Calibration and Filtering of Exponential Lévy Option Pricing Models

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Abstract

A well-studied topic in finance is fitting pricing models to available market information; this is the inverse of the option pricing problem. The accuracy of least squares calibration using option premiums and particle filtering of price data to find model parameters is determined. Derivative models using exponential Lévy processes are calibrated using regularized weighted least squares with respect to the minimal entropy martingale measure. Sequential importance resampling is used for the Bayesian inference problem of time series parameter estimation with proposal distribution determined using extended Kalman filter and sample impoverishment is avoided with MCMC. The algorithms converge to their respective global optimums using a statistical optimization approach. Each of these methods follows a complementary path toward achieving the same goal: efficiently calibrate the model of choice to available information. Thus, they should agree on optimal parameters. We investigate this assertion.

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List of Notation

$\mathbb{E}X$	Expectation of random variable X
$d\mathbb{P}(\omega)$	Integration with respect to probability measure $\mathbb{P}$
$\mathbb{F}(f(x))$	Fourier transform of function $f(x)$
$\mathbb{F}^{-1}(f(x))$	Fourier inverse of function $f(x)$
$\mathcal{F}(t) \equiv \mathcal{F}_t$	Filtration at time t
$\varphi_X(\nu)$	Characteristic function of random variable X
$p_X(t)$	Probability density function of random variable X
$X \sim N(\mu, \sigma^2)$	X is normally distributed with mean μ and variance σ^2
$X \sim \Gamma(k, \theta)$	X is gamma distributed with shape k and scale θ
$S(t) \equiv S_t$	Stochastic process at time t
S_T	Stochastic process at maturity time (of a derivative)
$c(t, S(t))$	Price of a European call option at time t with underlier $S(t)$
$\varepsilon(\mathbb{P}, \mathbb{Q})$	Relative entropy between $\mathbb{P}$ and $\mathbb{Q}$
CGMY	Carr-Geman-Madan-Yor
VG	Variance Gamma
VGSA	Variance Gamma with Stochastic Arrival
COS	Fourier Cosine Transform
LSS	Least Squares Solution
MEMMC	Minimal Entropy Martingale Measure Calibration
RMELSC	Regularized Minimal Entropy Least Squares Calibration

Contents

1 Introduction

This paper presents the most common methods of fitting stock price observations to the mathematical models which capture their stochastic behavior. We first give an introduction to quantitative finance and the necessity of having a fast, accurate method of pricing derivatives. Multiple processes of Exponential Lévy form are presented. These include both jumps and diffusion components and thus capture a wider range of empirical asset behavior. These generalized processes are then used to price vanilla path-independent options in sections 3 and 4. We show how to empirically fit these models to observed price data. The model fitting methodologies discussed in sections 5 and 6 are backtested to determine their accuracy. These methods are two sides of the same coin, they both retrieve the parameters for a given model which best describe current market conditions, but through complementary means. As such, their predicted parameters should agree. This notion is investigated in section 7.

Arguably the father of the quantitative theory of finance, Harry Markowitz, in his 1952 PhD thesis *Portfolio Selection* described the basis of an entirely new scientific discipline. He developed the concept of treating investment management as an optimization problem based on the mean and variance of each constituent stock. He argued that investors, given their own personal aversion to risk, should hold only those portfolios whose variance is minimized against a specified mean rate of return.

Stochastic calculus was introduced by Robert Merton in 1969 to accurately price financial securities. This laid the groundwork for Fischer Black and Myron Scholes to develop their famous Black-Scholes-Merton option pricing formula which won the 1997 Nobel prize in Economics. This formula provided a solution to a widely recognized problem: finding the fair market price of a European call option. An option is the right to buy one share of a given stock at a specified price and time.

1.1 Option pricing in the binomial model

Consider a stock XYZ whose price is given by a stochastic process $S(t)$, where S_0 denotes the price of $S(t)$ at time $t = 0$. Set the current price of $S_0 = \$30$ in a one-period binomial model such that the space of all possible events is defined as $\Omega = \{\omega_u, \omega_d\}$. In the binomial model, the stock price has two possible trajectories, it may go up or down. Let $S_t(\omega_u)$ denote the price of the stock at time t after going up, and $S_t(\omega_d)$ be the price after going down. Here we consider the factors u, d which denote the change in the stock price given the events that took place. We let $u = 2$ and $d = 1/2$ so that the stock doubles during event ω_u and halves during event ω_d. The binomial model of the stock price from time $t = 0$ to $t = 1$ follows

$$S_0 = 30 \diagup\diagdown \begin{array}{l} S_1(\omega_u) = S_0 \cdot u = 30 \cdot 2 = 60 \\ S_1(\omega_d) = S_0 \cdot d = 30/2 = 15. \end{array}$$

Consider a portfolio $X(t)$ with Δ shares of $S(t)$ and a money market account $M(t) = M_0 \cdot (1 + r)^t$. $M(t)$ grows at a constant interest rate r with initial capital M_0. At time $t = 1$ the value of the portfolio $X(t)$ is

$$X(0) = \Delta S_0 + M(0) \diagup\diagdown \begin{array}{l} X_1(\omega_u) = \Delta S_1(\omega_u) + M(1) \\ X_1(\omega_d) = \Delta S_1(\omega_d) + M(1). \end{array}$$

An option is a type of derivative, or contingent claim, which is a financial instrument that *derives* its price from an underlying stock, referred to as the *underlier*. A European call is a type of option that gives the buyer the right but not the obligation (i.e. the option) to buy the stock at an expiration time T and for a

given price K. We refer to K as the *strike price.* We will price a European call option based on the stock described above. Our example uses expiration time $T = 1$ and strike price $K = \$40$.

The *terminal condition* of the derivative is its value at expiration, where time $t = T$. If the stock price $S(t)$ is greater than strike price K, then the call option value $c(t)$ is equal to $c(T) = (S(T) - K)$, where $S(T)$ is the value of the stock when the call expires. In our example it will be either $S_0 \cdot u = \$60$ or $S_0 \cdot d = \$15$. If the stock price is below the strike price K, then buying the stock using the call option contract would be more expensive than buying the stock on the market, so the option is worthless and we set $c(T) = 0$. Combing the two equations yields $c(T) = \max\{S(T) - K, 0\} := (S(T) - K)^+$. Our goal is to price the derivative $c(t)$ at time 0. Our option at the time of expiration $t = T = 1$ follows the model

$$c_0 \underset{}{\overbrace{}} \begin{array}{l} (S_T(\omega_u) - K)^+ = (S_1(\omega_u) - K)^+ = (60 - 40)^+ = 20 \\[2mm] (S_T(\omega_d) - K)^+ = (S_1(\omega_d) - K)^+ = (15 - 40)^+ = 0. \end{array}$$

To determine the price of any derivative security, we replicate its payoff. We determine a specific number of shares of $S(t)$ and value of our money market account $M(t)$ which perfectly replicates the value of the derivative $c(t)$ at any time. This is called the *replicating portfolio* and is exactly the portfolio $X(t)$ we developed earlier. Using the value at time $t = 1$ of $X(t)$ presents the following system of equations:

$$c_1(\omega_d) = \Delta S_1(\omega_d) + M(1) = \Delta S_0 \cdot d + M_0 \cdot (1 + r)^1$$
$$c_1(\omega_u) = \Delta S_1(\omega_u) + M(1) = \Delta S_0 \cdot u + M_0 \cdot (1 + r)^1.$$

Solving for Δ and M_0 yields

$$\Delta = \frac{c_1(\omega_u) - c_1(\omega_d)}{S_0 \cdot (u - d)} \quad \text{and} \quad M_0 = \frac{u \cdot c_1(\omega_u) - d \cdot c_1(\omega_d)}{(1 + r)(u - d)}.$$

The constant Δ is the *delta hedge* which is the number of shares of $S(t)$ one must hold which, along with M_0, will hedge against the risk of holding the derivative. In conclusion, given the delta hedge at time $t = 0$, we can determine the value of the replicating portfolio $X(0) = \Delta S_0 + M_0$. This is equal to the value of the derivative at time 0:

$$\begin{aligned} c_0 = X(0) &= \Delta S_0 + M_0 \\[2mm] &= \frac{c_1(\omega_u) - c_1(\omega_d)}{S_0 \cdot (u - d)} S_0 + \frac{u \cdot c_1(\omega_u) - d \cdot c_1(\omega_d)}{(1 + r)(u - d)} \\[2mm] &= \frac{1}{1 + r} \left[\frac{1 + r - d}{u - d} c_1(\omega_u) + \frac{u - 1 - r}{u - d} c_1(\omega_d) \right] \\[2mm] &= \frac{1}{1 + r} \left[\widetilde{p}_u c_1(\omega_u) + \widetilde{p}_d c_1(\omega_d) \right]. \end{aligned}$$

The vectors $\widetilde{p}_d = \dfrac{u - 1 - r}{u - d}$ and $\widetilde{p}_u = \dfrac{1 + r - d}{u - d}$ are *risk neutral probabilities* satisfying $\widetilde{p}_d + \widetilde{p}_u = 1$ and $\widetilde{p}_u, \widetilde{p}_d \in \mathbb{R}^{\geq}$. Here we see that the price of a derivative security at time $t = 0$ is a weighted combination of replicating portfolios representing every possible state price at time $t = 1$. Taking the limit of this is exactly the motivation behind the continuous time application of stochastic calculus for solving this problem, as we will see later. Given our example with $u = 2, d = 1/2$ and $R = 1 + r = 1$, the risk neutral probabilities are

$$\widetilde{p}_u = \frac{2 - 1}{2 - 1/2} = \frac{2}{3} \qquad \text{and} \qquad \widetilde{p}_d = \frac{1 - 1/2}{2 - 1/2} = \frac{1}{3} = 1 - \widetilde{p}_u.$$

We can solve for the initial price of the derivative,

$$c(0) = \frac{1}{1 + r} \left[\widetilde{p}_u c_1(\omega_u) + \widetilde{p}_d c_1(\omega_d) \right] = \frac{2}{3} 20 + \frac{1}{3} 0 = \frac{40}{3} \approx \$13.33.$$

\$13.33 is thus the fair market price of the European call with strike price \$40 on a stock with a current price of \$30 which we know for certain will, at the time of expiration, have a value of either \$60 or \$15. The binomial model is the simplest way to conceptualize the process of option pricing and, as thus, it has serious shortcomings. Our example relied on the following assumptions:

1. Shares of stock can always be subdivided
2. The interest rates for investing and borrowing are the same
3. There is zero bid-ask spread, i.e. the purchase price and selling price of a stock are the same
4. There are only two possibilities of a stock's value in the subsequent period

The first three assumptions are required by the Black-Scholes-Merton formula, which we will describe later. The fourth assumption is required by the binomial model. The application of stochastic calculus in describing the stock price as a geometric Brownian motion will overcome this last assumption.

2 Financial Modelling

In many continuous-time models of finance, stocks are represented by geometric Brownian motion given by the stochastic differential equation (SDE) $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$ where $W(t)$ is a standard Brownian motion, α and σ are the drift and volatility factors, respectively. Similarly to the discrete case explored earlier, the replicating portfolio is defined as $X(t) = S(t)\Delta(t) + M(t)$ with differential

$$dX(t) = \Delta(t)dS(t) + (1 + r)(X(t) - \Delta(t)S(t))dt$$
$$= \underbrace{(1 + r)X(t)dt}_{1} + \underbrace{\Delta(t)(\alpha - r)S(t)dt}_{2} + \underbrace{\Delta(t)\sigma S(t)dW(t)}_{3}.$$

The numbered portions can be thought of as:
1) Portfolio rate of return
2) The risk premium associated with investing in $S(t)$
3) The volatility of $S(t)$

We use this basic concept of replication to describe how derivatives can be evaluated.

2.1 Evolution of Option Value

Given the assumption that the option can be represented in continuous time by some stochastic differential equation, the evolution of its value can be calculated in continuous time quite simply. There are two major principles behind the valuation of these derivatives. The first is the notion that any derivative can be hedged, which is equivalent to replicating its payoff, or terminal condition. This results (and is inferred) from the market model being *complete*. The second is the concept of risk-neutral pricing which asserts that, for some risk-free rate r, discounting any derivative by r results in the process satisfying the martingale property. The discounted stochastic process defining the behavior of the derivative can then be evaluated by referring to the *risk-neutral measure*. These concepts are revisited in section 2.6.

Consider a function $c(t, S(t))$ that depends only on time t and the price of the underlying asset $S(t)$ that gives us the value of the option at time t. By Ito's formula, its differential is:

$$dc(t, S(t)) = \left[c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S(t)^2 c_{xx}(t, S(t))dW(t)\right].$$

Equating the discounted portfolio value differential $d(e^{-rt}X(t))$ with the discounted option price differential $d(e^{-rt}c(t, S(t)))$ yields the *Black-Scholes-Merton partial differential equation* (BSM pde)

$$c_t(t, S(t)) + rxc_x(t, S(t)) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t, S(t)) = rx.$$

Solving the BSM pde with the terminal condition $c(T, S(t)) = (S(T) - K)^+$ and boundary conditions

$$c(t, 0) = 0 \quad \text{and} \quad \lim_{x \to \infty}\left[c(t, x) - (x - e^{-r(T-t)}K)\right] = 0, \quad \forall t \in [0, T]$$

yields the Black-Scholes-Merton pricing formula for a European call:

$$c(t, S(t)) = S(t)N(d_+(\tau, S(t))) - Ke^{-r\tau}N(d_-(\tau, S(t)))$$

where $\tau = T - t$ is the time to maturity, $N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{y}^{-\infty} e^{-\frac{x^2}{2}} dz$ is the standard normal distribution and

$$d_{\pm}(\tau, S(t)) = \frac{1}{\sigma\sqrt{\tau}} \left[\log \frac{S(t)}{K} + \left(r \pm \frac{\sigma^2}{2} \right) \tau \right].$$

Intuitively, $N(d_+)$ is the risk-adjusted probability that the option will be exercised and $N(d_-)$ is the factor by which the present value of the exercised option exceeds the current stock price.

The Black-Scholes formula is still widely used within the financial industry. The model's constant volatility assumption is inconsistent with observed market prices, however. The simplest addition to the Black-Scholes model is to let volatility be a function of time and price of the underlier. This gives us the *Local Volatility Model* by Derman and Kani [25]. Here we modify the standard geometric Brownian motion so that volatility is a function of time and $S(t)$ while the drift term is a function of time. This is given by

$$dS(t) = \alpha(t)S(t)dt + \sigma(t, S(t))dW(t).$$

This addition results in the generalized BSM pde which prices derivatives based on the local volatility model

$$c_t(t, S(t)) + \alpha(t)S(t)c_t(t, S(t)) + \frac{1}{2}\sigma(t, S(t))^2 S(t)^2 c_{xx}(t, S(t)) = r(t)c(t, S(t)).$$

While incorporating a volatility function that changes over time more closely approximates observed prices, volatility in reality is often unpredictable in nature, and many models incorporate a stochastic element to the calculation of volatility. One extension of the local volatility model incorporating stochastic volatility results in the aptly named Stochastic Volatility Model. The evolution of the underlying asset price is given by

$$dS(t) = \alpha S(t)dt + V(t)S(t)dW_1(t),$$
$$dV(t) = \alpha_v(t, V(t))dt + \sigma(t, V(t))dW_2(t).$$

The two Brownian motions $W_1(t)$ and $W_2(t)$ are correlated by factor ρ. The terms α_v and σ are the drift and variance of the volatility itself.

The most popular is the Heston stochastic volatility model which is described by the following SDEs:

$$dS(t) = \alpha S(t)dt + \sqrt{V(t)}S(t)dW_1(t),$$
$$dV(t) = \kappa(\theta - V(t))dt + \sigma V(t)dW_2(t),$$

where κ represents the volatility's speed of mean reversion, θ is the long term variance and σ is the volatility of the variance [38].

What has been presented so far are known as pure diffusion models. They have attempted to capture the mean-reverting behavior of stocks by adding a time-dependent drift component while modifying the volatility function of the underlier to capture the dynamic volatility phenomena we observe empirically. The drawback to these models is they assume a smooth transition between the changes of stock prices. What is actually observed is that stock prices often jump instantaneously. It has been proposed by Geman, Madan, and Yor (2001) that stochastic pricing processes need not have a diffusion component but must incorporate jumps. The use of the Poisson process to capture jump characteristics while removing the element of diffusion results in the Variance Gamma process.

2.2   Variance Gamma process (VG)

Developed by Madan, Carr, and Chang [51], the VG model was sought to ascertain consistency with the observation that the local log-price movements of stock prices are long-tailed relative to the normal distribution, but approach normality over time [53,55,54,31,11]. Stock prices have been observed to ignore smooth transitions through time and jump to different prices instantaneously. The importance of producing a process capable of admitting Poisson-type jumps is necessary. The VG process is of Lévy type with finite moments which accounts for high activity by allowing an infinite number of jumps in any interval of time. Since VG

produces finite variation, it can be represented as the sum of two independent and increasing processes, one representing price increases and the other price decreases. It generalizes Brownian motion and models the dynamics of the log stock price. Lacking a diffusion component it is known as a pure jump process.

To construct the VG process $X(t; \sigma, \nu, \theta)$ we take a Brownian motion with drift θ and variance σ represented by

$$b(t, \sigma, \theta) = \theta t + \sigma W(t),$$

where $W(t)$ is a standard Brownian motion, $X(t; \sigma, \nu, \theta)$ can be defined in terms of $b(t, \sigma, \theta)$ as:

$$X_{VG}(t; \sigma, \nu, \theta) = b(\gamma(t; 1, \nu), \sigma, \theta).$$

The time change is given by an independent random variable $\gamma(t; 1, \nu)$ with unit mean, positive variance, and follows a gamma density. The characteristic function of the VG model is

$$\mathbb{E}(e^{iuX_t}) = \left(\frac{1}{1 - iu\theta\nu + \sigma^2 u^2 \nu/2} \right)^{\frac{t}{\nu}},$$

and the asset price is given by

$$\ln S(t) = \ln S(0) + (r - q + \omega)t + X(t; \sigma, \nu, \theta) \quad \equiv \quad S(t) = S(0) \exp\{(r - q + \omega)t + X(t; \sigma, \nu, \theta).\}$$

The term $\omega = \frac{1}{\nu} \ln(1 - \theta\nu - \sigma^2 \frac{\nu}{2})$ is the martingale correction which ensures risk-neutrality, ie.

$$\mathbb{E}S_t = S_0 e^{(r-q)t}.$$

2.3   Analytical Formula for VG Option Price

Unlike many models in finance there exists an analytical formula for a European option price following VG law [53]. The price of a European option can be determined simply by evaluating the risk-neutral expectation of the terminal condition,

$$C_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(S(T) - K)^+],$$

where $\mathbb{Q}$ is an equivalent measure under which $e^{-rt}S(t)$ is a martingale. We define the unit-time log characteristic function of VG as

$$\varphi(u) = \frac{1}{\nu} \ln \left(\frac{1}{1 + \nu u^2/2} \right).$$

Defining the VG process $N = \{N(t) = b(G(t)), t \in [0, T]\}$ for Brownian motion $b(t)$ and right-continuous process of independent gamma increments $G(t)$, the change of measure density process has the form

$$\lambda(t) = \exp \left(\sum_{s \leq t} \alpha(\omega, s) \Delta N_s - \int_0^t \varphi \left(\frac{\alpha(\omega, s)}{i} \right) ds \right).$$

The term $\alpha(\omega, s)$ is chosen so that $\lambda(t)$ is a $\mathbb{Q}$-martingale. The authors of [53] show that $e^{-rt}\lambda(t)S(t)$ is a martingale only if $\alpha(\omega, s)$ satisfies

$$\mu - r = \varphi(\alpha/i) + \varphi(\sigma/i) + \varphi((\alpha + \sigma)/i).$$

The change of measure density is $\lambda(t) = e^{\alpha N(t) - \varphi(\alpha/i)t}$. It is sufficient to determine the option price at time $t = 0$ with maturity at $t = T$. The asset price under a VG process is defined as

$$S(t) = S(0) \exp \left(\sigma N(t) + \left[r + \left(\frac{1}{\nu} \ln \frac{1 - \nu(\alpha + \sigma)^2/2}{1 - \nu\alpha^2/2} \right) t \right] \right),$$

so the option price can now be written as

$$C(t) = \mathbb{E}^{\mathbb{P}}[e^{-rt}\lambda(t)(S(t) - K)^+].$$

Noting that for large t, the VG log characteristic function takes the form

$$\varphi(u) = \frac{t}{\nu} \ln \left(\frac{1}{1 + \nu u^2/2(t/\nu)} \right).$$

Then, since $N(t)/\sqrt{t} \sim \mathcal{N}(0,t)$ for large t/ν where $\mathcal{N}$ is the normal cumulative density function, we integrate the following with respect to $\mathcal{N}(0,t)$:

$$\left[S(0) \exp \left(\sigma N(t) + \frac{t}{\nu} \ln \left[\frac{1 - \nu(\alpha+\sigma)^2/2}{1 - \nu\alpha^2/2} \right] \right) - Ke^{-rt} \right]^+ e^{\alpha N(t) + (t/\nu)\ln(1-\nu\alpha^2/2)}$$

which yields the price of a VG European call:

$$C(t) = S(0)e^{(\alpha+\sigma)^2 t/2}(1 - \nu(\alpha+\sigma)^2/2)^{t/\nu}\mathcal{N}(d_1) - Ke^{-rt+\alpha^2 t/2}(1 - \nu\alpha^2/2)^{t\nu}\mathcal{N}(d_2),$$

$$d_1 = \frac{\ln(S_0/K)}{\sigma\sqrt{t}} + \left[\frac{r + (1/\nu)\ln\left(\frac{1 - \nu(\alpha+\sigma)^2/2}{1 - \nu\alpha^2/2} \right)}{\sigma} + (\alpha+\sigma) \right]\sqrt{t},$$

$$d_2 = d_1 - \sigma\sqrt{t}.$$

The following Matlab implementation of the VG approximation will be used to determine the accuracy of subsequent numerical techniques:

```
1    function c = VGCall(S0, K, r, T, theta, sigma, nu)
2    alpha = (-theta/sigma);
3    a    = (alpha + sigma)^2;
4    num = 1 - nu*a/2;
5    den = 1 - nu*alpha^2/2;
6    d1 = log(S0/K)/(sigma*sqrt(T)) + ((r + 1/nu*log(num/den))/sigma ...
7       + alpha+sigma)*sqrt(T);
8    d2 = d1 - sigma*sqrt(T);
9    c = S0 * exp(a*T/2) * (1 - nu*a/2)^(T/nu)*normcdf(d1) ...
10      - K * exp(-r*T + alpha^2*T/2) * (1-nu*alpha^2/2)^(T/nu)*normcdf(d2);
11   end
```

2.4   Variance Gamma with Stochastic Arrival (VGSA)

Developed by Carr, Geman, Madan and Yor [16], the VGSA model is a modified VG process which allows for volatility clustering through a mean-reverting time change. Volatility clustering has been shown to be present in many markets and so a generalized process capable of accounting for this feature is favorable. The clustering phenomena is achieved through persistent random time changes, which must be mean-reverting. The typical example of a mean-reverting process is the square root model by Cox-Ingersoll-Ross (CIR) [22]. In order to implement VGSA, we evaluate VG at a continuous stochastic time change given by the integral of the CIR process representing the instantaneous stochastic clock. The mean reversion introduced by the CIR process accounts for clustering, also known as volatility persistence, and because of this we are able to calibrate across both strike and maturity simultaneously, unlike VG [39].

