinian Singularities

Tomoki Oda
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Advisor:

James McKernan
Professor of Mathematics

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Foreword

1   Introduction

In 1880, Klein employed polynomial equations coming from the Invariant theory of a finite subgroup of $SL(2,\mathbb{C})$ to solve the quintic equation. The invariant polynomials that appear in this way are singular at the origin. We will call the singularities arising from a quotient of $\mathbb{C}^2$ by a finite subgroup $SL(2,\mathbb{C})$ Kleinian singularities. At the time, Klein did not fully recognize the importance of these singularities.

A half-century later, Du Val reincarnated Klein's work. He discovered that Kleinian singularities are classified by Dynkin diagrams. Already by that time Dynkin diagrams had been used to successfully classify several different mathematical objects. Nevertheless it required a further half century to fully reveal the deep connections between Dynkin diagrams and Kleinian singularities.

In 1978, McKay discovered new connections between irreducible representations of finite subgroups of $SL(2,\mathbb{C})$ and Dynkin diagrams. This discovery proved very inspirational. Eventually, it was conjectured that a similar phenomena would happen with the quotient of $\mathbb{C}^n$ by a finite subgroup of $SL(n,\mathbb{C})$. This phenomenon is now referred to as the McKay correspondence. This thesis aims to explain the development of this theory.

2   Invariant Theory

Group theory tells us there are 5 types of finite subgroup of $SO(3)$

Proposition 2.1. *Any finite subgroup of SO(3) is isomorphic to one of these 5 groups,*
(1) C_n(Cyclic group of order n)
(2) D_n (Dihedral group of order 2n)
(3) T(Tetrahedral group)
(4) O (Octahedral group)
(5) I (Icosahedral group)

We will denote finite subgroups of $SO(3)$ by Γ. We can lift elements of $SO(3)$ to $SU(2)$, because $SU(2)$ is a double cover of $SO(3)$, $Spin(3) \cong SU(2)$. The following theorem tells that finite subgroups of $SL(2,\mathbb{C})$ correspond one-to-one to finite subgroups of $SU(2)$, and by taking preimages by the double-

cover map we have a one to one correspondence between finite subgroups of $SO(3)$ and $SU(2)$

Theorem 2.2. *Every finite subgroup of $SL(n, \mathbb{C})$ is conjugate to a finite subgroup of $SU(n)$*

Proof. This statement holds for any n. We will use two facts.
(a) For any finite subgroup G of $SL(n, \mathbb{C})$, there is a G-invariant Hermitian inner product $< -, - >_G$ on $\mathbb{C}^n$. It can be obtained from the canonical inner product by averaging

$$< x, y >_G = \frac{1}{|G|} \Sigma_{g \in G} < gx, gy > \tag{1}$$

(b) Given any two Hermitian inner products $< -, - >$ and $< -, - >_*$ on $\mathbb{C}^n$. There is a $T \in GL(n, \mathbb{C})$ such that for $x, y \in \mathbb{C}^n$

$$< x, y >_* = < Tx, Ty > \tag{2}$$

For (b) consider two orthonormal bases $\mathscr{B}$ with respect to $< -, - >$, and $\mathscr{B}_*$ for $< -, - >_*$. There is a $T \in GL(n, \mathbb{C})$ such that $T\mathscr{B}_* = \mathscr{B}$. Then (2) holds for any elements x and y of $\mathscr{B}$ and hence arbitary $\mathbb{C}^n$
Given a finite subgroup $G < SL(n, \mathbb{C})$. We chose a G-invariant inner products $< -, - >_G$ and find a $T \in GL(n, \mathbb{C})$, such that $< x, y >_G = < Tx, Ty >$ with $< -, - >$ being the canonical product. Then TGT^{-1} is in $SU(n)$ $\qquad \square$

Based on the above theorem, we can classify finite subgroups of $SL(2, \mathbb{C})$ in a fairly simple manner. All finite subgroups of $SL(2, \mathbb{C})$ are isomorphic to one of the following 5 groups.

Proposition 2.3. *(1) C_n(cyclic group of order n)*
(2) $\tilde{D}_n$ (double cover of dihedral group)
(3) $\tilde{T}$(double cover of tetrahedral group)
(4) $\tilde{O}$ (double cover of binary octahedral Group)
(5) $\tilde{I}$ (double cover of binary icosahedral group)

We will denote $\tilde{\Gamma}$ to be a finite subgroup of $SL(2, \mathbb{C})$. We are interested in finding invariant polynomial of $\tilde{\Gamma}$.

In this section we take generator of $C_n < SL(2, \mathbb{C})$ to be $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, with $\zeta = exp(\frac{2\pi i}{n})$.

Let $G < GL(n, \mathbb{C})$ be a group. We define actions on polynomial rings, where $g \in G, z \in \mathbb{C}^n, f \in \mathbb{C}[z_1 \ldots z_n]$, in the following way.

$$(g \cdot f)(z) = f(g^{-1}(z))$$

Definition 2.4. Polynomials $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called G- invariant polynomial if for any element of $g \in G$ we have

$$g \cdot f = f \tag{3}$$

Sets of G-invariant polynomial form a ring. This ring is called G-invariant ring. We will denote that ring to be $\mathbb{C}[z_1, \ldots, z_n]^G$.

$SO(3)$ is the symmetry group of S^2. We can identify S^2 as a Riemann sphere $\mathbb{P}^1$, so the action of $SO(3)$ can be regarded as subgroup of $Aut(\mathbb{P}^1) \cong SL(2, \mathbb{C})/ < \pm 1 >$. Thus for each $\gamma \in \Gamma$ we can choose a lift to an element $\tilde{\gamma} \in SL(2, \mathbb{C})$. For each $(p_1, p_2) \in \mathbb{C}^2$ we define the functions $\tilde{\gamma}_1, \tilde{\gamma}_2$ by the equation $\tilde{\gamma} \cdot (p_1, p_2) = (\tilde{\gamma}_1(p_1, p_2), \tilde{\gamma}_2(p_1, p_2))$. For each point $(p_1, p_2) \in \mathbb{C}^2 \setminus \{0\}$ we define a function

$$F_p(z_1, z_2) = \prod_{\gamma \in \Gamma} (\tilde{\gamma}_2(p_1, p_2)z_1 - \tilde{\gamma}_1(p_1, p_2)z_2) \tag{4}$$

For another point $q = (q_1, q_2) \in \mathbb{C}^2 \setminus \{0\}$, if there is some $\gamma \in \Gamma$ such that $\gamma \cdot (q_1 : q_2) = (p_1 : p_2)$ then $F_p(z_1, z_2) = \chi(\gamma)F_q(z_1, z_2)$ for some constant $\chi(\gamma) \in \mathbb{C}$. Therefore for each orbit $\mathcal{O}$ of Γ acting on $\mathbb{P}^1$ this construction gives a function $F_{\mathcal{O}}$ which is well defined up to constants. Note that F_p has repeat roots when the orbit has a non-trivial stabilizer. In the case of D_n, T, O and I there are 3 such orbits, e, f, and v. We will choose functions F_v, F_f and F_e that correspond to those orbits. Klein showed that we can multiply the equations F_v, F_f and F_e by suitable constants so that the generators of the $\tilde{\Gamma}$-invariant ring can be written as the following.

$\tilde{\Gamma}$	Generators	Relation u
$\tilde{C}_n$	$x = z_1 z_2, y = \frac{z_1^n - z_2^n}{2}, z = \frac{z_1^n + z_2^n}{2\sqrt{-1}}$	$x^n + y^2 + z^2 = 0$
$\tilde{D}_{2k+1}$	$x = F_f, y = 2iF_eF_v, z = i(F_e^2 F_f - F_v^2 F_f)$	$x^{n+1} - xy^2 - z^2 = 0$
$\tilde{D}_{2k}$	$x = F_f^2, y = i(F_e^2 - F_v^2), z = i(2F_eF_vF_f)$	$x^{n+1} - xy^2 - z^2 = 0$
$\tilde{T}$	$x = F_e, y = \frac{1}{3}F_vF_f, z = F_v^3 - F_f^3$	$x^4 + y^3 + z^2 = 0$
$\tilde{O}$	$x = F_f, y = F_v^2, z = F_eF_v$	$x^3 y + y^3 + z^2 = 0$
$\tilde{I}$	$x = F_v, y = F_f, z = F_e$	$x^5 + y^3 + z^2 = 0$

Proof. [14] □

Three generators of above table determine a homomorphism $\Phi : \mathbb{C}[w_1, w_2, w_3] \to \mathbb{C}[z_1, z_2]$ by $\Phi(w_1) = x, \Phi(w_2) = y, \Phi(w_3) = z$. The image of Φ is $\mathbb{C}[z_1, z_2]^{\tilde{\Gamma}}$.