The CIR process $y(t)$ is defined as the solution to the SDE:

$$dy(t) = \kappa(\eta - y(t))dt + \lambda\sqrt{y(t)}dW(t),$$

where η is the rate of time change, κ is the rate of mean reversion, λ is the time change volatility. Since $y(t)$ represents the instantaneous rate of time change, we integrate to get the actual time change for all t:

$$Y(t) = \int_0^t y(u)du.$$

The characteristic function of the time change $Y(t)$ is

$$\mathbb{E}e^{iuY(t)} = \varphi(u, t, y(0), \kappa, \eta, \lambda) = A(u, t, \kappa, \eta, \lambda)e^{B(u,t,\kappa,\lambda)y(0)},$$

with

$$A(u, t, \kappa, \eta, \lambda) = \frac{\exp\left(\frac{\kappa^2 \eta t}{\lambda^2}\right)}{\left(\cosh(\gamma t/2) + \frac{\kappa}{\gamma}\sinh(\gamma t/2)\right)^{\frac{2\kappa\eta}{\lambda^2}}},$$

$$B(u, t, \kappa, \lambda) = \frac{2iu}{\kappa + \gamma\coth(\gamma t/2)},$$

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}.$$

The VGSA process is defined as:

$$Z_{VGSA}(t) = X_{VG}(Y(t); \sigma, \nu, \theta) = b(\gamma(Y(t); 1, \nu), \sigma, \theta).$$

where $\kappa, \lambda, \sigma, \nu, \theta, \eta$ are the six parameters defining the model. The characteristic function is given by

$$\mathbb{E}e^{iuZ_{VGSA}(t)} = \varphi(-i\Psi_{VG}(u), t, \nu^{-1}, \kappa, \eta, \lambda),$$

where $\Psi_{VG}(u)$ is the log characteristic function of VG at unit time defined as

$$\Psi_{VG}(u) = -\frac{1}{\nu}\log(1 - iu\nu\theta + \sigma^2\nu u^2/2).$$

We may now define the asset pricing process at time t as

$$S(t) = S(0)\frac{e^{(r-q)t + Z_{VGSA}(t)}}{\mathbb{E}e^{Z_{VGSA}(t)}}.$$

Since $\mathbb{E}e^{Z_{VGSA}(t)} = \varphi(-i\Psi_{VG}(-i), t, \nu^{-1}, \kappa, \eta, \lambda)$, the characteristic function of the log of the asset price is given by

$$\mathbb{E}e^{iu\log S_t} = e^{iu(\log S_0 + (r-q)t)} \cdot \frac{\varphi(-i\Psi_{VG}(u), t, \frac{1}{\nu}, \kappa, \eta, \lambda)}{\varphi(-i\Psi_{VG}(-i), t, \frac{1}{\nu}, \kappa, \eta, \lambda)^{iu}}.$$

2.5   CGMY process

Developed by Carr, Geman, Madan and Yor, the CGMY model accommodates behaviors represented by pure jumps and pure diffusions by allowing its arrival rates and variations to be either finite or infinite. Adjusting the parameters can model a variety of different behaviors. For instance, $Y < 0$ allows for finite activity, $0 \leq Y \leq 1$ allows for infinite activity with finite variation, while $1 \leq Y < 2$ allows infinite activity with infinite variation. The parameter C can be thought of as the measure of overall activity, while G and M are measures of skewness. The CGMY model is defined by its Lévy measure

$$\nu(x) = C\left[\frac{e^{-Gx}}{x^{1+Y}}\mathbb{I}_{x>0} + \frac{e^{-M|x|}}{|x|^{1+Y}}\mathbb{I}_{x<0}\right].$$

While the model cannot be represented by a single SDE, its characteristic function is

$$\mathbb{E}[e^{iuX_t}] = \exp\{Ct\Gamma(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y)\}.$$

The CGMY model is a special case of the tempered stable process, defined as

$$\nu(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha}}\mathbb{I}_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha}}\mathbb{I}_{x<0}.$$

The CGMY model is an interesting alternative to VG and VGSA since, while being a pure jump process, it allows very fine control over the intensity of the jumps, and can approximate a diffusion process by allowing an infinite number of jumps within any interval.

2.6 Risk-Neutral Pricing

In order to illustrate the concept of risk-neutral pricing, we first define Girsanov's theorem and the Martingale representation theorem on a multidimensional Brownian motion [67]. Letting $W(t)$ be a vector of Brownian motions

$$W(t) = \big(W_1(t), \ldots, W_d(t)\big),$$

Girsanov's theorem in multiple dimensions states, for a fixed positive time T with adapted stochastic process $\Theta(t) = \big(\Theta_1(t), \ldots, \Theta_d(t)\big)$ and defining

$$Z(t) = \exp\left\{-\int_0^\Theta (u)dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 du\right\},$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du,$$

then $\mathbb{E}Z(T) = 1$ and $\widetilde{W}(t)$ is a $d-$dimensional Brownian motion under probability measure $\widetilde{\mathbb{P}}$ defined by

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{F}(t),$$

where $\mathcal{F}(t)$ is the filtration associated with $W(t)$. The multidimensional Girsanov theorem shows that, given the adapted stochastic process $\Theta(t)$ which is path-dependent (adapted) on the Brownian motions $W(t)$, where the components of $W(t)$ are independent of each other, the Brownian motion $\widetilde{W}(t)$ defined using the dependent stochastic process $\Theta(t)$ is certainly dependent itself. However, under the probability measure $\widetilde{\mathbb{P}}$, $\widetilde{W}(t)$ is independent.

Finally, the Martingale representation theorem states that for a positive time T and a filtration $\mathcal{F}(t)$ associated with a $d-$dimensional Brownian motion $W(t)$, for a martingale process $M(t)$ defined on this filtration under the measure $\mathbb{P}$ of the Brownian motions, there exists an adapted, $d-$dimensional process $\Gamma(u)$ such that

$$M(t) = M(0) + \int_0^t \Gamma(u) \cdot dW(u), \qquad \text{for } 0 \le t \le T.$$

Concomitantly, by the results of the multidimensional Girsanov theorem explained previously, for a $\widetilde{\mathbb{P}}$-martingale process $\widetilde{M}(t)$, there is an adapted $d-$dimensional process $\widetilde{\Gamma}(u)$ satisfying

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u) \cdot dW(u), \qquad \text{for } 0 \le t \le T.$$

The impact of the previous theorems show that we can construct a multidimensional market model of m stocks, each with their own respective stochastic differential, based on a vector of Brownian motions $W(t)$, each component of which is correlated by some matrix ρ_{ij} which represents the instantaneous correlations between the Brownian motions, which are continuous martingales. Defining each stock to be some stochastic process labeled $S_i(t), i = 1 : m$, we may describe them in terms of their relative differentials, which combines the notion of the instantaneous correlation matrix along with the instantaneous standard deviations (or volatility processes) $\sigma_i, \; i = 1 : m$ as

$$\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_j(t)}{S_j(t)} = \rho_{ij}(t)\sigma_i(t)\sigma_j(t).$$

Then we may define a discount process

$$D(t) = e^{-\int_0^t R(u)du},$$

under which the instantaneous correlations and volatility processes remain unchanged, but the mean rates of return are discounted. The motivation behind the risk-neutral measure $\widetilde{\mathbb{P}}$ is that, a stochastic process under $\widetilde{\mathbb{P}}$ is a martingale, which permits us the use of Ito-calculus. Given some interest rate r, the discount process converts a probability measure $\mathbb{P}$ into the risk-neutral measure $\widetilde{\mathbb{P}}$ so that, should a mean rate of return of
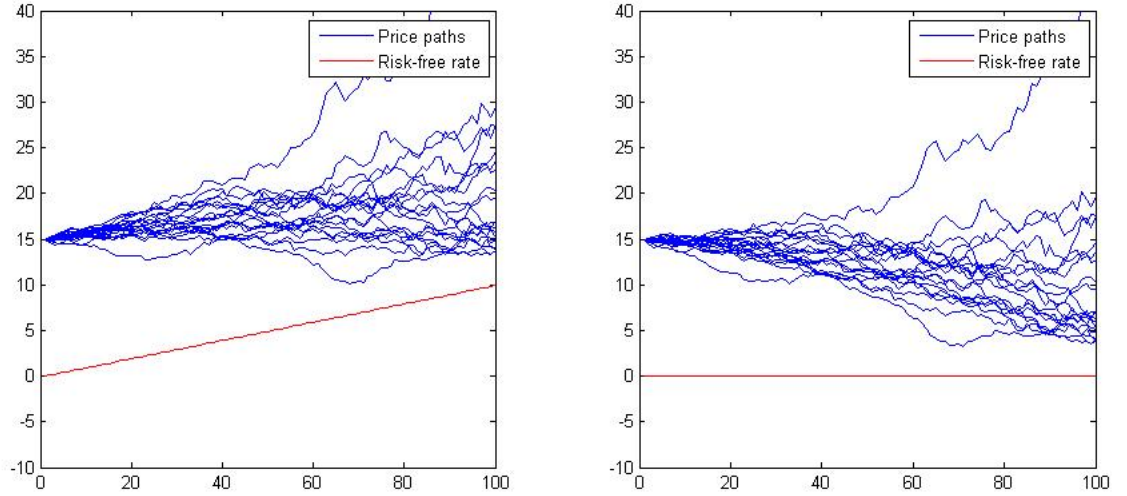
Figure 1: Change to Risk-Neutral Measure

Left: Random option price trajectories (blue) along with rate of risk-free asset (red) under the statistical measure
Right: Same asset price paths and risk-free rate transformed under the risk-neutral measure. The price paths become martingales.

some stock be equal to r, under the risk neutral measure its mean becomes 0, which validates the notion of it achieving the martingale property of constant expectation. Before we proceed, we define the precise definition or arbitrage (something for nothing). Given some portfolio process $X(t)$ with $X(0) = 0$ and for some positive time T satisfying

$$\mathbb{P}(X(T) \geq 0) = 1 \equiv \mathbb{P}(X(T) < 0) = 0 \qquad \text{and} \qquad \mathbb{P}(X(T) > 0) > 0,$$

then $X(t)$ is an arbitrage. Clearly, $X(t)$ represents some trading strategy that can start with zero capital and at some positive time T we are guaranteed to have not lost money while also having some positive probability of making money. Arbitrage is the notion around which our risk-neutral models fail.

This brings us to the first fundamental theorem of asset pricing. If a market model admits a risk-neutral measure, then it does not admit arbitrage. The proof of which is remarkably simple yet intuitively powerful. If a market model has a risk neutral measure $\widetilde{\mathbb{P}}$ then every discounted portfolio process $D(T)X(T)$ is a $\widetilde{\mathbb{P}}$-martingale $\implies \widetilde{\mathbb{E}}[D(T)X(T)] = 0$. Suppose we have some portfolio (hedging) process $X(t)$ with $X(0) = 0$ which is martingale and satisfies the characteristic of an arbitrage ($\mathbb{P}(X(T) < 0) = 0$), since $\widetilde{\mathbb{P}} \equiv \mathbb{P}$, we have $\widetilde{\mathbb{P}}(X(T) < 0) = 0$ also. Since $\widetilde{\mathbb{E}}[D(T)X(T)] = 0$ by the martingale property, this implies that $\widetilde{\mathbb{P}}(X(T) > 0) = 0$ by their equivalence. This also means that $\mathbb{P}(X(T) > 0) = 0$ which is a contradiction, therefore $X(t)$ is not an arbitrage. Moreover, since every portfolio process satisfies $X(0) = 0$, there can never be an arbitrage.

The second part of the fundamental theorem of asset pricing shows that a market is complete if and only if the risk-neutral probability measure is unique. Completeness in this sense refers to the concept that every possible derivative can be hedged.

3   Transform Methods

Efficient methods of numerically evaluating complex financial contracts are required. The Feynman-Kac theorem relates the expectation of a stochastic differential equation governing the behavior of the underlying price process with a analytically calculable PDE. Various methods of evaluating these complex derivative payoff functions fall into three major categories, partial integro-differential equation methods, Monte Carlo simulation methods, and numerical transform techniques. The numerical approximation techniques presented

in this chapter are among the many methods in the literature that require a transformation to the Fourier domain. The methods presented in this text are applied specifically to path-independent options but, as numerical integration techniques are a heavily studied topic in finance, there exists methods devoted toward their path-dependent counterparts. Efficient techniques for valuating options with early exercise, such as American options, have been developed [14,4,3,49,60]. The convolution (CONV) method can be used with the fast Fourier transform to achieve almost linear complexity on American and Bermudan options and uses the assumption that the probability density of a process can be seen as a transition density which is then written as a convolution integral [49]. The mechanics involve using the fact that the characteristic function of a convolution is the product of the constituting characteristic functions. Amongst others, the saddlepoint method of Carr and Madan is an alternative designed to price deep out-of-the-money options using a modified Lugannani-Rice saddlepoint approximation [15].

In this section we describe the foundation of deriving the Fourier-Cosine method for pricing path-independent derivatives for which the characteristic function is known. The first major development in this arena was by Carr and Madan [14] where they develop the Fast Fourier Transform (FFT) to valuate options efficiently. Given the characteristic function of the risk neutral density, a simple analytic expression of the Fourier transform of the option value can be developed. In every model used in this paper, the characteristic function is known. Since we focus on models with an explicit Lévy-Khintchine representation, the characteristic function arises naturally. Besides the VG, VGSA, and CGMY processes described earlier, the class of processes for which the characteristic function is known includes: the process of independent increments (McCulloch 1978); the inverse Gaussian (Barndorff-Nielson 1997); pure diffusion with stochastic volatility (Heston 1993); jump processes with stochastic volatility (Bates 1996); and jump processes with stochastic volatility and stochastic interest rates (Scott 1997). While the FFT method presented a considerable breakthrough in terms of computational speed and a complexity of $\mathcal{O}(N \log_2 N)$ (with N^2 the integration grid size), it has a few shortcomings. It can only be applied to path-independent derivatives with a European-style payoff with an explicit characteristic function. We first describe the concept of transforming the characteristic function of a derivative to determine the discounted expected value of its risk-neutral density.

3.1 Laplace transform

The use of the Laplace transform in option pricing is to transform a complex PDE into an ODE that is usually easier to solve. We illustrate the Laplace method as a precursor to the Fourier method for evaluating the characteristic function later. The Laplace transform $F(\gamma)$ of a function $F(\tau)$ is defined as

$$F(\gamma) = \mathcal{L}(f(\tau))) = \int_0^\infty e^{\gamma\tau} f(\tau) d\tau \tag{3.1.1}$$

where $\gamma \in \mathbb{C}$ and $f(\tau)$ is any function making the integral finite. Any function not satisfying the previous constraint does not have a Laplace transform. Convergence of the integral is satisfied if $\mathcal{R}(\gamma) > \gamma_0$ where γ_0 is the *abscissa of convergence*. The transform also satisfies linearity:

$$\mathcal{L}(af_1(\tau) + bf_2(\tau)) = a\mathcal{L}(t_f(\tau)) + b\mathcal{L}(f(\tau)).$$

Noting that the Laplace transform of a derivative is given by

$$\mathcal{L}(f'(\tau)) = \gamma\mathcal{L}(t(\tau))) - f(0^-),$$

when the Laplace transform for a function $f(\tau)$ is known, $f(\tau)$ can be recovered using the Bromwich inversion formula:

$$f(\tau) = \mathcal{L}^{-1}(F(\tau)) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{a-iR}^{a+iR} F(\gamma)e^{\tau\gamma} d\gamma.$$

The Laplace inversion is highly sensitive to round-off error, and so it is an ill-conditioned problem (Kwok and Barthez 1989). The standard inversion formula is a contour integral which is not a calculable expression. This can be avoided when the Laplace transform is known in closed form as a complex function. Instead of discretizing the forward Laplace integral, we operate the inversion using the Bromwich contour integral

while using the transform's values on the complex plane. Letting the contour be the vertical line $\gamma = a$ then the original function $f(\tau)$ is given by

$$f(\tau) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\gamma\tau} F(\gamma) d\gamma, \quad \tau > 0.$$

Letting $\gamma = a + iu$ we obtain,

$$f(\tau) = \frac{2e^{a\tau}}{\pi} \int_0^\infty \mathcal{R}(F(a+iu)) \cos(u\tau) du = -\frac{2e^{a\tau}}{\pi} \int_0^\infty \mathcal{I}(F(a+iu)) \sin(u\tau) du. \tag{3.1.2}$$

While there are many numerical integration methods which exploit the structure of the inversion formula (3.1.2), the commonality and recent attention of the Fourier method in finance is the most relevant.

3.2   Fourier Method

The Fourier series algorithm is a discretization method of solving (3.1.2) proposed by [26] and relies on the following trapezoidal approximation. With step size Δ,

$$f_\Delta(\tau) = \frac{\Delta e^{a\tau}}{\pi} \mathcal{R}\left(F\left(a\right)\right) + \frac{2\Delta e^{a\tau}}{\pi} \sum_{k=1}^\infty \mathcal{R}\left(F\left(a+ik\Delta\right)\right) \cos(k\Delta\tau).$$

Letting $\Delta = \frac{\pi}{2\tau}$ and $a = \frac{A}{2\tau}$ gives the alternating series

$$f_\Delta(\tau) = \frac{e^{\frac{A}{2}}}{2\tau} \mathcal{R}\left(F\left(\frac{A}{2\tau}\right)\right) + \frac{e^{\frac{A}{2}}}{\tau} \sum_{k=1}^\infty (-1)^k \mathcal{R}\left(F\left(\frac{A+2ki\pi}{2\tau}\right)\right)$$

which eliminates the cosine term. The choice of A must be made so that a falls to the left of the real part of the singularities of the function $F(\gamma)$. This method is surprisingly effective in the context of which the integrands are periodically oscillating, as the errors tend to cancel. The bound of the discretization error, shown by [1], is

$$|f(\tau) - f_\Delta(\tau)| < M \frac{e^{-A}}{1 - e^{-A}} \simeq Me^{-A},$$

for $f(\tau) < M$. If the real term has a constant sign for all k, the Euler accelerating algorithm can be convenient [1]. This consists of summing the first n terms of the series and then taking the weighted average of an additional m terms, ie.

$$f_\Delta(\tau) \approx E(\tau; n, m) = \sum_{j=0}^m \binom{m}{j} 2^{-m} s_{n+k}(\tau),$$

where $s_n(\tau)$ is the partial sum

$$s_n(\tau) = \frac{e^{\frac{A}{2}}}{2\tau} \mathcal{R}\left(F\left(\frac{A}{2\tau}\right)\right) + \frac{e^{\frac{A}{2}}}{\tau} \sum_{k=1}^{n+j} (-1)^k \mathcal{R}\left(F\left(\frac{A+2ki\pi}{2\tau}\right)\right).$$

Pricing a European Call via the Fast Fourier Transform

Following the work of Carr and Madan [14], we illustrate the most basic case of the method of Fourier inversion to price a European call option. We first represent the option pricing problem in terms of the log price density, which allows us to use the Fourier method to obtain the premium. For some security S_t with probability density function $f(S_t)$, log price density of the underlier $q(s_t)$ where $s_t = \ln(S_t)$ and $k = \ln K$ is the log strike. We obtain the log characteristic function:

$$\varphi(\nu) = \mathbb{E}[e^{i\nu s_t}] = \int_{-\infty}^\infty e^{i\nu s_t} p(s_t) dS_t.$$

The value of a European call $C_T(k)$ with strike $K = e^k$ can be expressed in terms of its risk neutral density

$$C\mathbb{E}\left[(S_T - K)^+\right] = C \int_k^\infty e^{-rt}(e^s - e^k)q(s)ds = C_T(k),$$

where the subscript T for the underlier process has been dropped for simplicity. The constant C is the discount value through which we convert to the equivalent martingale measure under which we take expectations. In general, to convert to the risk-neutral measure we would let $C = e^{-r(T-t)}$. We must modify this function as it is not square-integrable. We define the square-integrable call price by

$$c_T(k) = e^{\alpha K}C_T(k), \quad \alpha > 0$$

where α is the damping parameter ensuring convergence of the integral which results in the analytical tractability of the Fourier transform. We redefine the characteristic function of the modified option price $c_T(k)$ as

$$\Psi_T(\nu) = \int_{-\infty}^\infty e^{i\nu k}c_T(K)dk = e^{-rT}\int_{-\infty}^\infty q(s)\left(\int_{-\infty}^s e^{(\alpha+i\nu)k}(e^s - e^k)dk\right)ds,$$

for $\alpha > 0$. We can express the modified characteristic function of the option premium Ψ_T in terms of the log characteristic function of the asset price $\varphi(\nu)$. We develop this expression

$$\Psi_T(\nu) = \frac{e^{-rT}\varphi(\nu - (\alpha + 1)i)}{(\alpha + i\nu)(\alpha + i\nu + 1)},$$

then we can use Fourier inversion to retrieve the actual option premium. Since $C_t(k) \in \mathbb{R}$, its Fourier transform $\Psi_T(\nu)$ has even real part and odd imaginary part, so we can write the call option premium as:

$$C_t(K) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^\infty e^{-i\nu k}\Psi_T(\nu)d\nu = \frac{e^{-\alpha k}}{\pi}\int_0^\infty e^{-i\nu k}\Psi_T(\nu)d\nu.$$

Numerical integration of the above formula can be computed quite easily. First we define an upper bound B for the integration then numerically evaluate it using a quadrature method. Letting N be the number of grid intervals, $\Delta\nu = \frac{B}{N} = \eta$ be the spacing between those intervals and $\nu_j = (j-1)\eta$ be the endpoints for the integration intervals for $j = 1 : N + 1$, we can use the trapezoidal approximation method to obtain:

$$\begin{aligned}
C_T(k) &\approx \frac{e^{-\alpha k}}{\pi}\int_0^B e^{-i\nu k}\Psi_T(\nu)d\nu \\
&\approx \frac{e^{-\alpha k}}{\pi}\left[e^{-i\nu_1 k}\Psi_T(\nu_1) + 2e^{-i\nu_2 k}\Psi_T(\nu_2) + \cdots + 2e^{-i\nu_N k}\Psi_T(\nu_N)\right. \\
&\quad \left. + e^{-i\nu_{N+1}k}\Psi_T(\nu_{N+1})\right]\frac{\eta}{2}.
\end{aligned}$$

The terms are exponentially decaying so the elimination of the final term to satisfy FFT form does not significantly affect the accuracy of the approximation. This yields:

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi}\sum_{j=1}^N e^{-i\nu_j k}\Psi_T(\nu_j)\frac{\eta}{2}(2 - \delta_{j-1}).$$

For a more accurate approach we illustrate the application of Simpson's rule which incorporates different weights into the summation. This will allow us to increase the accuracy of integration with larger values of η:

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi}\sum_{j=1}^N e^{-i\nu_j k}\Psi_T(\nu_j)\frac{\eta}{3}(3 + (-1)^j - \delta_{j-1}).$$

The term δ_{j-1} is the Kronecker delta function defined as

$$\delta_i = \begin{cases} 0, & \text{if } i \neq 0 \\ 1, & \text{if } i = 0. \end{cases}$$

While the above direct integration method provides accurate results, it is far from efficient. The Fast Fourier Transform (FFT) originally developed by (Cooley and Tukey 1965) can efficiently compute the sum:

$$w(m) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(m-1)} x(j), \quad \text{for} \ \ m = 1 : N.$$

This can be reduced to $\mathcal{O}(N \ln N)$ multiplications using a divide and conquer algorithm to break down discrete Fourier transforms which greatly accelerates the speed at which we can evaluate option premiums. Converting the option pricing problem to FFT form requires us to create a range of strikes around the strike we are particularly interested in. For example, an at-the-money call with log strike k necessitates the definition of the range of log strikes

$$k = \beta + (m-1)\Delta k = \beta + (m-1)\lambda \qquad \text{for} \ \ m = 1 : N$$

where $\beta = \ln X_0 - \frac{\lambda N}{2}$. Thus our log strike of interest falls directly center. We will write the integral of $C_T(K)$ as an application of the above summation. The call premium can be written as

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\nu_j k} \Psi_T(\nu_j) w_j,$$

$$= \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(m-1)} e^{-i\beta\nu_j} \Psi_T(\nu_j) w_j.$$

Having the log characteristic function of the asset X_t we choose η and define the grid size as a power of 2, ie. let $N = 2^n$, $\lambda = \frac{2\pi}{N\eta}$, $\nu_j = (j-1)\eta$ and determine α. Then vector x is constructed as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \frac{\eta}{2} \frac{e^{-r(T-t)}}{(\alpha+i\nu_1)(\alpha+i\nu_1+1)} e^{-i(\ln X_0 - \frac{\lambda N}{2})\nu_1} \varphi(\nu_1 - (\alpha+1)i) \\ \eta \frac{e^{-r(T-t)}}{(\alpha+i\nu_2)(\alpha+i\nu_2+1)} e^{-i(\ln X_0 - \frac{\lambda N}{2})\nu_2} \varphi(\nu_2 - (\alpha+1)i) \\ \vdots \\ \eta \frac{e^{-r(T-t)}}{(\alpha+i\nu_N)(\alpha+i\nu_N+1)} e^{-i(\ln X_0 - \frac{\lambda N}{2})\nu_N} \varphi(\nu_N - (\alpha+1)i) \end{pmatrix},$$

where the discount factor $e^{-r(T-t)}$ is the constant C explained earlier which facilitates a measure change, and could be any constant. We input this vector $\mathbf{x}$ into the FFT routine which returns the vector $\mathbf{y} = \text{fft}(\mathbf{x})$ then the m call prices across the range of strikes k_m, $m = 1 : N$ is

$$\mathbf{y} = \begin{pmatrix} C_T(k_1) \\ C_T(k_2) \\ \vdots \\ C_T(k_N) \end{pmatrix} = \begin{pmatrix} \frac{1}{\pi} e^{-\alpha(\ln X_0 - \frac{N}{2}\lambda)} \mathcal{R}(y_1) \\ \frac{1}{\pi} e^{-\alpha(\ln X_0 - (\frac{N}{2}-1)\lambda)} \mathcal{R}(y_2) \\ \vdots \\ \frac{1}{\pi} e^{-\alpha(\ln X_0 - (\frac{N}{2}-(N-1))\lambda)} \mathcal{R}(y_N) \end{pmatrix}.$$

Before describing the FFT algorithm in detail we explain the necessity of the optimal damping parameter.

Optimal α

Our approximation using FFT relied on the calculation of damping parameter α which we now determine optimally. The damping parameter ensures that the call price is L^1 integrable which is sufficient for the Fourier transform to exist. The characteristic functions of the Black-Scholes and Variance Gamma models have a convenient analytical structure which enables the optimal payoff-independent α to be computed exactly. For Black-Scholes, the authors of [50] show that the optimal α satisfies

$$\alpha^* = \min_{\alpha \in \mathbb{R}} \left[-\alpha k + \ln(\varphi(-(\alpha+1)i)^2) \right] = -\frac{d_+}{\eta\sqrt{\tau}},$$

where $d_+ = \dfrac{1}{\sigma\sqrt{\tau}}\left[\log(\dfrac{S0}{K}) + (r + \dfrac{\sigma^2}{\sqrt{2}})\tau\right]$.

For the Variance Gamma model, the optimal α satisfies

$$\alpha^* = \min_{\alpha \in \mathbb{R}}\left[-\alpha k + \ln(\varphi(-(\alpha+1)i)^2)\right] = -\frac{\theta}{\sigma^2} - 1 + \frac{\tau}{\nu\widetilde{m}} - \text{sgn}(\widetilde{m})\sqrt{\frac{\theta^2}{\sigma^2} + \frac{2}{\nu\sigma^2} + \frac{\tau^2}{\nu^2\widetilde{m}^2}}$$

where $\widetilde{m} = f - k - \omega\tau$ is a quantity related to the log-moneyness of the option.

The code for implementing Fast Fourier Transform on the Black-Scholes model is as follows.

```matlab
S0 = 100;          %Spot
K = 100;           %Strike
r = 0.05;          %Risk-free rate
q = 0.0;           %Dividend rate
T = 5;
sigma = 0.3;
N = 2^9;           %Grid size: power of two
uplim = 50;        %upper limit for integration
eta = uplim/N;     %spacing of psi integrand

K = linspace(5,150,N);
lnS = log(S0);
lnK = log(K);

%Optimal damping
alpha = (-1/(sigma*T*eta))*(log(S0./K) + (r+sigma^2/sqrt(2))*T);

lambda = (2 * pi) / (N * eta);  %spacing for log strikes

%log strikes ranging from [lnS-b,lnS+b] (near the money)
b = (N * lambda) / 2;
ku = - b + lambda * (u - 1);

u = 1:N;
j = 1:N;
vj = (j-1) * eta;
%Fourier transform of the modified call price
phi = cf_BS(vj-(alpha+1).*1i,S0,r,sigma,T)./...
    (alpha.^2 + alpha - vj.^2 + 1i * (2 * alpha + 1) .* vj);

psi = exp(-r*T) * phi.* exp(1i * vj * (b)) * eta;
psi = (psi/3).*(3+(-1).^j-((j-1)==0));              %Simpson's rule
fft_psi = ones(1,N)*exp(-1i*2*pi/N)*(j-1).*(psi-1);  %Discrete FFT
cp = real(exp(-alpha.*ku).* fft_psi))/pi;           %call price vector

%Determine strikes near the money
strikeIdx = floor((lnK + b)/lambda + 1);
iset = max(strikeIdx)+1:-1:min(strikeIdx)-1;
xp = ku(iset);
yp = cp(iset);
call_price_fft = real(interp1(xp,yp,lnK));          %Linear interpolation
```

3.3 Fourier Cosine method

The FFT method of the previous section introduced significant advantages toward the problem of evaluating the price of a model with only a known characteristic function. Most notably is its ability to generate a series of option prices across a range of strikes in just a single iteration. Introduced by Fang and Oosterlee [32], the Fourier-Cosine (COS) method provides considerable improvements of FFT. While the amortized complexity $\mathcal{O}(n \log n)$ of FFT is certainly less than the COS method, the superiority of the cosine expansion arises when integrating non oscillating functions since the number of terms can be reduced significantly while maintaining the same level of accuracy. Consequently, empirically, the COS method is faster. Another significant advantage is that the derivation of the cosine expansion is disassociated from the terms dependent on the terminal condition, so that complex payoffs and path-dependent options can be priced. The FFT method described explicitly the Fourier transform of the option premium in terms of the log characteristic function of the asset price, whereas the COS method first represents the log density in terms of the Fourier cosine expansion whose coefficients can then be expressed in terms of the log characteristic of the asset. This can be reduced to an analytically calculable integral. The COS method suffers from similar drawbacks as FFT such as the inability to price highly out-of-the-money options. An interval bound must be explicitly determined to truncate the infinite integral to ensure the expansion has a finite number of terms. Consequently, the accuracy of the premium is highly sensitive to the choice of bounds.

We follow the notation of [32]. The Fourier cosine series expansion of a function $f(\theta) : \mathbb{R} \to [0, \pi]$ is

$$f(\theta) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos(k\theta) = {\sum_{k=0}^{\infty}}' A_k \cos(k\theta),$$

with Fourier cosine coefficient

$$A_k = \frac{2}{\pi} \int_0^\pi f(\theta) \cos(k\theta) d\theta.$$

The notation $\sum'$ indicates that the first term is weighted by one-half. The Fourier cosine series expansion can be obtained for functions defined on any finite interval $[a, b]$ with change of variables mapping a to 0 and b to π, ie.

$$\theta = \frac{x - a}{b - a}\pi \qquad \text{so that} \qquad x = \frac{b - a}{\pi}\theta + a.$$

The expansion then becomes

$$f(x) = {\sum_{k=0}^{\infty}}' A_k \cos\left(k\frac{x - a}{b - a}\pi\right) \quad \text{with} \quad A_k = \frac{2}{b - a} \int_a^b f(x) \cos\left(k\frac{x - a}{b - a}\pi\right) dx.$$

3.4 Cosine Coefficients in Terms of the Characteristic Function

We know that for any probability density function $p(x)$, the relevant characteristic function is obtained via the Fourier transform

$$\mathbb{E}e^{i\nu x} = \varphi(\nu) = \int_{-\infty}^{\infty} e^{i\nu x} p(x) dx.$$

By evaluating the characteristic function at $\nu = \frac{k\pi}{b-a}$ we define the truncated integral

$$\widehat{\varphi}\left(\frac{k\pi}{b - a}\right) = \int_a^b \exp\left\{ix\frac{k\pi}{b - a}\right\} f(x) dx \tag{3.4.1}$$

Multiplying (3.4.1) by $e^{i\frac{k\pi a}{b-a}}$ yields

$$\widehat{\varphi}\left(\frac{k\pi}{b - a}\right) e^{i\frac{k\pi a}{b-a}} = e^{i\frac{k\pi a}{b-a}} \int_a^b \exp\left\{ix\frac{k\pi}{b - a}\right\} f(x) dx$$

$$= \int_a^b \exp\left\{ik\pi\left(\frac{x - a}{b - a}\right)\right\} f(x) dx$$

$$= \int_a^b \left(\cos\left(k\pi\left[\frac{x - a}{b - a}\right]\right) + i\sin\left(k\pi\left[\frac{x - a}{b - a}\right]\right)\right).$$

Therefore,

$$\mathcal{R}\left[\widehat{\varphi}\left(\frac{k\pi}{b-a}\right)e^{i\frac{k\pi a}{b-a}}\right] = \int_a^b \cos\left(k\pi\left[\frac{x-a}{b-a}\right]\right)f(x)dx.$$

Choosing limits of integration so that $\widehat{\varphi}(\nu) \approx \varphi(\nu)$ gives the cosine coefficient defined as

$$A_k = \frac{2}{b-a}\mathcal{R}\left\{\widehat{\varphi}(\nu)\left(\frac{k\pi}{b-a}e^{-i\frac{ka\pi}{b-a}}\right)\right\}$$

so that

$$A_k \approx F_k = \frac{2}{b-a}\mathcal{R}\left\{\varphi(\nu)\left(\frac{k\pi}{b-a}e^{-i\frac{ka\pi}{b-a}}\right)\right\}. \tag{3.4.2}$$

We can now replace A_k with F_k in the series expansion of $f(x)$ within $[a,b]$ to get

$$f_1(x) = \overline{\sum}_{k=0}^{\infty} F_k \cos\left(k\pi\frac{x-a}{b-a}\right),$$

which is further truncated to obtain

$$f_2(x) = \overline{\sum}_{k=0}^{N-1} F_k \cos\left(k\pi\frac{x-a}{b-a}\right). \tag{3.4.3}$$

Keeping in mind that the error in $f_2(x)$ results from the error of approximating A_k with F_k and the truncation error of replacing the upper limit of summation by $N-1$. We are now able to use the derived COS formula to price a European call option. The option value at time t can be written as

$$v_1(x,t) = e^{-r\Delta t}\int_a^b v(y,T)f(y|x)dy.$$

We then replace the density function $f(y|x)$ by its cosine expansion in y

$$f(y|x) = \overline{\sum}_{k=0}^{\infty} A_k \cos\left(k\frac{x-a}{b-a}\pi\right),$$

so that the price of the option is now

$$v_1(x,t) = e^{-r\Delta t}\int_a^b v(y,T)\overline{\sum}_{k=0}^{\infty} A_k \cos\left(k\frac{x-a}{b-a}\pi\right)dy.$$

Interchanging the summation and integration and inserting the definition

$$V_k = \frac{2}{b-a}\int_a^b v(y,T)\cos\left(k\frac{x-a}{b-a}\pi\right)dy$$

yields

$$v_1(x,t) = \frac{b-a}{2}e^{-r\Delta t}\overline{\sum}_{k=0}^{\infty} A_k V_k.$$

We continue with the following approximations. The coefficients rapidly decay as $k \to \infty$ so,

$$v_1(x,t) \approx v_2(x,t) = \frac{b-a}{2}e^{-r\Delta t}\overline{\sum}_{k=0}^{N-1} A_k V_k.$$

The coefficients can be further approximated by F_k as defined earlier to obtain

$$v(x,t) \approx v_3(x,t) = e^{-r\Delta t}\overline{\sum}_{k=0}^{N-1}\mathcal{R}\left\{\varphi\left(\frac{k\pi}{b-a};x\right)e^{-ik\pi\frac{a}{b-a}}\right\}V_k. \tag{3.4.4}$$

3.5 Numerical solution to a vanilla European option

We first define the following:

   (i) S_t is the price of the underlying security at time t
   (ii) K is the strike price
  (iii) $x = \ln\left(\frac{S_t}{K}\right)$ is the log price of the underlier
  (iv) $y = \ln\left(\frac{S_T}{K}\right)$ is the log price at expiration

The log-asset price payoff for a European option follows

$$v(y,T) = [\alpha K(e^y - 1)]^+,$$

where $\alpha = 1$ for a call and $\alpha = -1$ for a put. We first define the cosine series coefficients for $g(y) = e^y$ on $[c,d] \subset [a,b]$,

$$\chi_k(c,d) = \int_c^d e^y \cos\left(k\pi \frac{y-a}{b-a}\right) dy \tag{3.5.1}$$

and the cosine series coefficients of $g(y) = 1$ on the same interval:

$$\psi_k(c,d) = \int_c^d \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{3.5.2}$$

The proof of which can be found in [32]. This gives us the analytical expression for a vanilla European call option

$$V_k^{call} = \frac{2}{b-a} \int_a^b K(e^y - 1)^+ \cos\left(k\pi \frac{y-a}{b-a}\right) dy = \frac{2}{b-a} K(\chi_k(0,b) - \psi_k(0,b)) \tag{3.5.3}$$

and a vanilla European put can be written as

$$V_k^{put} = \frac{2}{b-a} \int_a^b K(1 - e^y)^+ \cos\left(k\pi \frac{y-a}{b-a}\right) dy = \frac{2}{b-a} K(-\chi_k(0,b) + \psi_k(0,b)). \tag{3.5.4}$$

The authors of [32] propose the following calculation for the range of integration $[a,b]$ of the COS method:

$$[a,b] = \left[c_1 \pm L\sqrt{c_2 + \sqrt{c_4}}\right] \qquad \text{with} \ \ L = 10, \text{ elusively.} \tag{3.5.5}$$

The notation c_n denotes the n-th cumulant of the log asset price at expiration $X = \ln\left(S_T/K\right)$ defined by the cumulant generating function

$$G(w) = \ln \mathbb{E} e^{wX} = \ln(\varphi(-iw)),$$

with characteristic function φ. The n-th cumulant is the n-th derivative of the cumulant generating function evaluated at zero, ie.

$$c_n = G^{(n)}(0) = \frac{-i\varphi^{(n)}(0)}{\varphi(0)}$$

The following table describes all cumulants which will be used in this paper, where w is the martingale correction term satisfying $e^{-wt} = \varphi(-i,t)$.