Theorem 2.5. *The homomorphism Φ induces an isomorphism*

$$\mathbb{C}[w_1, w_2, w_3]/u \cong \mathbb{C}[z_1, z_2]^{\tilde{\Gamma}} \tag{5}$$

Proof. It is clear that Φ is homeomorphism and image of Φ is $\mathbb{C}[z_1, z_2]^{\tilde{\Gamma}}$. Since $u \subset ker\Phi$ we have a surjection $\Phi : \mathbb{C}[w_1, w_2, w_3]/u \to \mathbb{C}[z_1, z_2]^{\tilde{\Gamma}}$. u is irreducible polynomial, so both $\mathbb{C}[w_1, w_2, w_3]$ and $\mathbb{C}[z_1, z_2]^{\tilde{\Gamma}}$ are integral domain. To have an isomorphism over integral domain over $\mathbb{C}$, we need to have same transcendental degree. We want to compare the transcendental degree of both polynomial ring.
First we see transcendental degree of $\mathbb{C}[z_1, z_2]^{\tilde{\Gamma}}$ is 2. For any $g \in \mathbb{C}[z_1, z_2]$ we can introduce the polynomial $h(X) = \prod_{\gamma \in \tilde{\Gamma}}(X - \gamma \cdot g)$. $h(X)$ has coefficient of $\mathbb{C}[z_1, z_2]^{\tilde{\Gamma}}$ and $h(g) = 0$ so $\mathbb{C}[z_1, z_2]$ is an algebraic extension of $\mathbb{C}[z_1, z_2]^{\tilde{\Gamma}}$ means $trdeg\mathbb{C}[z_1, z_2]^{\tilde{\Gamma}} = trdeg\mathbb{C}[z_1, z_2] = 2$
On the other hand, we have $trdeg\mathbb{C}[w_1, w_2, w_3]/u \leq trdeg\mathbb{C}[w_1, w_2, w_3] = 3$ but this is not an isomorphism so $trdeg\mathbb{C}[w_1, w_2, w_3]/u \leq 2$. But from surjection we have $trdeg\mathbb{C}[w_1, w_2, w_3]/u \geq 2$ thus we shows an isomorphism. □

Our results suggests $\mathbb{C}[z_1, z_2]^{\tilde{\Gamma}}$ are finitely generated. In general $\mathbb{C}[z_1, \ldots, z_n]^G$ is finitely generated when G is finite group $G < GL(n, \mathbb{C})$. This fact implies this ring is the coordinate ring of an affine variety.

Proposition 2.6. *If finite group G is acting on $\mathbb{C}[z_1, \ldots, z_n]$ then $\mathbb{C}[z_1, \ldots, z_n]^G$ is finitely generated.*

Proof. For $f \in \mathbb{C}[z_1, \ldots, z_n]$, we consider the polynomial $\prod_{g \in G}(X - g \cdot f) = X^N + a_{N-1}(f)X^{N-1} \ldots a_0(f)$ with coefficients $a_i \in \mathbb{C}[z_1, \ldots, z_n]^G$. Let $A \subset \mathbb{C}[z_1, \ldots, z_n]^G$ be a $\mathbb{C}$-subalgebra generated by $a_i(z_j), 1 \leq j \leq n$ and $1 \leq i \leq N - 1$. Since A is finitely generated, it is noetherian. As A-module $\mathbb{C}[z_1, \ldots, z_n]$ is finitely generated by finitely $z_1^{t_1} \ldots z_n^{t_n}$ with $t_i < N$. Therefore A-submodule $\mathbb{C}[z_1, \ldots, z_n]^G$ is also finitely generated. $\qquad \square$

We may assume that finitely many forms $1, f_1 \ldots f_r$ of degrees $0, d_1 \ldots d_r$ respectively generate the $\mathbb{C}$-algebra $\mathbb{C}[z_1, \ldots, z_n]^G$. The polynomial map

$$F : \mathbb{C}^n \to \mathbb{C}^r \tag{6}$$

with components $f_1 \ldots f_r$ induces the surjection

$$\mathbb{C}[w_1, \ldots, w_r] \to \mathbb{C}[z_1, \ldots, z_n]^G, h \to h \circ F \tag{7}$$

Let u be a kernel of (7) and define $V = \{x \in \mathbb{C}^r : h(x) = 0 \text{ for every } h \in u\}$. This setting allow us to discuss geometric interpretation of finite group invariant ring.

Proposition 2.7. *(1) $F(\mathbb{C}^n) = V$*
(2)$F(z) = F(w)$ if and only if $z = g \cdot w$ for some $g \in G$

Proof. (1) It is clear that $F(\mathbb{C}^n) \subset V$. We want to prove $V \subset F(\mathbb{C}^n)$. Let $c = (c_1, \ldots, c_n) \in V$ be given. Let I be the ideal generated by $< f_1 - c_1, \ldots, f_r - c_r >$ and we will prove that $I \neq \mathbb{C}[z_1, \ldots z_n]$. If we can prove this fact, then from Nullstellensatz there is a common zero for $f_1 - c_1 \ldots f_r - c_r$ and we can see $F(\mathbb{C}^n) = V$. Suppose $I = \mathbb{C}[z_1, \ldots, z_n]$ then there is a polynomial $p_1, \ldots, p_r \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\sum_1^r p_i(f_i - c_i) = 1$. We acts $g \in G$ and add up, using the invariance of f_i

$$(\sum_{g \in G} g \cdot p_1)(f_1 - c_1) + \ldots (\sum_{g \in G} g \cdot p_r)(f_r - c_r) = |G| \tag{8}$$

Since $\sum g \cdot p_i \in \mathbb{C}[z_1, \ldots z_n]^G$ there exist polynomial $\tilde{p}_i \in \mathbb{C}[w_1, \ldots, w_r]$ with $\tilde{p}_i \circ F = \sum g \cdot p_i$

$$h(x) = \tilde{p}_1(z_1 - c_1) + \ldots p_r(z_r - c_r) - |G| \in u \tag{9}$$

Therefore $h(x) = 0$ for every $x \in V$. In particular $h(c) = 0 \to |G| = 0$ contradiction.

(2) If $z = g \cdot w$, then $F(z) = F(w)$.

Assume that $z \neq g \cdot w$ for all $g \in G$, then there is an $f \in \mathbb{C}[z_1, \ldots, z_n]$ with $f(z) = 0$ and $f(g \cdot w) = 1$ for all $g \in G$. Introduce the polynomial $h := \prod g \cdot f$, and this is an $G-$invariant polynomial i.e $h \in \mathbb{C}[z_1, \ldots, z_n]^G$. Thus we can lift it to $\tilde{h} \in \mathbb{C}[w_1, \ldots, w_r]$. In this case $\tilde{h}(F(z)) = 0$ and $\tilde{h}(F(w)) = 1$. We showed separation. $\square$

Above two proposition shows that there is one to one correspondence between the orbit space $\mathbb{C}^n/G$ and V. This allows us to identify the coordinate ring of $\mathbb{C}^n/G$ as the ring $\mathbb{C}[z_1, \ldots, z_n]^G$.

3 Resolution

In previous section we have figured out that the invariant ring of a finite group forms an affine variety. Our main interest in this thesis is to discuss varieties represented as a $\mathbb{C}^2$ quotient by a finite group. In the case of a quotient of $\tilde{\Gamma}$, the variety has a singularity at the origin. We call this a **Kleinian singularity**.

Definition 3.1. If an affine variety $\mathbb{C}^n/G$ with a finite group $G < GL(n, \mathbb{C})$ has a singularity, we will call that point quotient singularity

We confirm that quotient of $\mathbb{C}^2$ by finite subgroup of $SL(2, \mathbb{C})$ has a singularity at the origin. How about other finite group? Some group will not contribute to create singularities. Other group will create worse singularities than Kleinian singularities.

Definition 3.2. Coxeter group G is a finite group, presented by $< r_1, r_2 \ldots r_n | (r_i r_j)^{m_{ij}} = 1 >$ with $m_{ij} \in \mathbb{N} \cup \{\infty\}$ $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$, $m_{ij} = \infty$ means no relation for $(r_i r_j)^{m_{ij}}$

Example 3.3. Symmetry group is a Coxeter group

Example 3.4. T, O, I is Coxeter group

Theorem 3.5. *[10] (Chevalley) The quotient of $\mathbb{C}^n$ by a Coxeter group $G < GL(n, \mathbb{C})$ is $\mathbb{C}^n/G \cong \mathbb{C}^n$.*

However in general, higher dimension of quotient singularities behave differently compared to $\tilde{\Gamma}$

Example 3.6. [9] Singularities of 3 dimension quotient by finite group $G \subset SL(3, \mathbb{C})$ of $\mathbb{C}^3$, $\mathbb{C}^3/G$ doesn't necessary be hyperplane and singularities is not isolated.

We want to define an appropriate notion of resolution of singularities for a quotient singularities of finite group. In $2-$dimension that resolution has connection to Dynkin diagrams.

Definition 3.7. For algebraic variety X, denote singular locus to be $X_{sing} \subset X$. We will call a morphism $\pi : \tilde{X} \to X$ a resolution of singularities if it satisfies the following conditions.
(1) The morphism π is proper
(2) π induces an isomorphism $\pi : \tilde{X} \setminus f^{-1}(X_{sing}) \to X \setminus X_{sing}$
(3) The variety $\tilde{X}$ is non-singular

Blow-up is one of the fundamental tools for the resolution of singularities. Nevertheless the calculation of blow-ups tends to be complicated calculation, so in this thesis, we only discuss the blow-up of variety by a point.