BS	$c_1 = (r - q)T$
	$c_2 = \sigma^2 T$
	$c_4 = 0$
	$w = 0$
VG	$c_1 = (\mu + \theta)T$
	$c_2 = (\sigma^2 + \nu\theta^2)T$
	$c_4 = 3(\sigma^4\nu + 2\theta^4\nu^3 + 4\sigma^2\theta^2\nu^2)T$
	$w = \frac{1}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2)$
CGMY	$c_1 = \mu T + CT\Gamma(1 - Y)(M_{Y-1} - G^{Y-1})$
	$c_2 = \sigma^2 T + CT\Gamma(2 - Y)(M^{Y-2} + G^{Y-2})$
	$c_4 = CT\Gamma(4 - Y)(M^{Y-4} + G^{Y-4})$
	$w = -C\Gamma(-Y)[(M - 1)^Y - M^Y + (G + 1)^Y - G^Y]$
VGSA	Compute using finite differences

3.6 Fourier-Cosine Algorithm

Here we implement the Fourier Cosine transform to price a call following the Variance Gamma model

```
1    S0 = 100; K = 90; T = 5; r = .1; sigma = 0.3; nu = 0.2; theta = -.14;
2
3    K = linspace(10,150,500);
4
5    chi = @(k,a,b,c,d) (1./(1+(k*pi/(b-a)).^2).*(cos(k*pi*(d-a)/(b-a))...
6        .*exp(d)-cos(k*pi*(c-a)/(b-a))*exp(c)+k*pi/(b-a).*sin(k*pi*(d-a)...
7        /(b-a))*exp(d)-k*pi/(b-a).*sin(k*pi*(c-a)/(b-a))*exp(c)));
8
9    psi = @(k,a,b,c,d) ( [d-c;(sin(k(2:end)*pi*(d-a)/(b-a))...
10       -sin(k(2:end)*pi*(c-a)/(b-a))).*(b-a)./(k(2:end)*pi)] );
11
12   %Variance Gamma characteristic function
13   cf_VG = @(u,S,K,t,r,sigma,nu,theta)  ((exp(u*(r+log(1-theta*nu-sigma^2...
14       *nu/2)/nu)*t*1i)).*((1-1i*theta*nu*u +sigma^2*nu*u.^2/ 2).^(-t/nu)));
15
16   N = 2^8;        %Grid size
17
18   %Variance Gamma cumulants
19   c1 = (r+theta)*T;
20   c2 = (sigma^2+nu*theta^2)*T;
21   c4 = 3*(sigma^4*nu+2*theta^4*nu^3+4*sigma^2*theta^2*nu^2)*T;
22
23   %Truncation range
```

```
24    L = 10;
25    a = c1−L*sqrt(c2+sqrt(c4));
26    b = c1+L*sqrt(c2+sqrt(c4));
27
28    %Compute Fourier Cosine transform
29    x = log(S0./K);
30    k = (0:N−1)';
31    Vk = 2/(b−a)*(chi(k,a,b,0,b) − psi(k,a,b,0,b)); %Cosine coefficients
32    w = [.5 ones(1,N−1)];                           %weights
33    ret = w*(cf_VG(k*pi/(b−a),S0, K, T, r, sigma, nu, theta)...
34        *ones(1,length(K)).*exp(1i*k*pi*(x−a)/(b−a)).*(Vk*ones(1,length(K))));
35    %Call price vector
36    cp =K*exp(−r*T).*real(ret);
```

3.7   VGSA Cumulants

Since we do not have the cumulants for VGSA in closed form, we must compute them using some discretization technique. We showed earlier that the characteristic function of the VGSA model is defined as a specific parameterization of the characteristic function of the CIR time-change:

$$\mathbb{E}e^{iuY(t)} = \varphi_{VGSA}(u,t,y(0),\kappa,\eta,\lambda) = A(u,t,\kappa,\eta,\lambda)e^{B(u,t,\kappa,\lambda)y(0)},$$

where

$$A(u,t,\kappa,\eta,\lambda) = \frac{\exp\left(\frac{\kappa^2\eta t}{\lambda^2}\right)}{\left(\cosh(\gamma t/2) + \frac{\kappa}{\gamma}\sinh(\gamma t/2)\right)^{\frac{2\kappa\eta}{\lambda^2}}},$$

$$B(u,t,\kappa,\lambda) = \frac{2iu}{\kappa + \gamma\coth(\gamma t/2)},$$

and $\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}$. The VGSA characteristic function is given by

$$\mathbb{E}e^{iuZ_{VGSA}(t)} = \varphi_{VGSA}(-i\Psi_{VG}(u),t,\nu^{-1},\kappa,\eta,\lambda),$$

where $\Psi_{VG}(u)$ is the log characteristic function of VG at unit time

$$\Psi_{VG}(u) = -\frac{1}{\nu}\log(1 - iu\nu\theta + \sigma^2\nu u^2/2).$$

The cumulant generating function satisfies:

$$G(w) = \ln\mathbb{E}e^{wX} = \ln(\varphi_{VGSA}(-iw))$$

Its n-th cumulant is the n-th derivative of the cumulant generating function with respect to u evaluated at zero:

$$c_n = G^{(n)}(0)$$

The first four cumulants are the first four finite differences of $G(w)$:

$$c_1 \approx \frac{G(h) - G(-h)}{2h}$$
$$c_2 \approx \frac{G(h) - 2G(0) + G(-h)}{h^2}$$
$$c_3 \approx \frac{G(2h) - 2G(h) + 2G(-h) - G(-2h)}{2h^3}$$
$$c_4 \approx \frac{G(3h) - 2G(2h) + 4G(0) - G(h) - G(-h) - 2G(-2h) + G(-3h)}{4h^4}$$

For some small value $h = 10^{-6}$.

3.8 Numerical Results of Cosine Method

Relative error of FFT using the Black-Scholes model with parameters $S_0 = 100, r = 0.1, q = 0, \sigma = 0.2, N = 2^7$, and varying strikes and times to maturity.
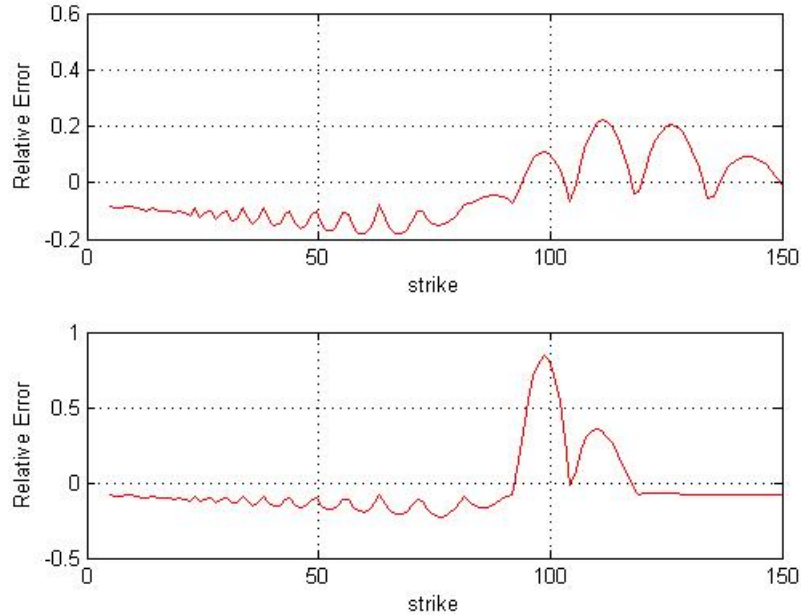

Figure 2: Top: T=1, Bottom, T=0.1

Reference option premium $c = 13.2697$ priced analytically with Black Scholes using parameters $S_0 = 100, K = 100, r = 0.1, q = 0, \sigma = 0.2, T = 1$

COS Method Black-Scholes					
		T=1		T=0.1	
K	N	Relative error	CPU time	Relative error	CPU time
80	2^5	-0.0028	0.002692	-1.02e-0.5	0.000292
100	2^5	-1.115e-05	0.002148	-2.88e-6	0.000255
120	2^5	-2.801e-07	0.001623	-2.20e-8	0.000275

Reference option premium priced analytically with Variance Gamma using parameters $S_0 = 100, K = 100, r = 0.1, q = 0, \theta = 0.15\sigma = 0.2, \nu = 0.1, T = 1$.

COS Method Variance Gamma							
		T=1			T=0.1		
K	N	premium*	Relative error	CPU time	premium*	Relative error	CPU time
80	2^5		0.8730	0.001167		0.1920	0.000635
	2^6	28.1547	0.1781	0.000548	20.8165	0.2375	0.000564
	2^7		0.1762	0.000593		0.2429	0.000586
100	2^5		0.00908	0.001223		0.0109	0.000675
	2^6	13.4251	0.0897	0.000576	3.0543	0.0166	0.000573
	2^7		0.0897	0.000586		0.0165	0.000578
120	2^5		0.2677	0.001294		0.0950	0.001272
	2^6	4.7984	0.2650	0.000592	0.0068	0.1012	0.000569
	2^7		0.2650	0.000593		0.1003	0.000580

*reference premiums determined analytically.

4   Monte Carlo Simulation Methods

Using Monte Carlo to evaluate the price of an option is conceptually the simplest yet the most computationally intensive method. It does not suffer from the mispricing phenomena of the Fourier Cosine method in pricing out of the money options, yet to get an accurate evaluation one must generally run Monte Carlo over a high number of asset pricing paths.

4.1   Antithetic Variates

When using Gaussian variables to drive a Monte Carlo simulation, we can take advantage of the fact that a standard normal random variable Z has an identical distribution to its reflection $-Z$. The central limit theorem allowing the standard error of Monte Carlo sampling to be determined required independent draws. If we view two random variates $v = Z$ and $\hat{v} = -Z$ as individual samples then we can take the pairwise average of the two $\bar{v} = \frac{1}{2}(v + \hat{v})$ and consider it as an individual sample.

4.2   Variance Gamma process

As explained earlier, the VG process can be obtained from evaluating a Brownian motion at a random time given by the gamma process $\gamma(t; 1, \nu)$ (called the subordinator) and follows

$$X(t; \sigma, \nu, \theta) = \theta\gamma(t; 1, \nu) + \sigma W(\gamma(t; 1, \nu)),$$

with characteristic function

$$\varphi(u) = \mathbb{E}e^{iuX_t} = \left(1 - iu\theta\nu + \sigma^2 u^2 \nu/2\right)^{-t/\nu}.$$

Then the log asset price at time t is given by

$$\ln S_t = \ln S_0 + (r - q + \omega)t + X(t; \sigma, \nu, \theta),$$

where $\omega = -\dfrac{1}{t}\log(\varphi(-i)) = \dfrac{1}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2)$ is the martingale correction. We assume N equidistant intervals of length $h = T/N$, then sample from a gamma distribution with mean k and variance νk. Recall a gamma density function with shape a and scale b follows

$$p_\Gamma(x) = \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\theta^k \Gamma(k)}.$$

Thus, its mean and variance are

$$\mu = ab = h,$$
$$\sigma^2 = ab^2 = \nu h,$$

which implies the scale and shape are $b = \nu, a = \frac{h}{\nu}$ respectively. So, given random variables $Z \sim N(0,1)$ and $G \sim \Gamma(\frac{h}{\nu}, \nu)$, we obtain a sample for the VG process

$$X_{VG}(h; \sigma, \nu, \theta) = \theta G + \sigma \sqrt{G} Z.$$

The following is our Matlab implementation of the Monte Carlo pricing of VG using subordinated Brownian motion.

```matlab
 1          S = 100; T = 5; r = .1; q = 0; sigma = 0.3; nu = 0.2; theta = -.14;
 2
 3          K = linspace(10,150,50);
 4          numPaths = 10000;
 5          N = 100;      %Number of time steps per path
 6
 7          %Martingale correction
 8          omega = 1/nu*log(1-theta*nu-sigma^2*nu/2);
 9          h = T/N; %Time step
10          lnS = 0;
11          payoff = 0;
12          %Gamma time changed Brownian motion
13          for j = 1:numPaths
14              lnS = log(S);
15              for i = 1:N
16                  Z = normrnd(0,1);
17                  G = gamrnd(h/nu,nu);
18                  %Cumulative log asset price
19                  lnS = lnS + (r-q+omega)*h + theta*G + sigma*sqrt(G)*Z;
20              end
21              %Terminal condition
22              payoff = payoff + max(exp(lnS)-K,0);
23          end
24          %Discounted average is the price of a European call
25          payoff = exp(-r*T)*payoff/numPaths;
```

4.3 CGMY tempered stable process

Madan and Yor [52] show that the CGMY process can be represented as a time-changed Brownian motion. The CGMY subordinator is absolutely continuous with respect to a one sided stable subordinator achieved by the truncation method proposed by Rosinski. Suppose the CGMY process $X(t)$ is obtained by a subordinated Brownian motion with measure $\nu(dy)$, then by Sato [64] the CGMY Lévy measure is given by

$$\mu(dx) = dx \int_0^\infty \nu(dy) \frac{1}{\sqrt{2\pi y}} e^{-\frac{(x-\theta y)^2}{2y}},$$

where $Y(t)$ is the independent subordinator. The CGMY process can be written as

$$X(t) = \theta Y(t) + W(Y(t)).$$

In particular, the authors of [52] show that the subordinator $Y(t)$ is absolutely continuous with the one sided stable $\frac{Y}{2}$ subordinator with the following Lévy measure

$$\nu(dy) = \frac{K}{y^{1+\frac{Y}{2}}} f(y) dy,$$

$$f(y) = e^{-\frac{(B^2-A^2)y}{2}} \mathbb{E}\left[e^{-\frac{B^2 y}{2} \frac{\gamma_{Y/2}}{\gamma_{1/2}}}\right],$$

$$B = \frac{G+M}{2}, A = \frac{G-M}{2},$$

$$K = \left[\frac{C\Gamma(\frac{Y}{2})\Gamma(1-\frac{Y}{4})}{2\Gamma(1+\frac{Y}{2})}\right],$$

with the independent gamma variates $\gamma_{Y/2} \sim \Gamma(Y/2,1)$ and $\gamma_{1/2} \sim \Gamma(1/2,1)$. We write the CGMY subordinator Lévy measure as

$$\nu_1(dy) = \nu_0(dy)\mathbb{E}\left[e^{-yZ}\right],$$

$$\nu_0(dy) = \frac{K}{y^{1+\frac{Y}{2}}},$$

$$Z = \frac{B^2}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}.$$

While the expression $\mathbb{E}e^{-yZ}$ can be simulated, it incurs unnecessary randomness, and can be avoided by evaluating explicitly the Laplace transform of Z:

$$\mathbb{E}\left[e^{-yZ}\right] = \frac{\Gamma(\frac{Y+1}{2})}{\Gamma(Y)\sqrt{\pi}}2^Y\left(\frac{B^2 y}{2}\right)^{\frac{Y}{2}} I(Y, B^2 y, B^2 y/2),$$

$$I(Y, a, \lambda) = (2\lambda)^{\frac{Y}{2}}\Gamma(Y)e^{\frac{a^2}{8\lambda}}D_{-Y}\left(\frac{a}{\sqrt{2\lambda}}\right),$$

where D is the parabolic cylinder function

$$D_p(z) = 2^{\frac{p}{2}}e^{-\frac{z^2}{4}}\left[\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1-p}{2})}U\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi}z}{\Gamma(-\frac{p}{2})}U\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right)\right],$$

and U is the confluent hypergeometric function of the first kind [35]. Since we have identified two Lévy measures satisfying

$$\frac{d\nu_1}{d\nu_0} = \mathbb{E}e^{-yZ} < 1,$$

it is shown in Rosinski [63] that we may simulate the paths of the CGMY subordinator ν_1 from the paths of ν_0 by rejecting all jumps smaller than a predetermined truncation level ϵ. Letting

$$A = \frac{G-M}{2} \qquad \text{and} \qquad B = \frac{G+M}{2},$$

we simulate from a one sided stable subordinator with measure:

$$\frac{1}{y^{\frac{Y}{2}+1}} \quad \text{and arrival rate} \quad \lambda = \int_\epsilon^\infty \frac{1}{y^{\frac{Y}{2}+1}}dy = \frac{2}{Y\epsilon^{\frac{Y}{2}}}.$$

In our implementation, we truncate the jumps below $\epsilon = 10^{-4}$ then replace them with their expected drift

$$d = \int_0^\epsilon y\frac{1}{y^{\frac{Y}{2}+1}}dy = \frac{\epsilon^{1-\frac{Y}{2}}}{1-\frac{Y}{2}}.$$

The jump times are simulated by the exponential random variable

$$t_i = -\frac{1}{\lambda}\log(1 - U_{1i}).$$

For uniform sequence of random variables U_i which gives cumulative jump times of

$$\Gamma_j = \sum_{i=1}^{j} t_j, \quad \text{and magnitude} \quad y_j = \frac{\epsilon}{(1 - U_{2j})^{\frac{2}{Y}}},$$

the stable subordinator is given by the process

$$S(t) = dt + \sum_{j=1}^{\infty} y_j \mathbb{I}_{\Gamma_j < t},$$

and the CGMY subordinator is given by

$$H(t) = dt + \sum_{j=1}^{\infty} y_j \mathbb{I}_{\Gamma_j < t} \mathbb{I}_{h(y) > U_3}.$$

For independent uniform sequence U_3, with:

$$h(y) = \exp\left(-\frac{B^2 y}{2}\right)\frac{\Gamma(\frac{Y+1}{2})}{\Gamma(y)\sqrt{\pi}}2^Y \left(\frac{B^2 y}{2}\right)^{\frac{Y}{2}} I(Y, B^2 y, B^2 y/2),$$

$$I(Y, a, \lambda) = (2\lambda)^{\frac{Y}{2}}\Gamma(Y)e^{\frac{a^2}{8\lambda}} D_{-Y}\left(\frac{a}{\sqrt{2\lambda}}\right).$$

Then, the CGMY random variable is

$$X = AH(t) + \sqrt{H(t)}z, \quad z \sim \mathcal{N}(0, 1).$$

Our algorithm for Monte Carlo simulation of CGMY using subordinated Brownian motion and Rosinski rejection is:

```
1   %Returns the laplace transform of the CGMY subordinator
2   function [ret] = CGMYSubLaplace(y,Y,B)
3       ret = (exp((-B^2.*y)/2).*gamma((Y+1)/2)*2.^Y...
4           .*(B^2.*y./2).^(Y/2).*I(Y,B^2.*y,B^2.*y./2))/(gamma(Y)*sqrt(pi));
5   %Parabolic cylinder function (Laplace change of variables)
6   function[ret] = ParCyl(p,z)
7       S=1;
8       c=1;
9       g=gamma(-p/2+0.5);
10      H=gamma(-p/2);
11      if z<40
12          u = Hypergeometric(-p/2,0.5,z.*z./2);
13          v = Hypergeometric(0.5-p/2,1.5,z.*z./2);
14          D = 2^(p/2).*exp(-z.*z./4).*((sqrt(pi).*u./g)-sqrt(2*pi).*z.*v./H);
15      else
16          for i=1:20
17              c = c * -(p-i+1)*(p-i)./(2*i.*z.^(2*i));
18              S = S+c;
19          end
20          D = exp(-z.^2./4).*z.^p.*S;
21      end
22      ret = D;
```

```matlab
23    end
24    %Confluent hypergeometric function of the first kind
25    function[ret] = Hypergeometric(a,b,z)
26        ret=1;
27        term=ones(size(z));
28        n=1;
29        while max(term)>1E-4 && n<100
30            term=term.*((a+n-1)*z/(n*(b+n-1)));
31            ret=ret+term;
32            n=n+1;
33        end
34    end
35
36    function[ret] = I(Y,a,lambda)
37        ret = (2.*lambda).^(-Y/2).*gamma(Y).*exp(a.^2./(8.*lambda))...
38            .*ParCyl(-Y,(a./sqrt(2.*lambda)));
39    end
40    end
41
42    %Monte Carlo simulation
43    C=10; G=10; M=10;
44    %Y < 2 higher values means small jumps have greater influence
45    Y=0.75;
46    numPaths=100;
47    T=1;
48    S0 = 100; strike = 100;
49    St = zeros(1,numPaths);
50    r = 0.08;
51    payoff = 0;
52    A=(G-M)/2;
53    B=(G+M)/2;
54    epsilon=1E-10; %Jump truncation level
55    d=epsilon^(1-Y/2)/(1-Y/2); %Expected drift of truncated jumps
56    lambda=2/(epsilon^(Y/2)*Y);   %Arrival rate of jumps
57    pathSize = ceil(lambda*(T)+1);   %Expected number of jumps before maturity
58    warning('off','all');
59    tic
60    for i =1:numPaths
61        %Vector of jump times
62        tj=0;
63        while tj(end)<T
64            U2=rand(1 ,pathSize);
65            ti=-log(1-U2)/lambda;
66            tj=[tj tj(end)+cumsum(ti)];
67        end
68        tj=tj(tj<T);    %Reject jumps occuring after maturity
69
70        U1=rand(1,length(tj)-1);
71        %Vector of jump magnitudes
72        yj =[0, epsilon./(1-U1).^(2/Y)];
73
74        U3=rand(size(yj));
75        %CGMY subordinator, Rosinski rejection
76        Ht = d*tj + cumsum(yj.*(CGMYSubLaplace(yj,Y,B)>U3));
```

```
77      %CGMY random variable
78      X=A*Ht+sqrt(Ht).*randn(size(Ht));
79      St(i) = S0*exp(r*T+X(end));
80      payoff = payoff + max(St(i)-strike,0);
81    end
82    toc
83    premium = exp(-r*T)*payoff/numPaths
```

4.4   Variance Gamma with Stochastic Arrival

Recall the CIR process $y(t)$ is the solution to the SDE:

$$dy(t) = \kappa(\eta - y(t))dt + \lambda\sqrt{y(t)}dW_t.$$

The time change is given by its integral,

$$Y(t) = \int_0^t y(s)ds,$$

which has characteristic function

$$\mathbb{E}e^{iuY(t)} = \varphi(u, t, y(0), \kappa, \eta, \lambda) = A(t, u)e^{B(t,u)y(0)},$$

where

$$A(t, u) = \frac{\exp\frac{\kappa^2\eta t}{\lambda^2}}{\left(\cosh(\gamma t/2) + \frac{\kappa}{\gamma}\sinh(\gamma t/2)\right)^{2\kappa\eta/\lambda^2}},$$

$$B(t, u) = \frac{2iu}{\kappa + \gamma\coth(\gamma t/2)},$$

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}.$$

The VGSA process is defined by the modified VG process

$$Z(t) = X_{VGSA}(Y(t); \sigma, \nu, \theta) = \theta\gamma(Y(t); 1, \nu) + \sigma W(\gamma(Y(t); 1, \nu)).$$

Its characteristic function is that of the time change $Y(t)$ with the following parameters:

$$\mathbb{E}e^{iuZ(t)} = \varphi(-i\psi_{VG}(u), t, \frac{1}{\nu}, \kappa, \eta, \lambda),$$

where $\psi_{VG}(u) = -\frac{1}{\nu}\log(1 - iu\theta\nu + \sigma^2\nu u^2/2)$ is the log of the characteristic function of the VG process. The characteristic function of the log of the stock price is:

$$\mathbb{E}e^{iu\log S_t} = e^{iu(\log S_0 + (r-q)t)} \cdot \frac{\varphi(-i\psi_{VG}(u), t, \frac{1}{\nu}, \kappa, \eta, \lambda)}{\varphi(-i\psi_{VG}(-i), t, \frac{1}{\nu}, \kappa, \eta, \lambda)^{iu}}$$

To simulate the VGSA process we again assume N equidistant time intervals of length $h = T/N$. Discretizing its differential gives:

$$\begin{aligned}
\Delta Z(t) &= Z(t) - Z(t - h) \\
&= \theta\gamma(Y(t); 1, \nu) + \sigma W(\gamma(Y(t); 1\nu)) - (\theta\gamma(Y(t - h); 1, \nu) + \sigma W(\gamma(Y(t - h); 1\nu)) \\
&= \theta(\gamma(Y(t); 1, \nu) - \gamma(Y(t - h); 1, \nu)) + \sigma\sqrt{\gamma(Y(t); 1, \nu) - \gamma(Y(t - h); 1\nu)}z,
\end{aligned}$$

where $z \sim N(0,1)$ and $\gamma(Y(t); 1, \nu) \sim \Gamma(\frac{h}{\nu}, \nu)$. Therefore,

$$\gamma(Y(t); 1, \nu) - \gamma(Y(t-h); 1, \nu) = \Gamma\left(\frac{Y(t)}{\nu}, \nu\right) - \Gamma\left(\frac{Y(t-h)}{\nu}, \nu\right)$$
$$= \Gamma\left(\frac{Y(t) - Y(t-h)}{\nu}, \nu\right).$$

So we can write the differential as

$$\Delta Z(t) = \theta \Gamma\left(\frac{Y(t) - Y(t-h)}{\nu}, \nu\right) + \sigma \sqrt{\Gamma\left(\frac{Y(t) - Y(t-h)}{\nu}, \nu\right)} z.$$

Milstein discretization of the CIR time change results in:

$$y_j = y_{j-1} + \kappa(\eta - y_{j-1})h + \lambda\sqrt{y_{j-1}h}z + \frac{\lambda^2}{4}h(z^2 - 1).$$

This transforms the stochastic clock as $\int_{t_{j-1}}^{t_j} y(u)du$ and by the trapezoidal approximation we obtain

$$\int_{t_{j-1}}^{t_j} y(u)du = \frac{h}{2}(y_{j-1} - y_j).$$

The change in the log asset price can be computed as follows. Given

$$\log S_t = \log S_0 + (r-q)t + Z(t) - \log \mathbb{E}e^{Z(t)},$$
$$\log S_{t-h} = \log S_0 + (r-q)(t-h) + Z(t-h) - \log \mathbb{E}e^{Z(t-h)},$$

subtracting yields the log stock price,

$$\log S_t = \log S_{t-h} + (r-q)h + \Delta Z(t) + \Delta \omega_t.$$

where $\omega = \log(\varphi_{VGSA}(-i\psi_{VG}(-i), (j-1)h, 1/\nu, \kappa, \eta, \lambda)) - \log(\varphi_{VGSA}(-i\psi_{VG}(-i), jh, 1/\nu, \kappa, \eta, \lambda))$ is the martingale correction. Our algorithm for simulating the modified VG to include mean reverting time change is:

```
1    S = 100; T = 5; r = .1; q = 0;
2    %VG parameters
3    sigma = 0.3; nu = 0.2; theta = -.14;
4    %CIR parameters
5    kappa = 0.01;      %Rate of mean reversion
6    lambda = 0.02;      %Volatility of time change
7    eta = 0.3;         %Long term rate of change
8
9    K =linspace(10,150,50);
10   numPaths = 1000;
11   N = 100;      %number of time steps per path
12   %VGSA Characteristic function
13   phi_VGSA = @(u,t,y0,kappa,eta,lambda) (exp(kappa^2*eta*t/lambda^2)...
14       /(cosh(sqrt(kappa^2-2*lambda^2*1i*u)*t/2)+kappa/sqrt(kappa^2...
15       -2*lambda^2*1i*u)*sinh(sqrt(kappa^2-2*lambda^2*1i*u)*t/2))...
16       ^(2*kappa*eta/(lambda^2))*exp(2*1i*u/(kappa + sqrt(kappa^2-2...
17       *lambda^2*1i*u)*coth(sqrt(kappa^2-2*lambda^2*1i*u)*t/2)))*y0);
18   %Log VG characteristic function
19   psi_VG = @(u) (-1/nu*log(1-1i*u*theta*nu-sigma^2*nu*u^2/2));
20   h = T/N;
21   payoff = 0;
```

```matlab
22        for i = 1:numPaths
23            X = 0;
24            y = ones(1,N);
25            lnS = log(S);
26            for j = 2:N
27                Z = normrnd(0,1);
28                %Discretize the CIR time change
29                y(j) = y(j-1) + kappa*(eta-y(j-1))*h+lambda...
30                    *sqrt(y(j-1)*h)*Z+lambda^2/4*h*(Z^2-1);
31                tj = h*(y(j)+y(j-1))/2;
32                G = gamrnd(tj/nu,nu); Z = normrnd(0,1);
33                %CIR time changed VG process
34                X = theta*G+sigma*sqrt(G)*Z;
35                %Martingale correction
36                omega = log(phi_VGSA(-1i*psi_VG(-1i), (j-1)*h, 1/nu, kappa,...
37                    eta, lambda))-log(phi_VGSA(-1i*psi_VG(-1i), j*h,...
38                    1/nu, kappa, eta, lambda));
39                lnS = lnS + (r-q)*h+omega+X;
40            end
41            payoff = payoff + max(exp(lnS)-K,0);
42        end
43        payoff = exp(-r*T)*payoff/numPaths;
```

4.5   Numerical Results of Monte-Carlo Simulation

Figure 3 shows the relative error between the individual Monte Carlo routines and the corresponding Fourier-cosine algorithms for VG, CGMY, and VGSA respectively. Each test involves 10,000 paths, uses a spot price of 100, strikes ranging from 50 to 150 and parameters $T = 1, r = 0.08, q = 0$. The Variance Gamma parameters are $\sigma = 0.41, \nu = 0.1, \theta = -0.1$. CGMY uses $C = G = M = 10, Y = 1.5$. VGSA uses the same parameters for VG, with $\kappa = \eta = \lambda = 0.001$ which brings VGSA very close to a VG process, as we can see by the similar errors.
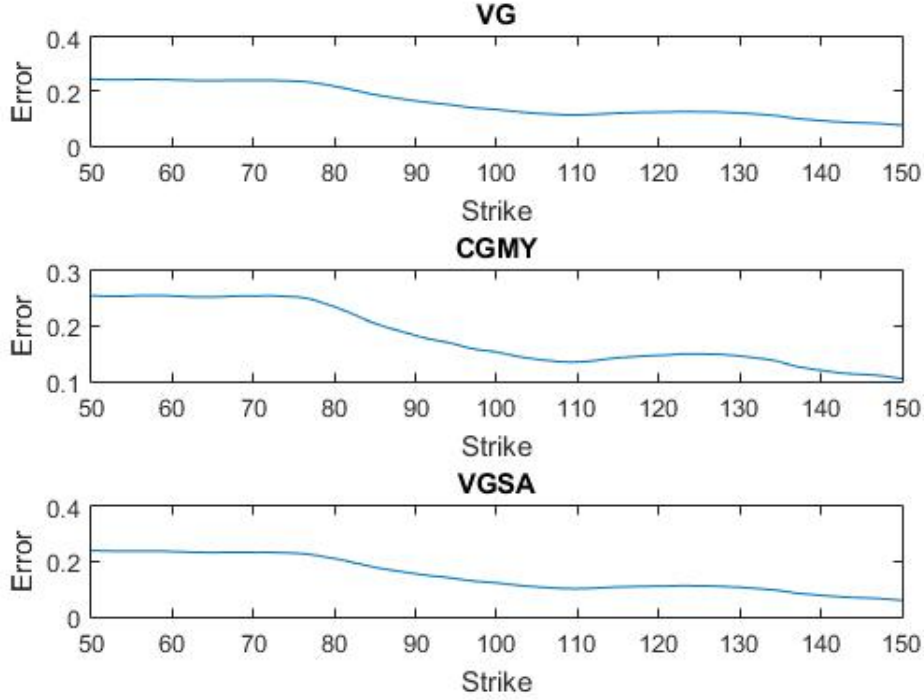
Figure 3: Error of Monte Carlo vs Fourier Cosine

5 Least-Squares Calibration

In this section we analyze the first (of two) major methods of fitting option pricing models to market information. Given a series of prices $\widehat{C}_i$ and strikes K_i, $i = 1 : n$, we search for the characteristic triplet of a risk-neutral exponential Lévy model Q describing the model parameters under which the discounted asset price $e^{-rt}S(t)$ is a martingale. The measure Q is chosen so that it minimizes the error with respect to the market prices. The calibration problem is the inverse of the option pricing problem. We construct Lévy process Q so that the call option prices are given by their risk-neutral expectations

$$\widehat{C}_i \approx C^Q = e^{-rT}\mathbb{E}^Q\big[(S(T_i) - K_i)^+\big] \quad \text{for each } i,$$

and the discounted process is a martingale.

The typical least squares problem is best explained by example. Given a series of market data (option prices and strikes) with the same maturity, risk-free rate, and dividend rate, we minimize the error between the observed prices and the ones predicted by our model Q. While there are many choices of error functional, throughout the subsequent examples we use the root mean square error, given by

$$RMSE(Q) = \frac{1}{\sqrt{N}}\sqrt{\sum_{i=1}^{N}(\widehat{C}(i) - C^Q(i))^2}.$$

In this section, we assume the underlier follows a VGSA process, and we construct an optimization problem to find the parameters which best fit the data. Recall the VGSA process and corresponding asset price is defined as follows:

$$Z_{VGSA}(t) = X_{VG}(Y(t); \sigma, \nu, \theta) = b(\gamma(Y(t); 1, \nu), \sigma, \theta),$$

where $\kappa, \lambda, \sigma, \nu, \theta, \eta$ are the six parameters defining the model. The characteristic function is given by

$$\mathbb{E}e^{iuZ_{VGSA}(t)} = \varphi(-i\Psi_{VG}(u), t, \nu^{-1}, \kappa, \eta, \lambda),$$

32

where $\Psi_{VG}(u)$ is the log characteristic function of VG at unit time:

$$\Psi_{VG}(u) = -\frac{1}{\nu} \log(1 - iu\nu\theta + \sigma^2 \nu u^2 / 2).$$

We may now define the asset pricing process at time t as

$$S(t) = S(0) \frac{e^{(r-q)t + Z_{VGSA}(t)}}{\mathbb{E}e^{Z_{VGSA}(t)}},$$

where T, r, q are obtained from market information, we must therefore determine the optimal $\sigma, \nu, \theta, \kappa, \eta, \lambda$. Any derivative-free nonlinear optimization method will suffice in determining a local minima. The images at the end of this section show the highly nonconvex error landscape for the VGSA naive least squares problem.

The typical least-squares optimization problem is ill-posed. We show a particularly interesting way to encourage convexity within the minimization functional and thus guarantee the existence of a solution. While the Black-Scholes model has to be replaced with models with finer structure such as those with jumps [27], the inverse problem is still ill-posed [65]. Various methods have been proposed which enforce some degree of stability, but restrict their domain to diffusion models [65,61,42]. Given the calibration problem's ill-posed nature, we must define extra criteria to ensure the models market price compatibility. Relative entropy as a criteria has solid foundations [24]. First we describe the notion of relative entropy.

5.1 Relative Entropy for Lévy Processes

As explained earlier, choosing an arbitrage free pricing model is equivalent to determining some measure $\mathbb{Q}$ satisfying the laws of the measure $\mathbb{P}$ under the constraint of being a martingale. There a many ways to determine the "distance" between these two measures. We will focus on relative entropy, or Kullback Liebler divergence, which is defined by choosing the distance function $f(x) = x \ln x$ so that

$$\varepsilon(\mathbb{Q}, \mathbb{P}) = \mathbb{E}^{\mathbb{Q}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] = \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right].$$

Including the above requirement within our optimization problem results in the minimal entropy martingale measure. The intuition behind this measure $\mathbb{Q}$, or MEMM, is that it satisfies the martingale requirement while adding the least amount of information to the prior model $\mathbb{P}$. In the case of exponential Lévy models, the MEMM does not always exist, but we can determine analytically the criterion for its tractability. The MEMM can also be related to the Escher transform [20]. Further reading on the concept of MEMM can be found in [33, 56, 23, 30].

The relative entropy between two Lévy processes can be thought of as the disparity of information contained within them and is thus an effective measure of their distance [17]. Relative Entropy for Lévy Processes is defined as follows [20]. First, let $\{X_t\}_{t\geq 0}$ be a real-valued Lévy process defined on spaces $(\Omega, \mathcal{F}, \mathbb{P}), (\Omega, \mathcal{F}, \mathbb{Q})$ with respective characteristic triplets $(A_P, \nu_P, \gamma_P), (A_Q, \nu_Q, \gamma_Q)$. Suppose that $\mathbb{Q} \ll \mathbb{P}$ (PIIS Thm [43]) which implies that $A_Q = A_P, \nu_Q \ll \nu_P$ and define $A \equiv A_Q = A_P$, then for every time horizon $T \leq T_\infty$, the relative entropy of $\mathbb{Q}|_{\mathcal{F}_T}$ with respect to $\mathbb{P}|_{\mathcal{F}_T}$ is computed as

$$\varepsilon(\mathbb{Q}, \mathbb{P}) = \varepsilon(\mathbb{Q}|_{\mathcal{F}_T}, \mathbb{P}|_{\mathcal{F}_T})$$

$$= \frac{T}{2A} \left[\gamma_Q - \gamma_P - \int_{-1}^{1} x(\nu_Q - \nu_P) dx \right]^2 \mathbb{I}_{A\neq 0} + T \int_{-\infty}^{\infty} \left(\frac{d\nu_Q}{d\nu_P} \log \frac{d\nu_Q}{d\nu_P} + 1 - \frac{d\nu_Q}{d\nu_P} \right) \nu_P dx. \quad (1)$$

The relative entropy criterion satisfies the following properties: [59]

1. Convexity: with two probability measures $\mathbb{Q}_1, \mathbb{Q}_2$ both equivalent to $\mathbb{P}$ then

$$\varepsilon(\alpha\mathbb{Q}_1 + (1-\alpha)\mathbb{Q}_2, \mathbb{P}) = \mathbb{E}^{\mathbb{P}} \left[f\left(\frac{d(\alpha\mathbb{Q}_1 + (1-\alpha)\mathbb{Q}_2)}{d\mathbb{P}} \right) \right]$$

$$= \mathbb{E}^{\mathbb{P}} \left[f\left(\alpha \frac{d\mathbb{Q}_1}{d\mathbb{P}} + (1-\alpha) \frac{d\mathbb{Q}_2}{d\mathbb{P}} \right) \right]$$

$$\leq \alpha\varepsilon(\mathbb{Q}_1, \mathbb{P}) + (1-\alpha)\varepsilon(\mathbb{Q}_2, \mathbb{P}).$$

2. Nonnegativity: $\varepsilon(\mathbb{P}, \mathbb{Q}) \geq 0$.

3. $\varepsilon(\mathbb{Q}, \mathbb{P}) = 0 \iff \frac{d\mathbb{Q}}{d\mathbb{P}} = 1$ a.s..

5.2 Modified Least-Squares Functional

The minimal relative entropy approach was extended to stochastic processes through the weighted Monte Carlo method by Avellaneda [6] and while Goll and Rüschendorf[34] describe calibration of minimal relative entropy, they do not propose an algorithm. We follow the method used by Cont and Tankov [20] which overcomes the shortcomings of the above approaches by defining the calibration problem to include jump processes and use relative entropy as a regularization criterion instead of a selection criterion. We identify the Lévy measure ν and volatility σ from observation of liquid call prices. Given adequate information, their determination is as follows:

1. Determine the risk-neutral density using the Breeden-Litzenberger formula [12]

$$p_X(t) = e^{r(T-t)} \frac{\partial^2}{\partial K^2} C(t, S(t)).$$

2. Compute characteristic function by Fourier transform of the density p_X.
3. σ, ν can be determined from the characteristic function through Fourier inversion

We first construct the norm

$$\|C\|_w^2 = \int_{\mathfrak{C}} C(T, K)^2 w(dT \times dK),$$

where the probability weighting measure $w = \sum_{i=1}^{N} w_i \delta_{(T_i, K_i)}(dT \times dK)$ corresponds to the weight of each individual constraint $1 : N$ and w is defined on the grid of strikes and maturities $\mathfrak{C} = [0, T_\infty] \times [0, \infty)$. Thus the quadratic pricing error for Q is

$$\|\widehat{C} - C^Q\|_w^2 = \sum_{i=1}^{N} w_i (\widehat{C}(T_i, K_i) - C^Q(T_i, K_i))^2.$$

The non-linear least-squares calibration problem can be formulated as

$$\|\widehat{C} - C^{Q^*}\|_w^2 = \min_{Q \in \mathcal{M} \cap \mathcal{L}} \|\widehat{C} - C^Q\|_w^2$$

where the model $\mathbb{Q}^*$ is the least squares solution that minimizes the squared pricing errors between the model and the market. $\mathbb{Q}^*$ lies in the set of all Lévy martingale measures $\mathcal{M} \cap \mathcal{L}$. This problem, however, is ill-posed since we cannot guarantee the existence of a solution, the least-squares functional is non-convex which prevents any gradient-based algorithm from finding the optimal solution, and it is highly sensitive to initial conditions.