Construction 3.8. The blow up of $\mathbb{C}^2$ at the $p := (p_1, p_2) \in \mathbb{C}$, $\tilde{\mathbb{C}^2}$, is defined in the following way.

$$\tilde{\mathbb{C}^2}_p = \{((z_1, z_2), (s : t)) \in \mathbb{C}^2 \times \mathbb{P}^1 | (z_1 - p_1)t = (z_2 - p_2)s\} \qquad (10)$$

We denote the projection to the $\mathbb{C}^2$ by φ. If $p' \in \mathbb{C}^2$ and $p \neq p'$ then $\varphi^{-1}(p)$ is one point so $\tilde{\mathbb{C}^2} \setminus \varphi^{-1}(0) \cong \mathbb{C}^2 \setminus 0$. On the inverse image of p is can be any point of $\mathbb{P}^1$, so $\varphi^{-1}(0) \cong \mathbb{P}^1$. We can take two open covering for $\tilde{\mathbb{C}}_{(0,0)}$. Notice when $s \neq 0$, $y = \frac{t}{s}x$, $t \neq 0$ $x = \frac{s}{t}y$. If we put $\frac{t}{s} = u$ and $\frac{s}{t} = v$ then $\tilde{\mathbb{C}^2}$ patched U_1 and U_2 with a $x = vy$ and $y = ux$

$$\tilde{\mathbb{C}^2} = U_1 \cup U_2, U_1 = \{(x, u) \in \mathbb{C}^2\} \cong \mathbb{C}^2, U_2 = \{(y, v) \in \mathbb{C}^2\} \cong \mathbb{C}^2$$

We define the blow up of subvariety $X \subset \mathbb{C}^2$ at point $p \in X$ by $\tilde{X} := \overline{\phi^{-1}(X - p)}$. The meaning of $\overline{X}$ is taking closure of X. Let's denote π to be the restriction of ϕ to X so that $\pi : \tilde{X} \to X$. The inverse image of p of π is called exceptional curve.

For a further discussion of exceptional curve, we need to develop intersection theory. Consider compact and smooth algebraic surface S and an irreducible curve $C \subset S$. Take open covering $\{U_i\}$ of S and suppose C is

defined by f_i on each U_i. Then $g_{ij} = \frac{f_i}{f_j}$ is a regular function non-vanishing on $U_i \cap U_j$. We can define a complex line bundle by letting the g_{ij} be transition functions. Let's this $\mathscr{L}(C)$.

For smooth compact algebraic curves $C_1, C_2 \subset S$, we get a line bundle of C_2 by restricting $\mathscr{L}(C_1)$ to C_2. Then we can define the intersection number to be the degree of the restriction of $\mathscr{L}(C_1)$ to C_2

$$(C_1, C_2) = deg\mathscr{L}(C_1)|_{C_2} \tag{11}$$

Let's clarify the meaning of degree of line bundle. For an open cover $\{U_i\}$ that $\mathscr{L}(C)$ is locally trivial and transition function to be g_{ij}. Then rational satisfying $s : C \to \mathscr{L}(C)$ satisfies $s_i(x) = g_{ij}s_j(x), x \in U_i \cap U_j$. Since $\{g_{ij}\}$ does not have pole and zeros on $U_i \cap U_j$ so s_i and s_j has same zeros and pole with multiplicity. We would define $degs$ as degree as degree of rational section s. We need to check well-definedness. Let $s' = \{s_i'\}$ be another rational section. We have $\frac{s_i}{s_i'} = \frac{s_j}{s_j'}$, so $\{\frac{s_i}{s_i'}\}$ defines a rational function over C. Since degree of rational function over C_2 is 0, we have $deg(s_i) = deg(s_j)$. This shows degree of rational section is independent of choice of the section. (C, C) is called self-intersection number. This quantity is a key when we classify resolution of Kleinian singularities.

Proposition 3.9. *If we blow up of smooth point, then self-intersection number of exceptional curve is -1*

Proof. By the definition of intersection number, it is enough to calculate around E. Consider blow-up of $\mathbb{C}^2$ around the origin. $\tilde{\mathbb{C}}^2$ has two open sets U_1 and U_2. Exceptional curve is defined by $x = 0$ on U_1 and $y = 0$ on U_2, so transition function of $\mathscr{L}(E)$ is

$$g_{12} = \frac{x}{y} = v = \frac{1}{u} \tag{12}$$

Next task is to compute $deg\mathscr{L}(E)|_E$. We chose rational function such that $s|_{U_1} = \frac{1}{u}$ and $s|_{U_2} = 1$ then this has order 1 pole on U_1 and no pole on U_2 so we have $deg(\mathscr{L}(E)|_E) = -1$ meaning $(E, E) = -1$ $\qquad \square$

In general, the resolution of singularities requires blow-ups along the subvarieties. Remarkably Kleinian singularity can be resolved only by a blow-up of points. Durfee [6] called this property to be absolute isolated singularity.

Conversely, if the an absolute singularity of -2 self-intersection number of exceptional curves is locally isomorphic to a Kleinian singularity.

Zariski and Walker proved that for an resolution of singularities of algebraic surface always exist. Suppose $\tilde{X}$ is a resolution of singularities of X, then we could create another resolution of singularities $\tilde{X}'$ by blowing up another point. In this sense, resolution of singularities are not defined uniquely. To define a proper notion of resolution of singularities, below statement is useful.Birational morphisms are only a composition of blow up.

Theorem 3.10. *[7] Let $f : X' \to X$ is a birational morphism of surfaces. Let $n(f)$ be the number of irreducible curves $C_i' \subset X'$ such that $f(C_i')$ is a points. Then $n(f)$ is finite and f can factored into a $n(f)$ blow up.*

In this sense, the resolution of singularities is not unique. Nevertheless, we want to define a concise resolution of singularities. For this purpose, we introduce notion of the minimal resolution of the algebraic surface.

Definition 3.11. Minimal resolution of algebraic surface $\pi : \tilde{X} \to X$ is a resolution of singularities such that for any other resolution of singularities $\psi : \tilde{X}' \to X$ there is morphism $\phi : \tilde{X}' \to \tilde{X}$ such that $\psi = \phi \circ \pi$

$$\tilde{X}' \xhookrightarrow{\phi} \tilde{X}$$
$$\psi \searrow \quad \downarrow \pi$$
$$X$$

We know clear answer how to judge $f : S' \to S$ is minimal or not.

Theorem 3.12. *[7] (Castelnuovo's Criteria) If $E \subset X$ is a curve of surface such that $E \cong \mathbb{P}^1$ and $(E, E) = -1$ then E is an exceptional curve of blow up.*

Theorem 3.13. *Assume $dim X = 2$. A resolution $\pi : \tilde{X} \to X$ of a singularities $X_{sing} \in X$ is the minimal resolution if and only if $\pi^{-1}(x)$ does not contain -1 curves for any $x \in X$.*

Proof. It is clear that minimal resolution $\tilde{X}$ does not contain -1 curve. Conversely, let $f : \tilde{X} \to X$ be a resolution of singularities without -1 curves. Let $g : \tilde{X}' \to X$ be an arbitrary resolution of singularities. By taking irreducible component of $Y := \tilde{X} \times_X \tilde{X}'$, we obtain resolution of singularities $\tilde{f} : Y \to X$.

$\tilde{f}$ is defined as $f \circ \phi = \tilde{f} = g \circ \varphi$ for some $\phi : Y \to \tilde{X}$ and $\varphi : \tilde{X}' \to X$. By theorem 3.10 we can decompose birational morphism as $\varphi = \sigma_1 \circ \ldots \sigma_{n(\varphi)}$ with σ_i is a blow up of a point. We assume $\tilde{f}$ achieve minimal $n(\varphi)$. If $n(\varphi) > 0$ then let E be irreducible exceptional curve respect to $\sigma_{n(f)}$. Suppose we have $\phi(E) = C$ algebraic curve then $(C, C) = (\phi^*(C), \varphi^*(C))$ [7], we have $-1 = (E, E) \leq (C, C)$ and the equality holds if and only if ϕ is isomorphic on a neighborhood of C. Since E is exceptional curve of f, we have $(C, C) = (E, E)$ and ϕ to be an isomorphism. It contradict definition of $\tilde{X}$ so $\phi(E)$ is a point. Thus ϕ factors by $\sigma_{n(\varphi)}$ contradicts the minimality of $n(\varphi)$. Hence $n(\varphi) = 0$ and we have $\phi : Y = \tilde{X}' \to \tilde{X}$. $\qquad \square$

4   Singularities

We will see Kleinian singularities are also normal singularities. Based on this fact we can see how we can characterize minimal resolution of Kleinian singularities.

Definition 4.1. If all local ring of the algebraic variety X are normal ring, then the X is called normal variety. A point in the singular locus, $p \in X_{sing}$, is called a normal singularity.

Proposition 4.2. *A quotient singularity is a normal singularity*

Proof. The coordinate ring of $X = \mathbb{C}^n / G$ is $\mathbb{C}[z_1, \ldots, z_n]^G$. $\mathbb{C}[z_1, \ldots, z_n]$ is also integrally closed ring. Take $h(z) \in \mathbb{C}(z_1, \ldots, z_n)^G$ that is integral over $\mathbb{C}[z_1 \ldots z_n]^G$. Since $h(z)$ is integral over $\mathbb{C}[z_1 \ldots z_n]$, $h(z) \in \mathbb{C}[z_1 \ldots z_n]$. As $g \cdot h(z) = h(z)$, $h \in \mathbb{C}[z_1 \ldots z_n]^G$. The local rings of a normal ring are also normal rings, so a quotient singularity is a normal singularity. $\qquad \square$

For a smooth variety there is a locally free sheaf Ω corresponding to the cotangent bundle. The canonical sheaf, ω_X, is defined as the determinant of ω_X. A divisor K_X satisfying $\omega_X \cong \mathcal{O}(K_X)$ is called a canonical divisor of X. For a normal variety, X, K_X is defined by taking the closure of $K_{X \setminus X_{sing}}$. It is a fact that the singular locus of a normal variety has codimension at least 2, and this implies that there is only one canonical divisor K_X that restricts to a given canonical divisor $K_{X \setminus X_{sing}}$.

Definition 4.3. For a birational morphism $f : Y \to X$, the divisors D in Y such that $f(D)$ has codimension at least 2 in X are called exceptional divisors. We write these with the notation E_i.

Theorem 4.4. *For the minimal resolution of 2-dimensional normal singularities $f : Y \to X$, we have*

$$K_Y = f^*K_X + \sum_{i=1}^{i=r} m_i E_i \tag{13}$$

For $m_i \in \mathbb{Q}$ $m_i \leq 0$ Moreover, if X has a Kleinian singularities, we have

$$K_Y = f^*K_X \tag{14}$$

Definition 4.5. If we have $K_Y = f^*K_X$ for a resolution of singularities $f : Y \to X$ we call f a crepant resolution.

We will prove (13) in this section and We will show (14) after we got minimal resolution of Kleinian singularities.
In order to prove (13), we will see an auxiliary theorem of Castelnuovo's theorem, which provides a criterion for -1 curve.

Theorem 4.6. *A compact curve E on a non-singular surface Y is a -1 curve if and only if*

$$(K_Y, E) < 0, (E, E) < 0 \tag{15}$$

Proof. If E is a -1-curve, then it satisfies $-2 = deg K_E = (E, E) + (K_Y, E)$, therefore it is clear that $(K_Y, E) < 0, (E, E) < 0$.
We will show that E is -1 curve and genus of E is 0.By Nagata's compactification theorem, we can assume Y as open subvariety of compact variety. Thus without loss of generality, we may assume that Y is a compact non-singular integral variety. By the exact sequence

$$0 \to \mathscr{O}_Y(-E) \to \mathscr{O}_Y \to \mathscr{O}_E \to 0 \tag{16}$$

We have $\chi(E, \mathscr{O}_E) = \chi(Y, \mathscr{O}_Y) - \chi(Y, \mathscr{O}(-E))$. Here, by the Riemann-Roch Theorem for the surface, we have $\chi(E, \mathscr{O}_E) = -\frac{1}{2}((K_Y, E) + (E, E))$. As $h^0(E, \mathscr{O}_E) = 1$, it follows that $h^1(E, \mathscr{O}_E) = \frac{1}{2}((K_Y, E) + (E, E) + 1$. By the assumption, we obtain that $h^1(E, \mathscr{O}_E) = 0$ and $(K_Y, E) + (E, E) = -2$ so $(E, E) = (K_Y, E) = -1$. Also $g = h^1(E, \mathscr{O}_E) = 0$
In order to show that $E \cong \mathbb{P}^1$, we want to show E is non-singular. Take the normalization $\pi : \overline{E} \to E$, so that there is a exact sequence.

$$0 \to \mathscr{O}_E \to \pi_*\mathscr{O}_{\overline{E}} \to \pi_*\mathscr{O}_{\overline{E}}/\mathscr{O}_E \to 0 \tag{17}$$

$$0 \to \Gamma(E, \mathscr{O}_E) \to \Gamma(E, \pi_* \mathscr{O}_{\overline{E}}) \to \Gamma(E, \pi_* \mathscr{O}_{\overline{E}}/\mathscr{O}_E) \to H^1(E, \mathscr{O}_E) = 0 \quad (18)$$

Since the first and second terms are both isomorphic to $\mathscr{C}$, we obtain $\pi_* \mathscr{O}_{\overline{E}}/\mathscr{O}_E = 0$. Therefore π is isomorphism and we see E is nonsingular By summing up all the facts, we showed $E \cong \mathbb{P}^1$ $\qquad\square$

Theorem 4.7. *For a resolution of singularities $f : \tilde{X} \to X$ of a normal variety $x \in X_{sing}$, we have $f^{-1}(x) = \sum r_i E_i$ of exceptional divisor. Then the intersection matrix $((E_i, E_j))_{ij}$ is negative definite.*

Proof. Let h be a regular function on X such that $h(x) = 0$. Consider the principal divisor $f^*h = D + \sum_{i=1}^{r} m_i E_i$ with D does not contain E_i on $\tilde{X}$. f^*h vanishes on E_i, therefore $m_i > 0$. To prove that the matrix $((E_i, E_j))_{ij}$ is negative definite, it is sufficient to prove that $((m_i m_j E_i, E_j))_{ij}$ is negative definite. We can denote $e_{ij} := m_i m_j (E_i, E_j)$. As $((f^*(h)), E_j) = 0$ [7], it follows that $0 = (D, E_i) + (\sum m_j E_j, E_i)$. Here by $(D, E_j) \geq 0$ we have $m_j(\sum m_i E_i, E_j)$. Therefore $\sum_{i=1}^{r} e_{ij} \leq 0$ for all j. In particular if j satisfies $(D, E_j) > 0$ then $\sum_{i}^{r} e_{ij} < 0$

Now for every $x = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$, the quadratic form ${}^t x e_{ij} x = \sum_{i=i...r, j=1...r} x_i x_j e_{ij} =$

$\sum e_{jj} x j^2 + 2 \sum_{i<j} 2 e_{ij} x_i x_j = \sum_{j=1}^{r} (\sum_{i=1}^{r} e_{ij}) x_j^2 - \sum_{i<j} e_{ij} (x_i - x_j)^2 \leq 0$ because $e_{ij} \geq 0$ for $i \neq j$. If $= 0$ holds, then for k such that $D \cap E_k = \emptyset$; it follows that $\sum e_{ik} < 0$ which requires $x_k = 0$. For k' such that $k' \neq k/$ such that $E'_k \cap E_k \neq$ we have $e_{kk'} > 0$, therefore by $(x_k - x'_k)^2 = 0$ we obtain $x'_k = 0$. As the exceptional set by the Zariski Main Theorem [7] we can make $x_i = 0$ for all i, shows $((E_i, E_j))_{i,j}$ is negative definite. $\qquad\square$

Proof. (Theorem 4.4) Normal surface X and it's minimal resolution of singularities $f : \tilde{X} \to X$, we want to define pull back of a Weil divisor D with the following form, $\tilde{D} \subset \tilde{X}$ as strict transformation of D

$$f^* D = \tilde{D} + \sum_{i=1}^{r} m_i E_i, m_i \in \mathbb{Q} \qquad (19)$$

This make sense because $f : \tilde{X} \setminus f^{-1}(X_{sing}) \cong X \setminus X_{sing}$ so that only difference between f^*D and $\tilde{D}$ should be exceptional curves. To finish defining the pull back of D, we need to find the coefficients m_i

If the pull back were defined, then by (Chapter 5.3.2 [7]) we would have a system of equations.

$$(f^*D, E_j) = 0 \Longleftrightarrow \sum_{i=1}^{r} m_i(E_i, E_j) = (D, E_j) \tag{20}$$

Since theorem 4.7 (E_i, E_j) is an invertible matrix, so we use this equation to define $m_i \in \mathbb{Q}$

We will show all m_i is negative. Suppose there is some $m_i > 0$, then by permuting m_i so that $m_i > 0 (i = 1 \ldots s)$ and $m_i \leq 0 (i = s + 1 \ldots r)$. As matrix $(E_i, E_j)_{ij}$ we have $(\sum_{i=1}^{s} m_i E_i)^2 < 0$. Then there exist $j (1 \leq j \leq s)$ such that $(\sum_{i=1}^{s} m_i E_i)(E_j)) < 0$. For this j we have

$$(K_Y, E_j) = (f^*K_X, E_j) + (\sum_{i=1}^{s} m_i E_i, E_j) + (\sum_{i=s+1}^{r} m_i E_i . E_j) < 0 \tag{21}$$

By theorem 4.6 this is -1 curve, contradict minimality. $\qquad \square$

5 Cyclic Groups

In this section, we will compute resolution of singularities on cyclic group.

Definition 5.1. For a two-dimensional normal variety X and its resolution of singularities Y and irreducible exceptional curves E_i, we can construct a corresponding graph. We will call it the dual graph

If there are n exceptional curves, then there are corresponding n vertices and each vertex is assigned a natural number $\leq n$.. If E_i and E_j are intersecting, then we a draw line between vertex i and j. If the self-intersection number of E_i is -2, then we don't write anything on that vertex i, but if it is not $b_i = -2$, then we write the self-intersection number on the vertex i.

We can understand the configuration of exceptional curves of the minimal resolution of Kleinian singularities by a dual graph.