Since models involving jumps do not always admit a unique measure, integrating prior views into the calibration procedure allows us to determine some martingale measure that expresses the observed option prices while maintaining equivalence to the historical measure or the preconceived notions of the investment manager. In this section we calibrate the problem with respect to a prior model which is the product of historical prices. Since the calibration procedure is likely to be executed daily, a good choice of prior is the optimal measure of the last time the calibration procedure was executed.

5.3 Regularization

We enforce uniqueness, existence, and stability into our calibration problem by introducing additional information through a prior model $\mathbb{P}$. We aim to find a solution to the least-squares calibration problem which minimizes the relative entropy with respect to our prior model $\mathbb{P}$. Given two probability measures $\mathbb{P}$ and $\mathbb{Q}$ with respective Levy triplets $(\sigma^2, \gamma_P, \nu_P)$ and $(\sigma^2, \gamma_Q, \nu_Q)$, we define our regularization term as the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$,

$$\varepsilon(\mathbb{Q}, \mathbb{P}) = \frac{T}{2\sigma^2} \left[\int_{-\infty}^{\infty} (e^x - 1)(\nu_Q - \nu_P)dx \right]^2 + T \int_{-\infty}^{\infty} \left[\frac{d\nu_Q}{d\nu_P} \ln \frac{d\nu_Q}{d\nu_P} + 1 - \frac{d\nu_Q}{d\nu_P} \right] \nu_P dx$$

We follow the approach outlined in [28] for regularization of ill-posed problems and add the minimal relative entropy $\varepsilon(\mathbb{Q}, \mathbb{P})$ as a penalization term so that our least squares problem takes the following form:

$$J_\alpha(\mathbb{Q}^*) = \min_{Q \in \mathcal{M} \cap \mathcal{L}} \|\widehat{C} - C^Q\|_w^2 + \alpha\varepsilon(\mathbb{Q}, \mathbb{P}).$$

If the regularization parameter α is large, the functional becomes convex and approaches the prior which enforces stability. If α is small, the functional is close to the least-squares problem which enforces precision. An appropriate choice of regularization parameter is necessary to the accuracy of the solution, as we will see later. The previous equation, through slight abuse of language, is the regularized minimal entropy martingale calibration problem (RMEMC).

5.4   Numerical Implementation of RMEMC

As we have shown, the initial calibration problem can be reformulated as an optimization problem determining the Lévy measure $\mathbb{Q}^*$ representing the minimum value of the regularization problem

$$J_\alpha(\mathbb{Q}^*) = \min_{Q \in \mathcal{M} \cap \mathcal{L}} \sum_{i=1}^{N} w_i(\widehat{C}(T_i, K_i) - C^Q(T_i, K_i))^2 + \alpha\varepsilon(Q, \mathbb{P}),$$

which can be solved once we obtain the

1. constraint weights w_i,
2. prior measure $\mathbb{Q}$, and
3. regularization parameter α.

Minimization Constraint Weights

The weights within the optimization problem identify our "confidence" in individual market data points. Options with higher trading volume (higher liquidity) will thus have a better estimation of their respective prices. As shown in [18], a reasonable solution is to minimize the squared differences between prices with respect to option vega, which is defined as the derivative of the Black Scholes option price with respect to volatility:

$$\text{vega} = |Ke^{-rT}N(d_-)\sqrt{T}|.$$

After letting each weight be the square reciprocal of vega, our calibration problem becomes

$$J_\alpha(\mathbb{Q}^*) = \min_{Q \in \mathcal{M} \cap \mathcal{L}} \sum_{i=1}^{N} \frac{(\widehat{C}(T_i, K_i) - C^Q(T_i, K_i))^2}{(K_i e^{-rT_i}N(d_-)\sqrt{T_i})^2} + \alpha\varepsilon(Q, P).$$

Determination of Prior Measure

We aim to construct the prior measure automatically from option price data. As we will see, while the initial prior model determines the speed of convergence, the solution will still converge to optimality regardless of our initial choice of prior. For this reason, we let the prior model $\mathbb{P}$ for the calibration problem of determining $\mathbb{Q}_n$ to be equal to the optimal measure generated previously, ie. $\mathbb{P} = \mathbb{Q}_{n-1}$. This gives us the recursive relation

$$J_\alpha(\mathbb{Q}_n^*) = \min_{Q \in \mathcal{M} \cap \mathcal{L}} \sum_{i=1}^{N} \frac{(\widehat{C}(T_i, K_i) - C^Q(T_i, K_i))^2}{(K_i e^{-rT_i}N(d_-)\sqrt{T_i})^2} + \alpha\varepsilon(\mathbb{Q}_n, \mathbb{Q}_{n-1}).$$

This is solved using a derivative free gradient based procedure such as Nelder-Mead [9].

Regularization Parameter

The authors of [20] suggest the optimal α be found using the Morozov discrepancy principle [28]. Briefly described, any parameter choice rule $\delta \mapsto \alpha(\delta)$, as the noise level $\delta \to 0$, must satisfy:

1. $\alpha(\delta) \to 0$,
2. $\frac{\delta^2}{\alpha(\delta)} \to 0$ given exactly attainable constraints,
3. or, $\frac{\delta}{\alpha(\delta)} \to 0$ if the model does not exactly replicate the data, as is often the case.

The discrepancy principle was developed by Morozov to use on least squares regularization in Banach spaces [57,58] and it has been shown that such a method of choosing the regularization parameter yields the most numerically favorable results [68]. Using the assumptions that:

1. The prior model $\mathbb{Q}^*$ is arbitrage free satisfying minimal entropy martingale measure criteria,

2. There exists a solution $\mathbb{Q}^+$ to the non regularized problem (MEMC) with finite entropy satisfying fully attainable constraints, ie $\|C^{Q^+} - \widehat{C}\|_w = 0$, and

3. There exists maximum noise level δ_0 such that the max error satisfies $\epsilon^{\max} := \inf_{\delta \le \delta_0} \|C^{Q^*} - \widehat{C}^\delta\|_w^2 > \delta_0^2$

Where C_0^δ represents perturbed price data within some noise level δ of $\widehat{C}$. Because this noisy data will not yield a very good solution from MEMC, we need to regularize the problem by defining a family of regularization operators $\{R_\alpha\}_{\alpha \ge 0}$ where α determines the intensity of the regularization [68]. If α is chosen appropriately the regularized, noisy problem converges to the MEMC problem admitting an exact solution, ie. $\text{RMEMC}_\alpha(\widehat{C}^\delta) \to \text{MEMC}(\widehat{C})$ as $\delta \to 0$. Denote Q_α^δ to be the solution to the noisy problem $\text{RMEMC}_\alpha(C_M^\delta)$ with regularization parameter α. The RMEMC a priori multivalued discrepancy function is thus

$$\epsilon_\delta(\alpha) := \|C^{Q_\alpha^\delta} - \widehat{C}^\delta\|_w^2.$$

We place the following constraints on the discrepancy principle. Given two constants c_1 and c_2 satisfying

$$1 < c_1 \le c_2 < \frac{\epsilon^{\max}}{\delta_0^2},$$

the discrepancy principle can be stated as, for a given noise level δ, choose parameter $\alpha > 0$ satisfying

$$\delta^2 < c_1 \delta^2 \le \epsilon_\delta(\alpha) \le c_2 \delta^2.$$

We aim to find a solution $\mathbb{Q}$ of $\text{MEMC}(\widehat{C})$ with noise level of order δ. We try to solve $\|C^{Q_\alpha^\delta} - \widehat{C}^\delta\|_w^2 \le \delta^2$. By sacrificing some precision we gain stability so we pick some constant $c \approx 1^+$ (e.g. $c = 1.1$) and search for the solution $\mathbb{Q}_\alpha^\delta$ in the level set defined by

$$\|C^{Q_\alpha^\delta} - \widehat{C}^\delta\|_w^2 \le c \cdot \delta^2,$$

where the highest stability is achieved when $\epsilon_\delta(\alpha) := \|C^{Q_\alpha^\delta} - \widehat{C}^\delta\|_w^2 = c \cdot \delta^2$ The noise level δ can be computed directly if the bid and ask prices are known, ie. given:

$$\widehat{C}^\delta(T_i, K_i) = \frac{\widehat{C}^{\text{bid}}(T_i, K_i) + \widehat{C}^{\text{ask}}(T_i, K_i)}{2}, \qquad \forall i = 1 : N,$$

the noise level is thus:

$$\delta := \frac{\|\widehat{C}^{\text{bid}} + \widehat{C}^{\text{ask}}\|_2}{2}.$$

While the theoretical foundations of using the discrepancy principle are solid, in practice it is unrealistic to determine the bid-ask spread of the options since liquidity may not offer reliable estimates. Instead, during the construction of our RMEMC algorithm, we opt for a similar approach of picking the regularization parameter based upon the size of vega, which as explained earlier is another method of estimating volatility of the markets. In fact, defining the regularization term as

$$\alpha = \mathcal{A} \cdot |Ke^{-rT} N(d_-)\sqrt{T}|, \quad \text{for some } \mathcal{A} > 0$$

has the advantage of speeding up convergence when the volatility is high and restricting the incorporation of new market information (noise) when volatility is low. The term $\mathcal{A}$ represents the investors preconceived

notions about the direction or fluidity of the market, and could just be set to 1. For instance, high volatility suggests that the market is responding to new information, and thus new price movements will likely have greater impact on the optimal pricing measure than historical price movements. When volatility is low, it is difficult to distinguish new information from noise, and so the regularization term acts as a dampening parameter which restricts change in the optimal measure.

Calibration Functional

We first discretize the calibration problem by taking Lévy process P with finite measure [68]:

$$\nu_P = \sum_{k=0}^{M-1} p_k \delta_{x_k}(dx),$$

which satisfies

$$\nu_Q = \sum_{k=0}^{M-1} q_k \delta_{x_k}(dx) \ll \nu_P.$$

Substituting the above expressions into the RMEMC problem yields, with for simplicity, $\mathbb{Q} = \mathbb{Q}_n$ and $\mathbb{P} = \mathbb{Q}_{n-1}$:

$$J_\alpha(\mathbb{Q}^*) := \|\widehat{C} - C^Q\|_w^2 + \alpha\varepsilon(\mathbb{Q}|\mathbb{P})$$

$$= \|\widehat{C} - C^Q\|_w^2 + \alpha\left(\frac{1}{2A}\left[\gamma_Q - \gamma_P - \int_{-1}^{1} x(\nu_Q - \nu_P)dx\right]^2 \mathbb{I}_{A\neq0} + \int_{-\infty}^{\infty}\left[\frac{d\nu_Q}{d\nu_P}\log\frac{d\nu_Q}{d\nu_P} + 1 - \frac{d\nu_Q}{d\nu_P}\right]\nu_P dx\right)$$

$$= \sum_{i=1}^{N}\frac{(\widehat{C}(T_i, K_i) - C^Q(T_i, K_i))^2}{(K_i e^{-rT_i} N(d_-)\sqrt{T_i})^2} + \frac{\alpha}{2A}\left(\frac{A}{2} + b_P + \sum_{j=0}^{M-1}(e^{x_j} - 1)q_j\right)^2 + \alpha\sum_{j=0}^{M-1}\left[q_j\log\left(\frac{q_j}{p_j}\right) + 1 - q_j\right]$$

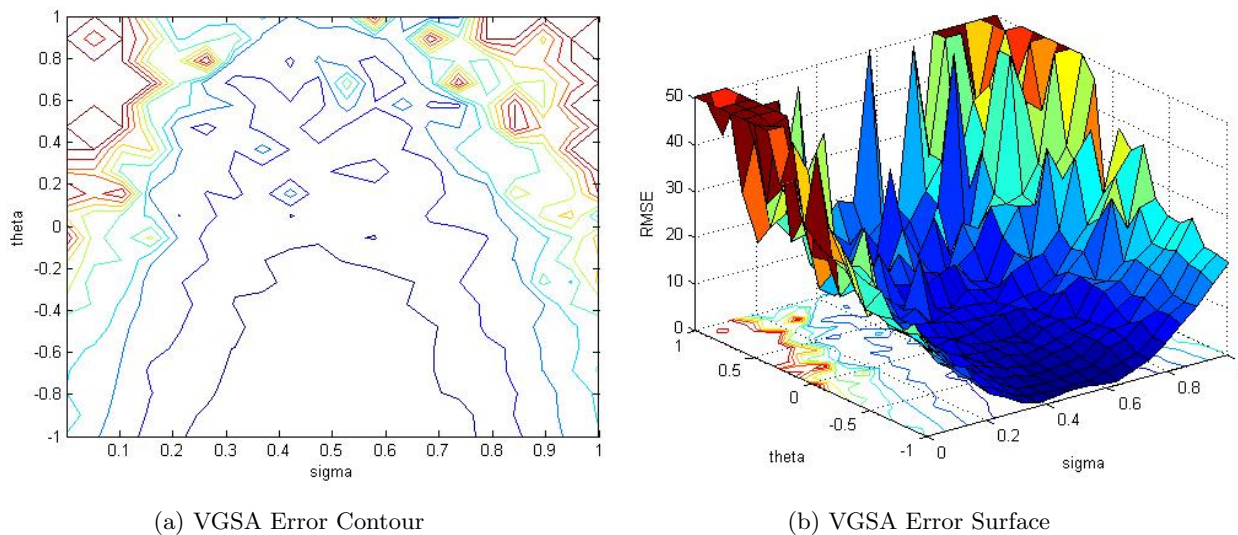Where $b_P = \gamma_P - \int_{-1}^{1} x\nu_P(dx)$ is the drift of the prior $\mathbb{P}$.

(a) VGSA Error Contour   (b) VGSA Error Surface

Figure 4: Naive Least Squares Error Surface

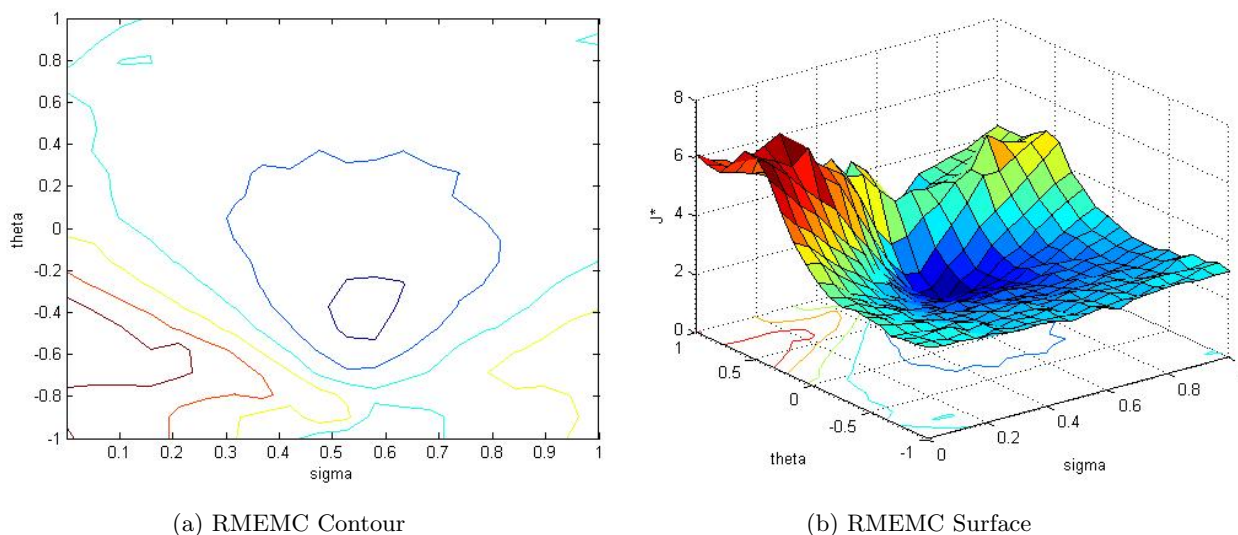(a) RMEMC Contour   (b) RMEMC Surface

Figure 5: Regularized Minimal Entropy Functional

The above images show exactly how nonconvex the naive least squares error surface is. The act of regularizing the error metric with respect to the minimal entropy martingale measure enforces convexity, but does not guarantee convexity. That is what the regularization parameter α is for. For $\alpha = 0$ we obtain the naive least squares problem. For some $\alpha > 0$ we obtain a function that is strictly convex, but our new parameter estimates will become closer to that of the prior measure. Careful choice of α must be made to ensure existence and stability while also reducing the error of the regularized pricing measure with respect to that of the prior. Obtaining a new measure with error greater than that of the prior is clearly undesirable, which will be avoided by limiting the size of α. Ensuring that the pricing error decreases with use of RMEMC, we may have to settle with partial regions of nonconvexity in the RMEMC surface, but this can be overcome. A statistical heuristic for determining multiple starting points lends the minimization problem especially well to parallelization and, combined with the increased convergence rate of RMEMC, will significantly increase the capacity of finding a global optimum.

5.5   RMEMC Algorithm

```matlab
data = xlsread('SPY_Options', 'Prices', 'A2:B69');
K = data(:,1);
MPrice = data(:,2);
n = length(K);


S0 = 90.692;
T = 0.194387;
r = 0.0179;
q = 0;

%Determine prior
start = [0 -1 -1];
options = optimset('MaxFunEvals', 1e10, 'MaxIter', 1e3, 'TolX', 1e-5);
[prior, ~] = fminsearch(@(b) cal_RMSE(b, S0, K, MPrice, r, q, T),...
    start, options);

theta = prior(1);
sigma = prior(2);
nu = prior(3);
%Use volatility derivative as weights
vega = @(S, T, K, sigma) (S*normpdf(1/sqrt(T)*(log(S./K)...
    +1/2*sigma^2*T))*sqrt(T));
w = 1./vega(S0, T, K, sigma);
alpha = 0.03;


charvec = [-50:50];
b = zeros(1,3);
%RMEMC functional
J = @(b, S0, K, MPrice, r, q, T) (cal_RMSE(b,S0,K,MPrice,r,q,T, w))+...
    alpha*RelativeEntropy(cf_VG(charvec,T,r,b(1),b(2),b(3)),...
    cf_VG(charvec,T,r,sigma,nu,theta));

[reg, ~] = fminsearch(@(b) J(b,S0,K,MPrice,r,q,T), start, options);
C = zeros(1,n);
D = zeros(1,n);
for i=1:n
    C(i) = cos_VG(S0, K(i), r-q, T, theta, sigma, nu);
    D(i) = cos_VG(S0, K(i), r-q, T, reg(1),reg(2),reg(3));
end

plot(K, C, 'd', K, MPrice, '.', K, D, '*');
```

6   Parameter Estimation

In the previous section we obtained deterministically the model which minimizes the error between estimated and actual market prices. The calibration procedure utilizes cross-sectional instruments while disregarding time series information. Parameter estimation develops the optimal pricing model in a supplemental fashion by incorporating historical asset movements. Realistic applications often require sample path generation of an unknown distribution inferred by some maximum likelihood or sequential Monte Carlo approach. Maximum likelihood is consistent and guaranteed to converge to the true distribution over time [66]. In cases of singular noise such as GARCH [29, 10, 21] the likelihood function is available in integrated form.

For partially observed processes, this is not the case, and a filtering technique is required. Partially observed processes do not have an explicit integrated density function and so we are forced to utilize the conditional density by calculating the hidden state (variable) on that day to best describe the observation.

6.1 Filtering

We first give an overview of the concept of filtering. It is an iterative process enabling us to determine model parameters using historical time-series data. The idea is to construct a *transition* equation connecting consecutive hidden states and a *measurement* equation connecting the hidden state to the observable data. We first estimate the hidden state at time t *a priori* using all prior information up to time $t-1$. We then construct a conditional *a posteriori* estimate using the prior measurement and the current time t observation. The main concept behind filtering is summarized by the following two steps which are executed recursively.

1. **Prediction (time) update:** Given observations up to time t_{k-1}, we apply the Chapman-Kolmogorov equation to determine the best prediction for x_k at time t_k described by the prior density

$$p(x_k|z_{1:k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|z_{1:k-1})dx_{k-1}.$$

2. **Measurement update:** Given observation z_k we apply Bayes rule to determine the probability of x_k to determine the posterior density

$$p(x_k|z_{1:k}) = \frac{p(z_k|x_k)p(x_k|z_{1:k-1})}{p(z_k|z_{1:k-1})}.$$

A simple yet very effective example of filtering is the Kalman filter. First developed for use in control engineering and signal processing, they have been vital toward the implementation of tracking systems, navigation, guidance systems, and most notably the trajectory estimation system of the Apollo program. The traditional Kalman filter is applicable only to linear systems. We will use the extended Kalman Filter (EKF) which is based upon a first order linearization of the transition and measurement equations which admits application to nonlinear systems.