Construction 5.2. *Case of Cyclic group*
We saw C_n invariant equation is $x^n + y^2 + z^2 = 0$ when we got invariant function. By change of coordinate, the variety defined by this equation is isomorphic to the variety defined by $x^n - yz = 0$. We will denote the variety defined by $x^n - yz = 0$ to be X

Let's define the algebraic surface $\tilde{X}_n \subset \mathbb{C}^3 \times (\mathbb{P}^1)^{n-1}$ with the following equations.

$$a_{n-1}x = b_{n-1}y \tag{22}$$

$$a_i b_{i+1} x = a_{i+1} b_i (1 \leq i \leq n-2) \tag{23}$$

$$a_1 z = b_1 x \tag{24}$$

Theorem 5.3. *Let π be the 1st coordinate projection map $\pi : \mathbb{C}^3 \times (\mathbb{P}^1)^n \to \mathbb{C}^3$. Then the restriction of π to $\tilde{X}_n$, $\phi := \pi|_{\tilde{X}_n} : \tilde{X}_n \to X_n$ is the minimal resolution of singularities of X_n. And $\phi^{-1}(0) = E$ is the union of $n-1$ projective lines. The exceptional curves E_i and E_j intersect transversely. The dual graph of this resolution of singularities is*

To show the above theorem we will prove 3 statements separately. (1) $\tilde{X}_n$ is a non-singular variety
(2)$\tilde{X}_n \setminus \pi^{-1}(0) \cong X_n \setminus 0$ and show properness
(3) Show dual graph is A_n type

Proof. (1)Take point of $p \in \tilde{X}_n$ to be

$$p = ((x, y, z)(a_1 : b_1) \ldots (a_{n-1}, b_{n-1})) \tag{25}$$

if $b_i \neq 0$ and $b_{i-1} = 0$ then $a_{i-1} \neq 0$ and $b_1 = \cdots = b_{i-2} = 0$. Hence we can take open set $W_i (i \leq i \leq n)$ of $\tilde{X}_n$ to be

$$W_i = \{((x, y, z), (a_1, b_1), \ldots, (a_{n-1}, b_{n-1}) \in \tilde{X} | a_1 \cdots a_{i-1} \neq 0, b_i \cdots b_{n-1} \neq 0\} \tag{26}$$

Clearly $\tilde{X} = \bigcup_{i=1}^n W_i$
Let's $u_i = \frac{a_i}{b_i} (b_i \neq 0)$ and $v_i = \frac{b_i}{a_i} (a_i \neq 0)$ then on each W_i, $\tilde{X}_n$ is defined by

$$u_{n-1}x = y \tag{27}$$

$$u_k x = u_{k+1} (k \geq i) \tag{28}$$

$$x = v_{i-1} u_i \tag{29}$$

$$v_{k+1} x = v_k (k \leq i-2) \tag{30}$$

$$v_1 x = z \tag{31}$$

Notice when $2 \leq i \leq n-1$ once we determine u_i and v_{i-1} then all other value are defined inductively. So it is $W_i \cong \mathbb{C}^2$ For W_i we can take coordinate to be (u_i, v_{i-1}). In the case of W_1 and W_n by applying same argument by taking it's coordinate to be (u_1, z) and (v_{n-1}, y) So we find $\tilde{X}$ nonsingular. $\square$

Proof. (2)
Since we are thinking on $\mathbb{C}$, being proper is same as being proper in analytic topology. Projection is proper map.
Prove if $(x, y, z) \neq (0, 0, 0)$ then $\phi^{-1}(0, 0, 0)$ is one point. If $x \neq 0$ then $y \neq 0$ and $z \neq 0$ by definition. We can see $a_1 b_1 a_{n-1} b_{n-1} \neq 0$. In addition to that, by induction $a_i b_i \neq 0 (1 \leq i \leq n-1)$ which tells that inverse image define uniquely from (x, y, z). If $x = 0$ then either $y = 0$ or $z = 0$ if $y = 0$ then from given equation, $a_1 = 0$ and $b_1 \neq 0$ so inductively $a_2 = \cdots = a_{n-1} = 0$ so $\pi^{-1}((0, 0, z)) = ((0, 0, z), (0, 1) \ldots (0, 1))$ inverse image is one point. $\square$

Proof. (3)
We want to see the behavior of exceptional sets. When $x = y = z = 0$ we have $a_2 b_1 = a_3 b_2 = \cdots = a_{n-1} b_{n-2} = 0$. Let define m be a smallest i such that $a_i = 0$. Since $a_2 a_3 \ldots a_{m-1} \neq 0$, $b_1 \ldots b_{m-2} = 0$. Also since $b_m \neq 0$ so $a_{m+1} = \ldots a_{n-1} = 0$. Now we have coordinate of E_{m-1} to be

$$E_{m-1} = ((0, 0, 0)(1 : 0), \ldots, (a_{m-1} : b_{m-1})(0 : 1) \ldots (0 : 1)) \subset \tilde{X} \qquad (32)$$

Then we can see $E = \cup_{i=1}^{n} E_i$ and also $E_i \cong \mathbb{P}^1$ and $E_i \cap E_j$ is one point if $|i - j| = 1$ and no intersection otherwise.
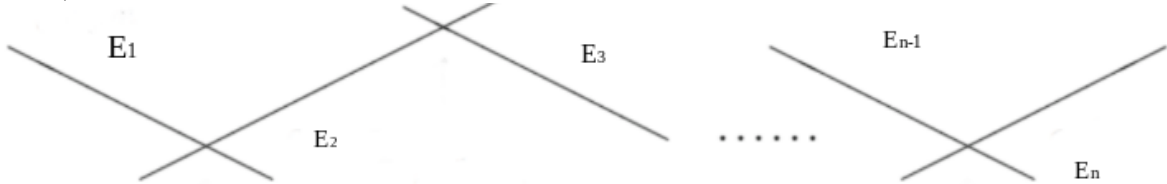
Next we calculate self-intersection number of E_i. When $2 \leq i \leq n-2$, E_i is defined to be v_{i-1} on W_i and u_{i+1} on W_{i+1} as $E_i \subset W_i \cup W_{i+1}$. Transition function is $g_{i,i+1} = \frac{v_{i-1}}{u_{i+1}}$ but again inductively we have $u_{i+1} = u_i x = u_i(v_{i-1} u_i) = v_{i-1} u_i^2$. This leads to $g_{i,i+1} = \frac{1}{u^2}$ so from same argument calculating self intersection number on section 3, we have $(E_i, E_i) = -2$. We can calculate in the same way when $i = 1$ and $i = n$. Since E does not contain -1 curve, this is minimal resolution. $\square$

We calculated minimal resolution of cyclic quotient singularities of cyclic

group generated by $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. However, general generator of cyclic subgroup of $SL(2, \mathbb{C})$ can be written as $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^q \end{pmatrix}$ with ζ is nth root of 1. We can construct non-smooth varieties with in a similar way. $\mathbb{C}^2/G$ can written as $X_{n,q}$. There is remarkable connection between continued fraction and dual graph.

Theorem 5.4. *Let*

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{\ddots}{b_{r-1} - \cfrac{1}{b_r}}}}$$

be a continuous fraction of $\frac{n}{q}$(called e Hirzebruch-Jung continued fraction with finite integer $b_i \geq 2$ then dual graph of minimal resolution of singularities of $X_{n,q}$ are described by the following dual graph
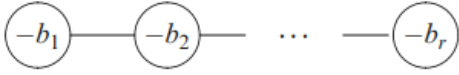
Lemma 5.5. Hirzebruch-Jung continued fraction can be written as uniquely with $(b_1 \ldots b_r)$

Proof. This similar proof to usual uniqueness of continued fraction. □

Lemma 5.6. The affine coordinate ring of $X_{n,q}$ is $\mathbb{C}[u^i v^j]$ for pair of $0 \leq i \leq n$ and $0 \leq j \leq n$ with $i + qj \equiv 0 (mod\ n)$

Proof. By the Noether's bound of degree on generating set, degree of generators $\mathbb{C}[u,v]^G$ are at most $|G|$. We can obtain all of the generators by the Reynolds operator. $R(f(u,v)) = \frac{1}{|G|} \sum_{g \in G} g \cdot f(u,v)$ then $R(u^i v^j) = \frac{1}{n} \sum_{a=1}^{a=n} \zeta^{a(i+qj)} u^i v^j = \frac{1}{m} \frac{1-(\zeta^{i+qj})^n}{1-\zeta^{i+jq}} u^i v^j$. Since $\zeta^n = 1$ sum This is vanish whenever denominator is not zero. Thus we shows that generator of $\mathbb{C}[u,v]^G$ is $u^i v^j$ with $i + qj \equiv 0$ □

Proof. (Proof of theorem 5.4) Let $Y_{n,q}$ is an affine variety of the coordinate ring $B_{n,q} = \mathbb{C}[u^n, u^{n-q}v, v^n]$. Since $\mathbb{C}[u,v]^G$ is integrally closed over $B_{n,q}$ and

since $\mathbb{C}[u,v]^G$ is normal, $X_{n,q}$ is a normalization of $\mu : X_{n,q} \to Y_{n,q}$. By the theory of normalization, if $f : M \to Y_{n,q}$ is surjective, then there exist unique $g : M \to X_{n,q}$ such that $f = \mu \circ g$.

Let $b_1, \ldots, b_r$ from Hirzebruch-Jung continued fraction and then non-singular variety $M(b_1 \ldots b_r)$ by patching $U_i \cong \mathbb{C}^2 (i = 0 \ldots r)$ as follows where we let (u_i, v_i) be the coordinate system of U_i

$$U_{2i} \cap U_{2i+1} = u_{2i} \neq 0, u_{2i+1} = \frac{1}{u_{2i}}, v_{2i+1} = u_{2i}^{b_{2i+1}} v_{2i} \tag{33}$$

$$U_{2i+1} \cap U_{2i+2} = v_{2i+1} \neq 0, v_{2i+2} = \frac{1}{v_{2i+1}}, u_{2i+2} = v_{2i+1}^{b_{2i+2}} u_{2i+1} \tag{34}$$

Then, the closed subset $E = \{v_0 = v_1 = 0\} \cup \{u_1 = u_2 = 0\} \cup \{v_2 = v_3 = 0\} \ldots$ is a complete closed subvariety isomorphic to $\mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1$. By looking at the images of an open subset $(U_0 \cap U_1) \setminus E = \{u_0 v_0 \neq 0\}$ by the isomorphisms of patching, we obtain that if r is odd, then $M = \{u_0 v_0 \neq 0\} \cup E \cup \{u_0 = 0\} \cup \{v_r = 0\}$. On the other hand, if r is even, we will replace v_r to u_r. For that reason we can assume r is odd.

Now we can see that $u_0 v_0, v_0, v_r$ are regular functions on M. Indeed it is sufficient to show that these are written as $u_i^a v_i^b (a, b \geq 0)$ on each U_i . Let us try on v_r. $v_r = u_{r-1}^{b_r} v_{r-1} = u_{r-2}^{b_r} v_{r-2}^{b_r b_{r-1}-1} = u_{r-3}^{b_{r-2}(b_r b_{r-1}-1)-b_r} v_{r-3}^{b_r b_{r-1}-1} \ldots$ Notice ratio of index of u_i and v_i is converging to Hirzebruch-Jung continued fraction so if we write $v_r = u_0^a v_0^b$ then $a = n$ and $b = q$ from continued fraction. Thus we obtain regular functions $v_0, u_0 v_0, u_0^n v_0^q$ on M and we can define $\Phi : M \to \mathbb{C}^3$ by $(u_i, v_i) \to (v_0, u_0, u_0^n v_0^q)$. Then $Im\Phi = Y_{n,q}$. Obviously Φ gives an isomorphism between $\{u_0 v_0 \neq 0\}$ and $\{(x_1, x_2, x_3) \in Y_{n,q} | x_1 x_2 x_3 \neq 0\}$. As g is isomorphic outside of singular point, M is a resolution of singularities of $X_{n,q}$

We can check weight of dual graph by checking transition function argument in the other case. This is minimal resolution by assumption of $b_i \geq 2$. There is no -1 curve.

$\square$

6   Non-Cyclic Groups

Construction 6.1. *Non-Cyclic Case*[15]
    Segre-Hirzebruch surface[3] of degree n, $\Sigma_n \subset \mathbb{P}^2 \times \mathbb{P}^1$ is defined by

$$\Sigma_n = \{(\zeta_0 : \zeta_1 : \zeta_2), (s : t) | t^n \zeta_0 = s^n \zeta_1\} \tag{35}$$

Σ_n is $\mathbb{P}^1$ bundle by projection to the 2nd coordinate $p : \Sigma_n \to \mathbb{P}^1$. We can take 4 local coordinates for Σ_n.

$$U_1 : (x_1, y_1) = (\frac{t}{s}, \frac{\zeta_0}{\zeta_2}) \qquad U_2 : (x_2, y_2) = (\frac{s}{t}, \frac{\zeta_1}{\zeta_2})$$

$$U_3 := (\frac{t}{s}, \frac{\zeta_2}{\zeta_0}) \qquad U_4 := (\frac{s}{t}, \frac{\zeta_2}{\zeta_1})$$

We have the following transition property

$$x_1 = x_3 = \frac{1}{x_2} = \frac{1}{x_4} \quad \text{and} \quad y_1 = \frac{1}{y_3} = x_2^n y_2 = \frac{x_4^n}{y_4}$$

$$y_2 = \frac{1}{y_4} \quad \text{and} \quad y_3 = \frac{1}{x_2^n y_2} = \frac{y_4}{x_4^n}$$

Namely Σ_n is covered by 4 open set,isomorphic to $\mathbb{C}^2$ patched by above relation. We can regard Σ_n to be $\mathbb{P}^1$ bundle over $\mathbb{P}$ by p. The section defined by $\zeta_2 = 0$ to be S_∞. The complement of S_∞ to be $\Sigma_n^* := \Sigma_n - S_\infty$. Σ_n^* are covered by U_1 and U_2, the change of coordinate between fiber is given by $y_1 = x_2^n y_2$.

We can see that many algebraic surface can be construct by Segre-Hirzebruch surface

Example 6.2. Σ_1^* is isomorphic to the blow-up of $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ with the origin. Similarly, if blow up one point on the $\mathbb{P}^2$ then it is isomorphic to Σ_1

Example 6.3. For $X = \mathbb{C}^2/C_n$, irreducible component of exceptional sets $E_i(2 \leq i - 2)$ were covered by 2 open cover W_i and W_{i+1} isomorphic to $\mathbb{C}^2$. The coordinate of W_i of (u_i, v_{i-1}), and W_{i+1} of (u_{i+1}, v_i) were patched by $u_i = \frac{1}{v_i}$ and $v_{i-1} = v_i^2 u_{i+1}$ this is same as change of coordinate of Σ_n^* so $W_i \cup W_{i+1} \cong \Sigma_2^*$.

Example 6.4. Cotangent bundle of $\mathbb{P}^1$ is $\Sigma_2^* \cong T^*\mathbb{P}^1$. This is confirmed in the following way. Let the homogeneous coordinates of $\mathbb{P}^1$ to be $(s : t)$ then when $s \neq 0$ each fiber of $T^*\mathbb{P}^1$ is the 1-dimensional vector space $\{\zeta_1 d(\frac{t}{s}) | \zeta_1 \in \mathbb{C}\}$ when $s \neq 0$ and when $t \neq 0$ $\{\zeta_0 d(\frac{s}{t}) | \zeta_0 \in \mathbb{C}\}$ and also $d(\frac{t}{s}) = -\frac{t^2}{s^2} d(\frac{s}{t})$. This means change of the coordinate between fiber can be described as $\zeta_0 = -(\frac{s}{t})^2 \zeta_1$. This shows above isomorphism

For finding resolution of singularities of non-cyclic Kleinian singularities, we want to we want to approximate by the cyclic quotient singularities. Denote $\tilde{\Gamma}$ as correspond finite subgroup of $SL(2,\mathbb{C})$, we blow up $\mathbb{C}^2$ at the origin and lift the action of $\tilde{\Gamma}$ to the action of $\tilde{\mathbb{C}}^2$. We saw $\tilde{\mathbb{C}}^2 \cong \Sigma_1^*$ on examples. $\tilde{\Gamma}$ acts the first coordinate of $\tilde{\mathbb{C}}^2$ as an element of $SU(2)$

$$\begin{array}{ccc} \tilde{\mathbb{C}}^2 & \longrightarrow & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \tilde{\mathbb{C}}^2/\tilde{\Gamma} & \longrightarrow & \mathbb{C}^2/\tilde{\Gamma} \end{array}$$

$\mathbb{C}^2/\tilde{\Gamma}$ has the singularity at the origin. $\tilde{\mathbb{C}}^2/\tilde{\Gamma}$ does not have same type of singularities. We will show that $\tilde{\mathbb{C}}^2/\tilde{\Gamma}$ has 3 cyclic group singularities, the orders of the cyclic groups are the orders of the stabilizers of the orbits of the action on $\mathbb{C}^2 \cup \{\infty\}$ by Γ. Since in 2-dimensions, singularities have a unique minimal resolution, it is enough to confirm that resolution of singularities of $\tilde{\mathbb{C}}^2/\tilde{\Gamma}$ does not contain -1 curve.

Proposition 6.5. *(1) Let the center of $\tilde{\Gamma}$ be Z. The orbit space of $\tilde{\mathbb{C}}^2$ by Z is isomorphic to Σ_2^*, so the action on $\tilde{\mathbb{C}}^2$ induces an action on Σ_2^**

$$\phi : \tilde{\mathbb{C}}^2 \to \tilde{\mathbb{C}}^2/Z \cong \Sigma_2^* \tag{36}$$

(2) $\tilde{\Gamma}$ act of exceptional curve E of $\tilde{\mathbb{C}}^2$ and this orbit space is isomorphic to $\mathbb{P}^1$. $E/\tilde{\Gamma} \cong \overline{E}/\Gamma \cong \mathbb{P}^1, \phi(E) = \overline{E}$

$$\begin{array}{ccccc} \tilde{\mathbb{C}}^2 = \Sigma_1^* & \xrightarrow{\phi} & \Sigma_2^* & \xrightarrow{\pi} & \Sigma_2^*/\Gamma \cong \Sigma_1^*/\tilde{\Gamma} \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \overline{p}} \\ \mathbb{P}^1 & \xrightarrow{/Z} & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1/\Gamma \cong \mathbb{P}^1 \end{array}$$

(3) Regard $\phi(E) = \overline{E}$ as the Riemann sphere. Consider the regular polyhedron corresponding to Γ which is inscribed on the sphere. Also consider the projection map from the center of the sphere to the surface of the sphere. Then let S_1 be the projection of the set of barycenters of the faces of the polyhedron, S_2 be the set of vertices, and S_3 be the projection of the set of midpoints of the edges. These sets S_1, S_2, and S_3 are orbits of Γ acting on

$\overline{E}$. We let $\pi(S_i) = P_i$ then Γ acts on $\overline{E} \setminus (S_1 \cup S_2 \cup S_3)$ *freely and there is an isomorphism*

$$\Sigma_2^*/\Gamma \setminus p^{-1}(\{P_1, P_2, P_3\}) \cong T^*(\mathbb{P}^1 \setminus \{P_1, P_2, P_3\}) \cong \Sigma_2^* \setminus p^{-1}(\{P_1, P_2, P_3\}) \quad (37)$$

Proof. (1) The center of $\tilde{\Gamma}$ is of order 2 and generated by $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and the action is $\tilde{\mathbb{C}}^2 \ni ((x_1, x_2), (s : t)) \to ((x_1^2, x_2^2), (s, t))$. Thus the orbit space is given by $\tilde{\mathbb{C}}^2/ <\sigma> = \{((y_1, y_2), (s : t)) \in \mathbb{C}^2 \times \mathbb{P}^1 | s^2 y_2 = t^2 y_1\} \cong \Sigma_2^*$

(2) This is from Chevalley's theorem

(3) The stabilizer $\tilde{\Gamma}_p$ of the action of $\tilde{\Gamma}$ on $p \in \mathbb{C}^2$ is trivial unless $p = (0,0)$. This is because chose coordinate to make $p = (1,0)$ and stabilizer to be $A = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \in \tilde{\Gamma}_p$. Since $\det A = 1$, $b = 1$ and $\tilde{\Gamma}$ is finite group, so there is n such that $A^n = \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix}$ is unit matrix. This means $a = 0$. Hence $\tilde{\Gamma}_p$ is trivial.