6.2 Extended Kalman Filter

The extended Kalman filter is used in the special case where the proposal density $p(x_k|z_{k-1})$ and observation density $p(x_k|z_k)$ are nonlinear and Gaussian [44]. They are modeled by a Markov chain built on operators perturbed by Gaussian noise. We assume a dynamic process x_k follows the nonlinear transition equation

$$x_k = f(x_{k-1}, w_k),$$
$$w_k \sim N(0, Q_k).$$

Suppose a measurement z_k follows the observation equation

$$z_k = h(x_k, u_k),$$
$$u_k \sim N(0, R_k).$$

The terms w_k, u_k are two mutually independent sequences of uncorrelated normal random variables representing the process noise and observation noise with covariance matrices Q_k, R_k respectively. Define the a priori estimate of the system given all except the current observation as

$$\widehat{x}_{k|k-1} = \mathbb{E}[x_k|x_{k-1}].$$

Define the a posteriori estimate given all current information as

$$\widehat{x}_{k|k} = \mathbb{E}[x_k|x_k].$$

Then, we can define the a priori and a posteriori error covariance matrices as

$$P_{k|k-1} = \text{cov}(x_k - \widehat{x}_{k|k-1}),$$
$$P_{k|k} = \text{cov}(x_k - \widehat{x}_{k|k}).$$

It is here where the EKF and traditional KF systems differ. We construct a linearization of the transition and measurement equations; the EKF reduces to the KF when the equations actually are linear. The Jacobian matrices of the transition equation with respect to the system process, A_k, and system noise, W_k, are defined as

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(\widehat{x}_{k|k-1}, 0) \qquad \text{and} \qquad W_{ij} = \frac{\partial f_i}{\partial w_j}(\widehat{x}_{k|k-1}, 0).$$

Similarly, the Jacobians of the observation equation with respect to system process H_k and measurement noise U_k are

$$H_{ij} = \frac{\partial h_i}{\partial x_j}(\widehat{x}_{k|k}, 0) \qquad \text{and} \qquad U_{ij} = \frac{\partial h_i}{\partial w_j}(\widehat{x}_{k|k}, 0).$$

This gives us the following time update equations for the a priori state estimate and error covariance

$$\widehat{x}_{k|k-1} = f(\widehat{x}_{k-1|k-1}, 0),$$
$$P_{k|k-1} = A_k P_{k-1} A_k^T + W_k Q_{k-1} W_k^T,$$

and measurement update equations for the respective a posteriori variables

$$\widehat{x}_{k|k} = \widehat{x}_{k|k-1} + K_k(z_k - h(\widehat{x}_{k-1}, 0)),$$
$$P_{k|k} = (I - K_k H_k) P_{k|k-1}.$$

The term K_k represents the optimal Kalman gain matrix found by minimizing the mean square error, or the trace of $P_{k|k}$, over all linear estimators

$$K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + U_k R_k U_k^T)^{-1}.$$

The Kalman gain corresponds to the mean of the conditional distribution of x_k given observation z_k.

Because of the first order linearization required by the EKF algorithm, it is known to fail in substantially nonlinear systems or if the state equations are highly non-Gaussian. An early example of an attempt to correct this shortcoming was to approximate the posterior by expansion in a prespecified function basis known as the Gaussian sum filter [2,47]. Several algorithms have been described which use a deterministic set of points to represent the posterior distribution such as the unscented Kalman Filter (UKF) [45,69] and Gaussian quadrature Kalman Filter (QKF) [40]. These methods have the advantage of not having to compute the Jacobian matrix which is often the most computationally expensive step within the EKF paradigm.

These methods all suffer if the posterior density can be represented by a Gaussian distribution, which is often not the case. Many attempts at successfully overcoming these limitations require the use of Monte Carlo methods to represent the posterior by a collection of random points. The uses of Monte Carlo for filtering of nonlinear processes can be traced to (Gordon et al 1993) [40,36] and were based off sequential importance sampling (SIS). This technique required simulating samples under some proposal distribution and then approximating the target distributions by applying appropriately defined importance weights. The sequential nature of SIS results from, in a nonlinear filtering context, defining the appropriate sequence of state distributions such that regenerating the sample population upon each new observation becomes unnecessary. SIS is a method described more generally as particle filtering.

6.3 Particle Filtering

Also known as sequential Monte Carlo (SMC), particle filtering is a recent alternative [5] for parameter estimation of nonlinear processes by discretizing the continuous density function $p(x_t|y_t)$. For N sample

points we obtain a sequence $\{x_t^{(i)}, w_t^{(i)}\}_{i=1}^N$ where $w_t^{(i)}$ is the weight associated to each particle $x_t^{(i)}$ at time t. The expectation with respect to the filter is

$$\mathbb{E}f(x_t) = \int f(x_t)p(x_t|y_t)dx_t \approx \sum_{i=1}^N w_t^{(i)}f(x_t^{(i)}).$$

The first step in filtering is determining the initial value. Our approach is to use the extended Kalman filter to converge quickly to the prior state value. The optimal proposal distribution is given by the target distribution

$$\pi(x_k|x_{0:k-1}, y_{0:k}) = p(x_k|x_{k-1}, y_k).$$

In practice, the transition prior is often used as the importance function which yields

$$\pi(x_k|x_{0:k-1}, y_{0:k}) = p(x_k|x_{k-1}).$$

We associate the signal noise with R_v and the probability function becomes

$$p(y_{j,k}|x_k) = \frac{1}{\sqrt{2\pi R_{v,j,j}}}e^{-\frac{(y_{j,k}-F_j(x_k))^2}{2R_{v,j,j}}}.$$

The Monte Carlo approximation of the likelihood at step k is:

$$l_k = \sum_{i=1}^N \frac{p(y_k|x_{i,k})p(x_{i,k}|x_{i,k-1})}{\pi(x_{i,k}|x_{i,k-1}, y_k)} = \sum_{i=1}^N w_k^{(i)}.$$

Then, to estimate the parameters we minimize the negative log likelihood

$$-\sum_{i=1}^N \log l_k.$$

The SIS method suffers from a major drawback not properly identified until [34]. The importance weights degenerate over time, known as *weight degeneracy*. The importance weights of most of the samples of the target distribution decrease over time to the point where they do not significantly contribute to the process. The bootstrap filter proposed by [34] solves this issue by regenerating the set of samples with importance weights above a prespecified threshold and disregarding those below it. This was the first successful attempt of applying SMC to nonlinear filtering [13]. Since then, various alternatives have been proposed such as the **con**ditional **dens**ity propag**ation** (condensation) filter [9], Monte Carlo filter [46] and sequential imputations [48]. The bootstrap filter, or sequential importance resampling (SIR) samples N draws from the current set of particles using the normalized weights as individual selection probabilities. Trajectories with small weights are eliminated, and those with larger weights are replicated. The standard particle filtering algorithm generally refers to the SIR method.

6.4   Sequential Importance Resampling

As mentioned earlier, to prevent weight degeneracy causing algorithmic divergence [5] we regenerate particles with higher weight and eliminate those with lower weight. The SIR particle filtering algorithm is described as follows.

1. Simulate the state from the prior by drawing N samples according to the model

$$x_k^{(i)} = f(x_{k-1}^{(i)}, u_{k-1}^{(i)}), \quad i = 1:N.$$

2. Associate the weights for each point by updating the importance function

$$w_k^{(i)} = w_{k-1}^{(i)} \frac{p(z_k|x_k^{(i)})p(x_k^{(i)}|x_{k-1}^{(i)})}{\pi(x_k^{(i)}|x_{k-1}^{(i)}, z_k)}.$$

Letting the proposal distribution $\pi(x_k^{(i)}|x_{k-1}^{(i)}, z_k)$ equal the transition probability $p(x_k^{(i)}|x_{k-1}^{(i)})$, the expression simplifies to

$$w_k^{(i)} = w_{k-1}^{(i)} p(z_k|x_k^{(i)}).$$

3. Normalize the weights

$$\widehat{w}_k(x_k^{(i)}) = \frac{w_k^{(i)}}{\sum_{i=1}^{N} w_k^{(i)}}.$$

4. Resample the weights: compare the CDF of the normalized weights to a uniform CDF, if

$$\frac{1}{N}(U(0,1) + j - 1) \geq \sum_{l=1}^{i} \widehat{w}_k(x_k^{(l)}),$$

then increment and skip i, else set the weight $\widehat{w}_k(x_k^{(i)}) = \frac{1}{N}$.

Therefore, the best estimate of x_k is the conditional expectation

$$\mathbb{E}[x_k|z_{1:k}] \approx \sum_{i=1}^{N} \widehat{w}_k(x_k^{(i)})x_k^{(i)}.$$

6.5   Integrated Densities

For a fully observed process, such as Variance Gamma, its probability density function is available in integrated form. Recall the VG process is defined as

$$X(t; \sigma, \nu, \theta) = \theta\gamma(t; 1, \nu) + \sigma W(\gamma(t; 1, \nu)),$$

with log asset price

$$\ln S_t = \ln S_0 + (r - q + \omega)t + X(t; \sigma, \nu, \theta),$$

where $\omega = \frac{1}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2)$ is the VG martingale correction. Given the following definitions:

$$x_h = z_k - (r - q)h - \frac{h}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2),$$

$$z_k = \ln\left(\frac{S_k}{S_{k-1}}\right),$$

$$h = t_k - t_{k-1}$$

we obtain the integrated VG density function

$$p(z_k|z_{1:k-1}) = \frac{2e^{\theta x_h/\sigma^2}}{\nu^{\frac{h}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{h}{\nu})}\left(\frac{x_h^2}{2\sigma^2/\nu + \theta^2}\right)^{\frac{h}{2\nu} - \frac{1}{4}} K_{\frac{h}{\nu} - \frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x_h^2(2\sigma^2/\nu + \theta^2)}\right)$$

where K_n is the modified Bessel function of the second kind [51].

A partially observed process, such as VGSA, has a density function that is not available in closed form, therefore we condition on a hidden parameter to determine its conditional density function which is available in integrated form.

Recall the log asset price following a VGSA process is defined as

$$d\ln S_t = (r - q + \omega)dt + X(h(dt); \sigma, \nu, \theta),$$
$$X(h(dt); \sigma, \nu, \theta) = B(\gamma(h(dt), 1, \nu); \theta, \sigma)$$

where the gamma cumulative distribution function is

$$F_\nu(h, x) = \frac{1}{\Gamma(\frac{h}{\nu})\nu^{\frac{h}{\nu}}}\int_0^x e^{-\frac{t}{\nu}}t^{\frac{h}{\nu} - 1}dt.$$

The gamma time change is modeled by an integrated CIR process so we set

$$h(dt) = y_t dt,$$
$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dW_t.$$

By conditioning on arrival rate (the hidden parameter) we determine the conditional likelihood function for VGSA

$$p(z_k|h^*) = \frac{2e^{\theta x_h/\sigma^2}}{\nu^{\frac{h^*}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{h^*}{\nu})}\left(\frac{x_h^2}{2\sigma^2/\nu + \theta^2}\right)^{\frac{h^*}{2\nu}-\frac{1}{4}} K_{\frac{h^*}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x_h^2(2\sigma^2/\nu + \theta^2)}\right)$$

for a given arrival rate $dt^* = y_t dt$ and $h^* = y_t h$ [39]. In order to give an intuition behind filtering, we follow the example outlined in [39]. We assume the hidden state (parameter set) evolves linearly, ie.

$$x_{t+1} = ax_t + w_{t+1},$$
$$w_{t+1} \sim N(0, \lambda^2), \qquad \text{f.s. } \lambda \in \Theta,$$

where we assume x_t (the prediction of the current state) is given along with the parameter set Θ. Given observation z_{t+1} at time $t+1$ we want to estimate x_{t+1}

$$\widehat{x}_{t+1} = \mathbb{E}(x_{t+1}|z_{t+1}).$$

Assume the model price is given by $h(x_{t+1}, \Theta)$ and assume the model price is related to the observation z_{t+1} by

$$z_{t+1} = h(x_{t+1}, \Theta) + u_{t+1},$$
$$u_{t+1} \sim N(0, \sigma^2), \qquad \text{f.s. } \sigma \in \Theta.$$

Both λ, σ belong to set Θ so are already known. We now generate M samples for x_{t+1}

$$x_{t+1}^{(i)} = ax_t + N(0, \lambda^2) \quad i = 1:M,$$

then generate M samples for u_{t+1}

$$u_{t+1}^{(i)} = y_{t+1} - h(x_{t+1}^{(i)}; \Theta), \quad i = 1:M.$$

The conditional likelihood function is

$$\mathcal{L}^{(i)} := \text{Likelihood}\left(u_{t+1}^{(i)}\Big|u_{t+1}^{(i)}\right) = \frac{\exp\left(-\frac{(u_{t+1}^{(i)})^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma},$$

so the best estimate for x_{t+1} is

$$\widehat{x}_{t+1} = \mathbb{E}(x_{t+1}|z_{t+1}) = \frac{\sum_{i=1}^{M}\mathcal{L}^{(i)} \times x_{t+1}^{(i)}}{\sum_{i=1}^{M}\mathcal{L}^{(i)}}.$$

6.6   VGSA Parameter Estimation via Particle Filtering

Using the integrated density for the VGSA process, we outline the particle filtering algorithm.

1. Initialize the states $x_0^{(i)}$ and weights $w_0^{(i)}$ for $i = 1:N$ where N is the number of price points available in the data.

2. Apply the extended Kalman filter to each state $x_k^{(i)}$ to obtain the transition update. Define $\widehat{x}^{(i)} = \kappa(\eta - x^{(i)})\Delta t$. The Gaussian approximation for the observation equation can be written as

$$z_k = h(x_k, B_k) = z_{k-1} + (\mu + \omega + \theta x_k)\Delta t + \sqrt{(\theta^2\nu + \sigma^2)x_k\Delta t}B_k.$$

We then determine the Jacobians as follows:

$$A_{ij} = (1 - \kappa)\Delta t,$$
$$W_{ij} = \lambda\sqrt{x^{(i)}\Delta t},$$
$$H_{ij} = \theta\Delta t,$$
$$U_{ij} = \sqrt{\theta^2\nu + \sigma^2},$$

The time update (prior estimate) is

$$\widehat{x}_{k|k-1} = x_k^{(i)} + \kappa(\eta + x_k^{(i)}\Delta t).$$

The measurement update (posterior estimate) is

$$\widehat{x}_k = \widehat{x}_{k|k-1} + K_k(z_k - h_{k|k-1}(\widehat{x}_{k|k-1})),$$
$$= \widehat{x}_{k|k-1} + K_k(z_k - (z_{k-1} + (\mu + \omega + \theta)\widehat{x}_{k|k-1})),$$

which gives us the simulated state

$$\widetilde{x}_k^{(i)} = \widehat{x}_k^{(i)} + \sqrt{P_k^{(i)}}Z, \quad Z \sim N(0,1).$$

3. We can now calculate the weights

$$w_k^{(i)} = w_{k-1}^{(i)} \frac{p(z_k|x_k^{(i)})p(x_k^{(i)}|x_{k-1}^{(i)})}{p(x_k^{(i)}|x_{k-1}^{(i)}, z_k)},$$

where $p(z_k|x_k^{(i)})$ is the integrated VGSA density defined earlier, $p(x_k^{(i)}|x_{k-1}^{(i)}) = N(x_{k-1}^{(i)} + \kappa(\eta - x_{k-1}^{(i)}\Delta t, \lambda\sqrt{x_{k-1}^{(i)}\Delta t})$ and $q(x_k^{(i)}|x_{k-1}^{(i)}, z_k) = N(\widehat{x}_k^{(i)}, \sqrt{P_k^{(i)}})$.

4. We then proceed through the rest of the particle filtering algorithm as explained earlier.

The following VGSA particle filter algorithm will determine the log likelihood to be maximimized via some gradient descent optimization method:

```
function [logl, estimates, errors] = VGSAParticleFilter(log_stock_prices,...
        mu, N, kappa, eta, lambda, sigma, theta, nu)
    Nprices = N;
    Nsims = 100;      %Number of particles
    x = zeros(1,Nsims);
    xsim = zeros(1,Nsims);
    w = zeros(1,Nsims);
    u = zeros(1,Nsims);
    c = zeros(1,Nsims);
    dt = 1/252; %Trading days per year
    eps = 1e-10;
    Pkk1 = zeros(1,Nsims); %a priori error (P_{k|k-1})
    Pkk = zeros(1,Nsims); %a posteriori error (P_{k|k})
    U = zeros(1,Nsims); %Gradient of h WRT measurement noise
    Kk = zeros(1,Nsims); %Kalman gain
    W = zeros(1,Nsims);
    xhat = zeros(1,Nsims);
    xk = zeros(1,Nsims);
    omega = log(1-theta*nu-sigma^2*nu/2)/nu;
    x0 = 1;
    P0 = 0.000001;
```

```
22     x = x0 + sqrt(P0)*randn(1,Nsims);
23     Pkk(:) = P0;
24     A = 1-kappa*dt; %Jacobian WRT system process
25     H = theta*dt; %Gradient of h WRT measurement noise
26     logl = 0; %Log likelihood
27     estimates = zeros(1,Nprices);
28     errors = zeros(1,Nprices);
29     wprev = ones(1,Nsims);
30     for k=1:Nprices-1
31         z = log_stock_prices(k+1)-log_stock_prices(k);
32         xh = z-mu*dt-dt/nu*log(1-theta*nu-sigma^2*nu/2);
33         x1_sum = 0;
34
35         for i=2:Nsims
36             %Simulate the state via extended Kalman filter
37             %Time update
38             xhat(i) = max(eps,x(i)+kappa*(eta-x(i))*dt); %prior transition update
39             W(i) = lambda*sqrt(x(i)*dt); %Jacobian WRT system noise
40             Pkk1(i) = A*Pkk(i)*A + W(i)^2; %prior error estimate
41
42             U(i) = sqrt(theta^2*nu+sigma^2)*sqrt(xhat(i)*dt);
43             %Optimal gain
44             Kk(i) = Pkk1(i)*H/( H*Pkk1(i)*H + U(i)*U(i));
45             %Measurement update
46             xk(i) = xhat(i) + Kk(i) * (z - (mu+omega+theta*xhat(i))*dt);
47             Pkk(i) = (1.0-Kk(i)*H)*Pkk1(i); %Posterior error covariance matrix
48
49             x1_sum = x1_sum + xhat(i);
50
51             %Simulate the state
52             xsim(i) = max(xk(i) + sqrt(Pkk(i))*randn(1), eps);
53             %Calculate weights
54             m = xk(i);
55             s = sqrt(Pkk(i));
56             %Normal density with mean m and stddev s
57             q = 1/(s*sqrt(2*pi))*exp( -0.5*(xsim(i) - m)^2/(s^2));
58
59             m = x(i-1) + kappa*(eta - x(i-1))*dt;
60             s = lambda*sqrt(x(i-1) * dt);
61             %Normal density
62             px = 1/(s*sqrt(2*pi))*exp( -0.5*(xsim(i) - m)^2/(s^2));
63
64             h = dt*xsim(i);
65             %Arguments of Bessel function
66             Kx = max(eps, 1.0/(sigma^2)*sqrt(xh^2*(2*sigma^2/nu+theta^2)));
67             Knu = max(eps, (h/nu-0.5));
68
69             %gammaln/besselk requires appropriate error handlers omitted here
70             %VGSA integrated density
71             pz = 2.0*exp(theta*xh/(sigma^2))...
72             /(nu^(h/nu)*sqrt(2*pi)*sigma*gammaln(h/nu))...
73             *(xh^2/(2*sigma^2/nu+theta^2))^(0.5*h/nu-0.25)...
74             *besselk(Kx,Knu);
75             %weights
```

```
76        w(i) = wprev(i) * pz * px / max(q, eps);
77      end
78      sumweights = sum(w);
79      logl = logl + log(sumweights);
80      % estimates[i1+1] for z[i1] => error term
81      estimates(k+1) = log_stock_prices(k+1)-(log_stock_prices(k) +...
82          (mu+omega+theta*x1_sum/Nsims)*dt);
83      errors(k) = (theta*theta*nu + sigma*sigma)*x1_sum/Nsims*dt;
84
85      w = w./sumweights; %Normalize
86      wprev = w;
87
88      %Resample and reset weights
89      c(1)=0;
90      for i=2:Nsims
91          c(i) = c(i-1) + w(i);
92      end
93      i = 1;
94      for j=1:Nsims
95          u(j) = 1.0/Nsims*(rand+j-1);
96          while u(j) > c(i) && i < numel(c)
97              i = i+1;
98          end
99          xsim(j) = x(i);
100         w(j) = 1.0/Nsims;
101     end
102  end
103  logl = -logl;
104  end
```

7   Optimal Parameter Set

> *Whether cross-sectional option prices are consistent with the time-series properties*
> *of the underlying asset returns is probably the most fundamental of tests.*

- D.S. Bates

The path shown in Fig. 6 will be used for backtesting the calibration and filtering procedures. It was generated with variance gamma using the following parameters: $\sigma = 0.28, \nu = 0.41, \theta = 0.1, T = 1, r = 0.1, q = 0$ over $N = 2520$ corresponding to 10 price changes per trading day for one full year. We aim to fit VGSA to a VG path, here we describe the affect of the extra parameters of VGSA to gain some intuition behind path behavior.