That means if there is non-trivial stabilizer of $\tilde{\Gamma}$ on $\tilde{\mathbb{C}}^2$ then it is on exceptional curve E. Means that for the stabilizer on $\Sigma_2^* = \tilde{\mathbb{C}}^2/Z$ is on $\overline{E}$. If we regard $\overline{E}$ as Riemann sphere, then only points with nontrivial stabilizer are $p \in S_1 \cup S_2 \cup S_3$ so Γ acts freely on $\overline{E} - S_1 \cup S_2 \cup S_3$. So map of orbit space by the action of Γ

$$\pi : \overline{E} \setminus S_1 \cup S_2 \cup S_2 \to \overline{E}/\Gamma \setminus \{P_1, P_2, P_3\} \quad (38)$$

is local isomorphism. So by the local isomorphism of cotangent bundle was induced as

$$\pi_* : T^*(\overline{E} \setminus S_1 \cup S_2 \cup S_3) \to T^*(\overline{E}/\Gamma \setminus \{P_1, P_2, P_3\}) \quad (39)$$

On the other hand we see Σ_2^* is isomorphic to the cotangent bundle $T^*\mathbb{P}^1$ and this π_* corresponds to the quotient by the action by Γ on $\Sigma_2^* \setminus p^{-1}(S_1 \cup S_2 \cup S_3)$, which gives the isomorphism of (37) $\qquad \square$

Proposition 6.6. $P_1, P_2, P_3 \in \overline{E}/\Gamma \cong \mathbb{P}^1$ *are all of the singularities of* $\tilde{\mathbb{C}}^2/\tilde{\Gamma} = \Sigma_2^*/\Gamma$. *They are the same type of singularities as* $\mathbb{C}^2/C_p, \mathbb{C}^2/C_q, \mathbb{C}^2/C_r$, *cyclic Kleinian singularities. The numbers* p, q, r *are the orders of the stabilizers of the action of* Γ *corresponding to the orbits* S_1, S_2, *and* S_3.

Proof. Elements of Γ are rotations around an axis passing through the center of an edge, a vertex, or the barycenter of a face. These order are same as order of stabilizer p, q, r of Γ. Consider an intersection of axis and sphere $P \in S_1 \cup S_2 \cup S_3$ and their stabilizer Γ_P. Also $\tilde{\Gamma}_P$ to be the preimage in $SU(2)$. We have $\Gamma_P \cong C_n, \tilde{\Gamma}_P \cong C_{2n}$ with $n = p, q, r$. The action of $\tilde{\Gamma}_P$ on $\tilde{\mathbb{C}}^2$ is defined by $((x_1, x_2)(s : t)) \to ((\mu x_1, \mu^{-1} x_2), (\mu s : \mu^{-1} t))$ with $\mu = exp(\frac{2\pi i}{2n})$ so the action on Σ_2^* is $((y_1, y_2)(s : t)) \to ((\mu^2 y_1, \mu^{-2} y_2), (\mu s : \mu^{-1} t))$. The action on the open sets U_1 and U_2 is

$$(\frac{t}{s}, y_1) \to (\zeta^{-1}\frac{t}{s}, \zeta y_1) \tag{40}$$

$$(\frac{s}{t}, y_2) \to (\zeta\frac{s}{t}, \zeta^{-1} y_2), \zeta = \mu^2 \tag{41}$$

Choose the coordinate chart U_2, and generator $\Gamma_P = < \gamma >$ so that the action is $\gamma(x_2, y_2) = (\zeta x_2, \zeta^{-1} y_2)$. This is nothing but the action of C_n on $\mathbb{C}^2$. This means that the orbit space $\tilde{\mathbb{C}}^2/\tilde{\Gamma} \cong \Sigma_2^*/\Gamma$ has a singularities of the same type of $\mathbb{C}^2/C_p$, $\mathbb{C}^2/C_q$, $\mathbb{C}^2/C_r$ on the $\overline{E}/\Gamma \cong \mathbb{P}^1$ $\square$

We need to define a resolution of the singularities $P_i(i = 1, 2, 3)$ on $\overline{E}$. Let $F_n : \mathbb{C}^2 \to X_n = \mathbb{C}^2/C_n$ be the quotient of the cyclic group of order n.

Proposition 6.7. Σ_2^*/C_n *is isomorphic to* $((\Sigma_2^* \setminus p^{-1}(P_i)) \amalg X_n)/ \sim$ *with*

$$\Sigma_2^* \setminus p^{-1}(P_i) \ni (x_2, y_2) \sim F_n(z_1, z_2) \in X_n \iff x_2 = z_1^n, y_2 = \frac{1}{n}z_1^{1-n}z_2 \tag{42}$$

Proof. The action of Γ is induced by $\Sigma_2^* \cong T^*\mathbb{P}^1$. $\pi : \mathbb{P}^1 \to \mathbb{P}^1/C_n \cong \mathbb{P}^1$ was given by $z \to z^n$. $d(z_1^n) = nz^{n-1}dz_1$. Dividing by nz^{n-1}, we get the desired relation. $\square$

Proposition 6.8. *Let $\tilde{S}$ be a surface patching resolution of singularity $\tilde{X}_n$ and Σ_2^* in the following way*

$$\Sigma_2^* \ni (x_2, y_2) \sim (z, u_1) \in W_1 \iff x_2 = z, y_2 = u_1 \tag{43}$$

The W_1 and z, u_1 is from proof of 5.3. Then define a map $\tilde{\phi} : \tilde{S} \to S$ to be

$$\tilde{\phi}(x_2, y_2) = (x_2, y_2), (x_2, y_2) \in \Sigma_2^* \setminus p^{-1}(P_i) \tag{44}$$

$$\tilde{\phi}(z, u_1) = \phi(z, u_1), (z, u_1) \in W_1 \tag{45}$$

Then this is a resolution of singularities of S. The dual graph is same as theorem 5.3

Proof. We can confirm that $\tilde{\phi}$ coincides on $\Sigma_2^*\backslash p^{-1}(P_i)\cap W_1$ with the following way.

$$\phi(z, u_1) = (x, y, z) = (zu_1, z^{n-1}u_1^n, z) \tag{46}$$

$F_n(z_1, z_2) = (x, y, z)$ with $x = z_1 z_2$ and $y = z_2^n$, $z = z_1^n$ and so $z = z_1^n$ and $u = z_1^{1-n}z_2$. This make $\tilde{\phi}$ is well defined and since ϕ defines resolution of singularity of X_n so $\tilde{\phi}$ defines the resolution of singularity $\qquad\square$

By proposition 6.7 and 6.8 we can define a resolution of singularities by patching Σ_2^* and $\tilde{X}_p$, $\tilde{X}_q$ and $\tilde{X}_r$ so we can find the minimal resolution $\tilde{X}$ of $\tilde{\mathbb{C}}^2/\tilde{\Gamma}$. All vertices of the dual graph will connect to two edges except for a one vertex which connects to three. If you remove that vertex, the graph will split into three parts. Each part has length $p-1, q-1, r-1$.
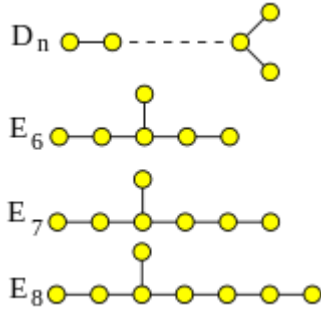
As this resolution of singularities will not contain -1 curve, this is a minimal resolution. Also, exceptional curves are -2 curves, so we can formulate Kleinian singularities with the following characterization.

Proposition 6.9. *For 2-dimensional normal singularities, the following statements are equivalent.*
1. Dual graph is ADE Dynkin diagram with no weight
2. Every exceptional prime divisor E_i satisfies $(E_i, E_i) = -2$ and $E_i \cong \mathbb{P}^1$
3. Hypersurface equation is one of the table on section 2

Proof of Proposition 6.9 is provided on [10]. Proposition 6.9 and proposition 6.10 will provide enough foundation to prove (14).

Proposition 6.10. *For a prime divisor of $E \subset X$ of nonsingular surface X, we have*

$$\frac{(K_X, E) + (E, E)}{2} + 1 \geq 0 \tag{47}$$

$\frac{(K_X,E)+(E,E)}{2}$ *is an integer, and equality hold when $E_i \cong \mathbb{P}^1$*

Proof. This is a consequence of Riemann-Roch theorem, see detail in [10] □

Proof. of (14) By the above proposition proposition , the number $(E_i, E_i) + (K_{\tilde{X}}, E_i) = -2$. For a minimal resolution of Kleinian singularities, we know that $(E_i, E_i) = -2$ so $K_{\tilde{X}}$ for resolution of Kleinian singularities we have $(K_{\tilde{X}}, E_i) = 0$. This means that $K_{\tilde{X}} = f^* K_X$

□

This shows the minimal resolution of Kleinian singularities is crepant. Moreover, according to [17] $K_{\tilde{X}} = 0$ for quotient singularity of $\mathbb{C}^n/G$

7 McKay Correspondence

We saw dual graph of quotient variety of $\tilde{\Gamma}$. Amazingly these graphs appear ubiquitously in mathematics. This type of graph is called a Dynkin diagram and they are usually constructed by a Cartan matrix.

Definition 7.1. A Cartan matrix C is a symmetric $\mathbb{Z}$ coefficent matrix satisfying the following properties
(a) $(a_{ii}) = 2$
(b) $(a_{ij}) \leq 0$ if $i \neq j$
(c) $(a_{ij}) = 0$ then $a_{ji} = 0$
(d) $C = DS$ with diagonal matrix D and symmetric matrix S

We can construct a Dynkin diagram from a $n \times n$ Cartan matrix in the following way. There are n vertices and each vertices are assigned natural number $\leq n$. If $|a_{ij}| = 0$ then there is no edges between i and j. If $|a_{ij}| = n \neq 0$ then we can draw n edges between i and j

Example 7.2. There is a Cartan matrix for finite Coxeter group. We can classify finite Coxeter group with Dynkin diagrams.
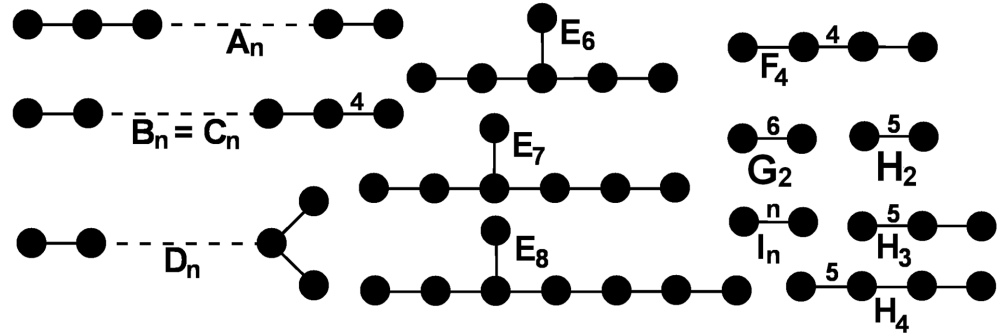
In 1978,John McKay found that Dynkin Diagrams from the irreducible representation of binary polyhedral group. Let $\rho_N : \tilde{\Gamma} \to SL(2,\mathbb{C})$ be trivial representation. Then for $\rho_1...\rho_n$ to be distinct irreducible representation of binary polyhedral group on finite dimensional complex vector space then we have

$$\rho_N \otimes \rho_i = \sum_{k=0}^{n} a_{jk}\rho_k \qquad (48)$$

We can define matrix $A(G) := (a_{ij})_{ij}$ from (48). McKay discovered $2I - A(G)$ is a Cartan matrix. He constructed a Dynkin diagram from the $2I - A(G)$. Graph from C_n looks like $\tilde{A}_n$, $\tilde{D}_n$ looks like $\tilde{D}_n$, $\tilde{T}$ looks to be $\tilde{E}_6$, $\tilde{O}$ looks like $\tilde{E}_7$ and $\tilde{I}$ looks like $\tilde{E}_8$.
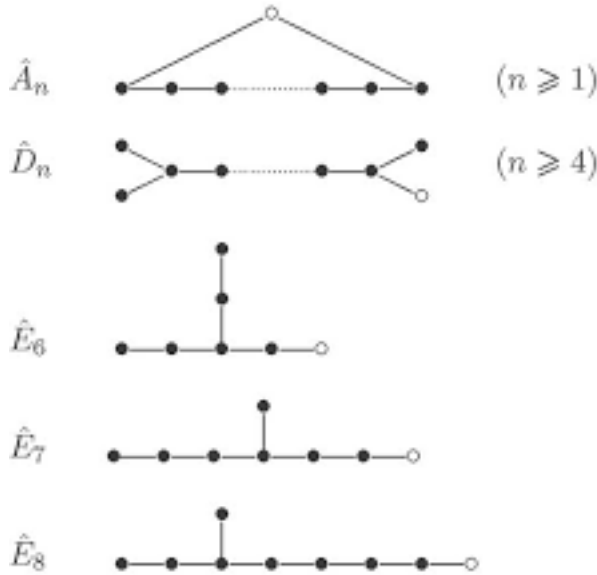
Curious similarity between McKay's Dynkin diagrams and dual graphs allow us to develop mathematics that intersects representation theory and algebraic geometry. Reid proposed the following conjecture based on McKay's above observation.

Conjecture 7.3. *Geometric McKay correspondence[16]*
Let $G < SL(n,\mathbb{C})$ is a finite subgroup. Assume that the quotient $X = \mathbb{C}^n/G$ with $K_X = 0$ has a crepant resolution $f : \tilde{X} \to X$. Then there exist natural bijections
(1){Irreducible representation G} $\Longleftrightarrow$ basis of $H^*(\tilde{X}, \mathbb{Z})$
(2){conjugacy classes of G} $\Longleftrightarrow$ basis of $H_*(\tilde{X}, \mathbb{Z})$

As we saw in the previous section, Kleinian singularities have crepant

resolution. We can confirm above conjecture is hold in two dimension.

Theorem 7.4. *For a minimal resolution of Kleinian singularities, the $\mathbb{Z}$-singular Homology is defined by*

$$H_0(\tilde{X}, \mathbb{Z}) = \mathbb{Z} \tag{49}$$

$$H_i(\tilde{X}, \mathbb{Z}) = 0 (i \neq 0, 2) \tag{50}$$

$$H_2(\tilde{X}, \mathbb{Z}) = \mathbb{Z}^n \tag{51}$$

with n to be number of irreducible component of excecptional sets $E = E_1 \cup \cdots \cup E_n$.

Proof. $\tilde{X}$ is path connected so $H_0(\tilde{X}, \mathbb{Z}) = \mathbb{Z}$.
We proved that $\tilde{X}$ is patched by Σ_2^*. We can write $\tilde{X} = \cup_{i=1}^n U_i$ with $U_i \cong \Sigma_2^*$. We will prove (50)(51) by induction. Let $\tilde{X} = Y \cup U_n$ with $Y = \cup_{i=1}^{n-1} U_i$, when $n = 1$ (50)(51) is true because zero section of Σ_2^* is $E_n \cong \mathbb{P}^1$. So $\mathbb{P}^1$ is deformation retract of U_1. Suppose statement is true for Y, then since $Y \cap U_n$ so $H_i(Y \cap U_n, Z) = \mathbb{Z}$ for $i = 0$ and otherwise $H_i(Y \cap U_n, Z) = 0$. By mathematical induction we have $H_0(Y, \mathbb{Z}) \oplus H_0(U_n, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ so as $H_2(Y, \mathbb{Z}) \oplus H_2(U_n, \mathbb{Z}) = \mathbb{Z}^{n-1} \oplus \mathbb{Z}$ and $H_i(Y, \mathbb{Z}) \oplus H_i(U_n, \mathbb{Z}) = 0$ otherwise. Mayer-Vietoris sequence shows that given statement. $\qquad \square$

The above theorem tells that $H_2(\tilde{X}, \mathbb{Z})$ is generated homology class of exceptional curve E_i. Exceptional curves correspond one to one on the vertices of the dual graph. On the other hand, by regarding a dual graph as a McKay graph, a number of vertices correspond to a number of an irreducible representation of $\tilde{\Gamma}$. By the theory of representation of the finite group, this corresponds to a number of the conjugacy class. Thus the similarity between the dual graph and McKay graph could naturally suggest Conjecture 7.3.

Accordinig to [12] this McKay correspondence is a special case of so-called Vafa's conjecture in string theory. Vafa's conjecture can calculate the topological invariant of a quotient of Kähler manifold by a finite group.

In this thesis, we have been exclusively discussing Kleinian singularities, but we want to conclude by exploring the properties of 3-dimensional quotient singularities. The classification of finite group $G < SL(3, \mathbb{C})$ was complete in 1903 by Blichfeldt [1]. There are A to L types. There is no as simple classification as a 2-dimensional case.
In terms of Conjecture 7.3, Ito and Reid [17] proved McKay conjecture holds

for a $\mathbb{C}^3/G$ when G has a crepant resolution. The next question was that what type of finite group $G < SL(3,\mathbb{C})$ can have crepant resolutions? Marku-shevich, Roan, and Ito [17] showed that all finite subgroups $G < SL(3,\mathbb{C})$ have a crepant resolution.

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