1. CIR time change process: $dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dW_t$,
2. κ: rate of mean reversion,
3. η: long-term rate of change,
4. λ: volatility of the time change.

For all optimization configurations, we impose non-negativity constraints on the above variables.
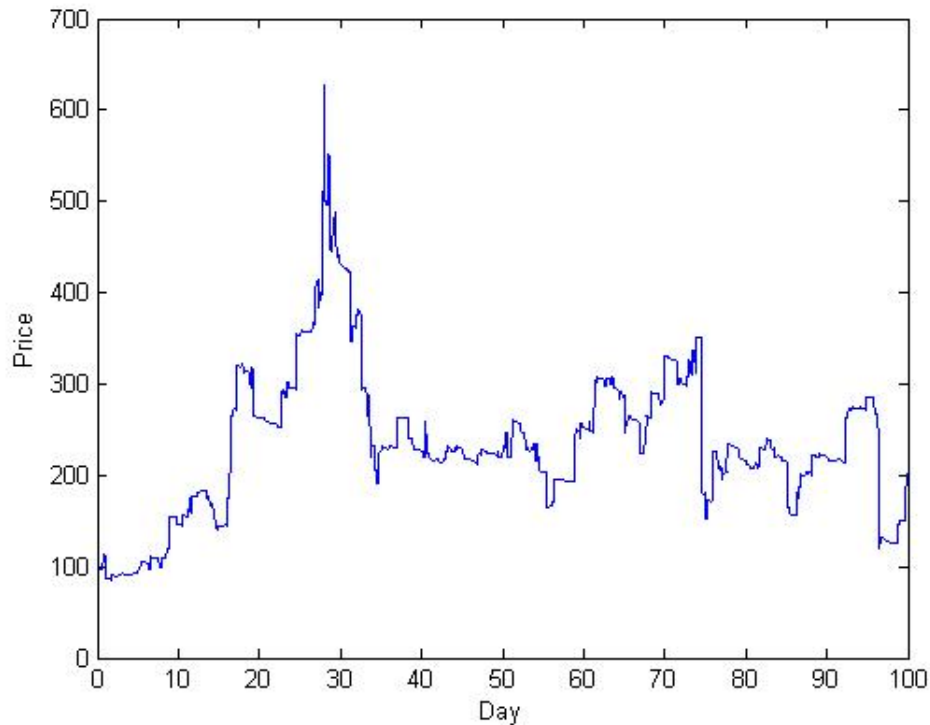
Figure 6: A typical path generated under VG law

7.1   Backtesting RMEMC

We backtest the calibration procedure of section 5 by determining the set of option premiums associated with a stock following a VG process with arbitrary parameters then try to recover the original underlying process. Given random initial conditions satisfying nonegativity constraints for σ, ν we begin by calculating a series of option prices using the asset price at day 1. We use these estimated option prices and the actual observed option prices to determine the Kullback Leibler divergence between the estimated and actual price measures. Throughout the algorithm, at the end of each iteration the estimated pricing measure is used as the regularization prior for the next iteration. Each successive iteration generates the optimal pricing measure by adding the minimum amount of information to the optimal pricing measure for the previous day. Figure 7 shows the sum of squares option pricing errors (SSE) of two estimated pricing measures, with either high or low relative entropy (RE) with respect to the true distribution. The pricing errors are determined across a range of 30 option premiums calculated each trading day, calculated via Monte Carlo with both the true pricing measure and the estimated pricing measure.

Relative entropy is approximate since we used Monte Carlo simulation for VGSA. An advantage of using Monte Carlo to compute option prices as opposed to a transform or PDE method is that way can safely use out of the money options. While Monte Carlo does not suffer from out of the money price skewness, it requires high computational power to obtain a satisfactory level of accuracy.

The minimal entropy martingale measure adds a degree of stability to the estimated pricing measure. Setting the initial condition at time t equal to the optimal parameters of time $t - 1$ generates the source of stability, but convergence to the true distribution will take longer depending on choice of regularization parameter. A sound question is whether to calculate the inter-day pricing measure using information gleaned from all time-series information of stock since time $t = 0$. This is similar to a maximum likelihood procedure. Using only the optimal information from time $t - 1$ we generate the next likely pricing measure as a sequence of approximating measures to the movement of the asset price. This sequence of approximating measures is Markovian.
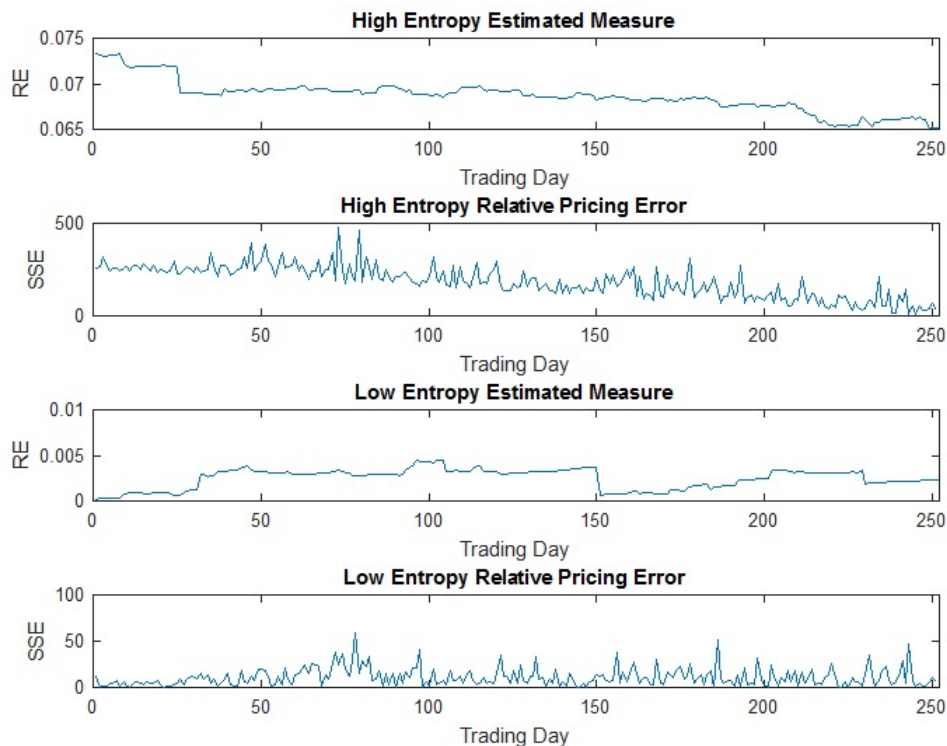
Figure 7: Performance of two predicted pricing measures where the relative entropy represents its 'distance' from the true distribution

Beginning the calibration procedure with parallelized gradient descent algorithms iterating over a grid of initial points yielded the best outcome, similar to that of other global optimization schemes such as simulated annealing. How do we determine which initial condition has the lowest relative entropy with respect to the true distribution? The ability of the calibration procedure to converge to the true distribution is theoretically sound, but in practice it is very slow. In order to make substantial progress we need large stock price observations, which often is not available. As a stock price changes, so may its underlying distribution, and so we are effectively chasing the parameters of a dynamic distribution that's changing faster than the number of observations allow us to estimate.

Minimal entropy least squares calibration is very effective effective at resisting change in parameters however. If we manage to obtain the parameters of the true distribution, then for an appropriate choice of regularization parameter we can take new price movements into account while maintaining close proximity to the measure which accurately predicted historical prices. The notion of minimizing relative entropy is mathematically very similar to maximum likelihood. We see in the next section that non-Gaussian time series information can be accurately characterized via particle filtering, and so a joint method of particle filtering *accounting* for option pricing information will likely produce much better results.

7.2   Backtesting the Particle Filter

We backtest the time series parameter estimation approach by first simulating via Monte Carlo a typical path of a stock following the VGSA process. We then try to recover the original parameters using our model of section 6. Where the nonzero value of λ may account for the large swings of the estimate asset path corresponding to smaller swings of the true asset. Recall λ is the volatility of the time change, the subordination process is what controls the mean reverting behavior of VGSA, and thus larger shifts in the

subordinator may result in more intense price jumps. The filtering procedure appears to be very sensitive to volatility, as represented in Figure 8 which shows the true VG path along with the predicted paths.
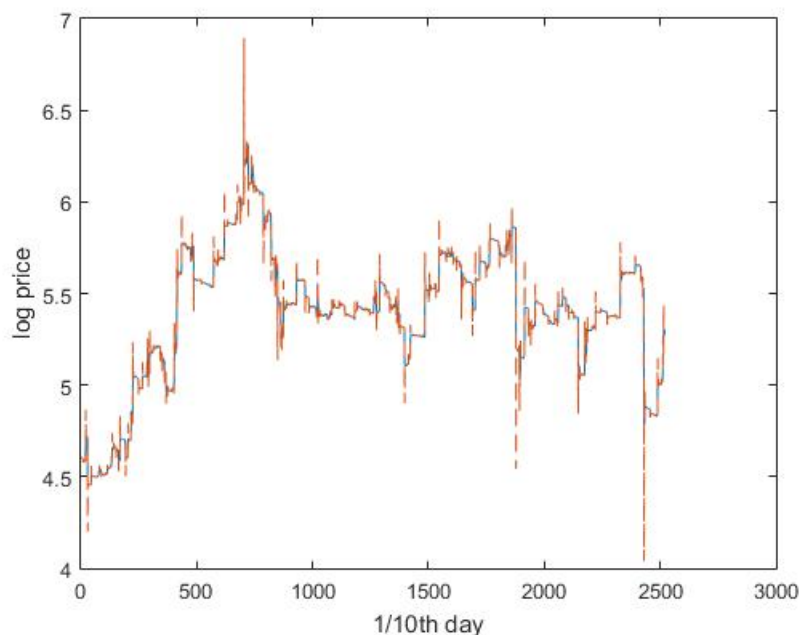

Figure 8: Estimated and True Asset Paths

7.3   Which one is correct?

We have shown two different methods for extracting model parameters from market information. The cross sectional approach of least squares estimation of available option premiums should indeed agree with the parameters obtained from the time-series filtering approach. The filtering approach recovers the statistical parameter set from time-series information. The least squares calibration procedure recovers risk-neutral parameter set from cross-sectional information. Figure 9 shows two paths of the VGSA and VG variety using the parameter sets obtained from times-series and cross sectional data. The estimated path recovered from time-series information is discounted under the risk-neutral measure.
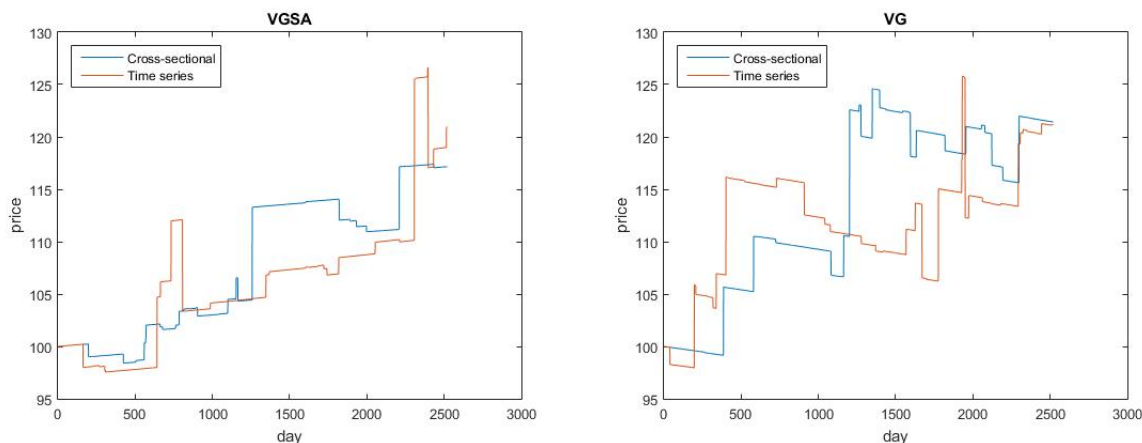

Figure 9: Cross-sectional (RMEMC) and Time series (particle filter) price paths
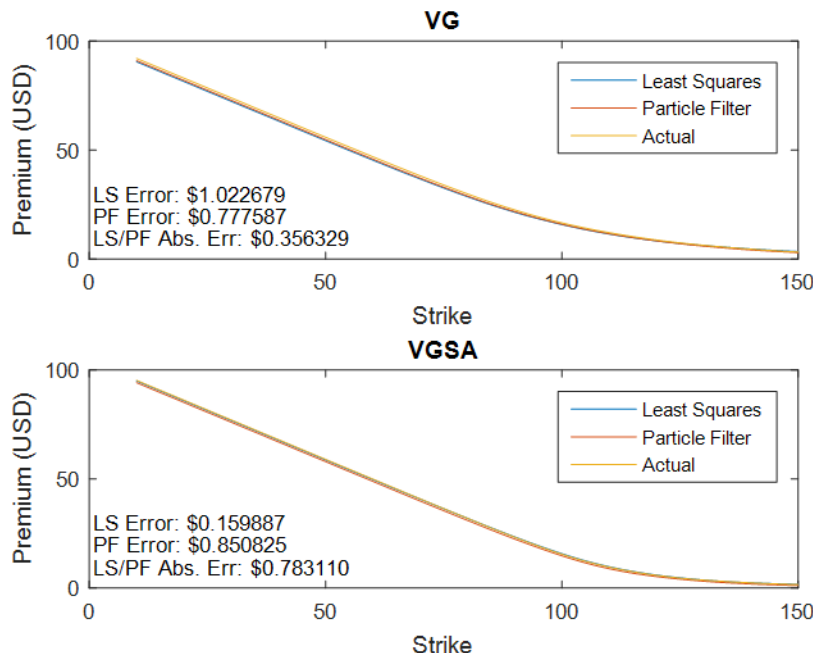
Figure 10: Least Squares/Particle Filter Estimated Option Premiums and Actual Premiums

LS/PF Error: The absolute error between estimated and actual option premiums for a given strike. For each strike, 30 options were calculated, and so the error is absolute average.
LS/PS Abs. Err.: The average absolute error between least squares and particle filter estimated option premiums. This is, on average, how much the RMEMC and PF algorithms agree.

   While it is meaningless to directly compare the optimal parameters from particle filtering and least squares calibration, we can measure the discrepancy between their estimated option premiums. Since RMEMC estimated option premiums do not have to be discounted but PF estimated option prices do, a simple modification to the Monte Carlo pricing algorithm can compare the estimated prices from both. We can see in Figure 10, that these algorithms do in fact agree on their estimated option premiums. Interestingly, the VG estimated prices from RMEMC and PF agree with each other more than they agree with the actual option prices. Much of this error is due to Monte Carlo. It is difficult to determine the effect on Monte Carlo pricing error on these estimates, it is however safe to assert that these methods do in fact agree with each other.

7.4   Future Work

While both RMEMC and particle filtering yield parameter sets that agree with each other quite well, they are both very sensitive to initial conditions, and are restricted to the behavioral capacity of the stochastic process upon which we are modeling. This could be overcome if we can construct a filtering algorithm which does not rely on an explicit density function. Bates [7] shows that direct filtration of latent affine processes is feasible.

Direct Filtration using the Conditional Characteristic Function

Basic affine jump diffusion processes have the form

$$dZ_t = \kappa(\theta - Z_t)dt + \sigma\sqrt{Z_t}dB_t + dJ_t,$$

for $t \geq 0, Z_t \geq 0$. Analysis of transform techniques for asset pricing using variations of this model have been studied in Duffie, Pan, Singleton (2000). Processes for which the conditional characteristic function is exponentially affine in y_t are represented by

$$F(U, V | x_t) = \exp\left(C(U, V) + D(U, V)x_t\right).$$

The time update equation then becomes in terms of the conditional characteristic function

$$F(U, V, z, x | t) = \exp\left(C(U, V)G_{t|t}\left[D(U, V)\right]\right),$$

where $G_{t|s}$ is the moment generating function of x_t conditional on the first s observations. The measurement update equation can also be written in terms of $F(U, V, z, x | t)$. The likelihood function becomes

$$L_{1|T} = \prod_{t=0}^{T-1} \frac{1}{\pi} \int_0^\infty \mathcal{R}\left\{\exp[C(iU, 0) - iU z_{t+1}]G_{t|t}[D(iU, 0)]\right\} dU.$$

We are able to provide direct filtration of any process which has an analytical conditional characteristic function.

Subordination of Levy Processes

The following is due to Sato [64]. Let $\{Z_t\}$ be a subordinator (an increasing real-valued Lévy process) with Lévy measure ρ, drift β_0 and $P_{Z_1} = \lambda$, for instance,

$$\mathbb{E}[e^{-u Z_t}] = \int_{[0,\infty)} e^{-us} \lambda^t(ds) = e^t \Psi(-u), \qquad u \geq 0,$$

where for any complex w s.t. $\mathcal{R}(w) \leq 0$,

$$\Psi(w) = \beta_0 w + \int_{0,\infty} (e^w s - 1)\rho(ds),$$

with

$$\beta_0 \geq 0 \qquad \text{and} \qquad \int_{(0,\infty)} (1 \wedge s)\rho(ds) < \infty.$$

Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ independent of $\{Z_t\}$ with generating triplet (A, ν, γ) and let $\mu = P_{X_1}$. Define

$$Y_t(w) = X_{Z_t(w)}(w), \qquad t \geq 0.$$

Then, $\{Y_t\}$ is a Lévy process on $\mathbb{R}^d$ where

$$P[Y_t \in B] = \int_{[0,\infty)} \mu^s(B)\lambda^t(ds), \qquad B \in \mathcal{B}(\mathbb{R}^d) \text{ (Banach space)}$$

and

$$\mathbb{E}e^{i(z, Y_t)} = e^{t\Psi(\log \hat{\mu}(z))}, \qquad z \in \mathbb{R}^d.$$

The generating triplet $(A^\#, \nu^\#, \gamma^\#)$ of $\{Y_t\}$ is defined as

$$A^\# = \beta_0 A,$$

$$\nu^\#(B) = \beta_0 \nu(B) + \int_{(0,\infty)} \mu^s(B)\rho(ds), \qquad B \in \mathcal{B}(\mathbb{R}^d / \{0\}),$$

$$\gamma^\# = \beta_0 \gamma + \int -(0, \infty)\rho(ds) \int_{|x| \leq 1} x\mu^s(dx).$$

$\{X_t\}$ is transformed to $\{Y_t\}$ by subordination using the subordinator (or directing process) $\{Z_t\}$.

The question remains whether an implicit Lévy measure can be determined via direct particle filtering. This would certainly increase the accuracy of the pricing model, while the affect on generalizability remains uncertain. Another question is can we converge to the optimal density function using a sequence of subordinators? This eliminates the need for an explicit Lévy measure, and however computationally expensive it may be, the sequence can be truncated to obtain a desired level of accuracy.

8   Appendix

8.1   Stochastic calculus

Here we give a brief overview of stochastic calculus as it applies to finance. We define the notion of probability theory, Brownian motion, Ito calculus, and how they allow us to describe the stochastic processes relevant to asset pricing.

Theory of Infinite Probability Spaces

Definition: Let Ω be a nonempty set, let $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is a σ-algebra of Ω iff:

1. The empty set $\emptyset$ is in $\mathcal{F}$,
2. For any $A \in \mathcal{F}$, its complement $A^c \equiv \Omega/A \in \mathcal{F}$ also, and
3. A sequence of sets $A_n \in \mathcal{F}$ implies their union $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Definition: A probability measure $\mathbb{P}$ is a function which, to every set A in the σ-algebra $\mathcal{F}$, assigns a value from $[0,1]$, written as $\mathbb{P}(A)$ and is the combined probability of every event in set A of occurring. It satisfies:

1. $\mathbb{P}(\Omega) = 1$, and
2. Given a sequence A_n of disjoint sets in $\mathcal{F}$,

$$\mathbb{P} \bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

The probability of a sequence of events is equal to the sum of the probabilities of each individual event, provided they are disjoint. We refer to $(\Omega, \mathcal{F}, \mathbb{P})$ as a *probability space.*

Definition: Given probability space $(\Omega, \mathbb{P}, \mathcal{F})$, a *random variable* is a function $X : \Omega \to \mathbb{R}$ such that for every Borel subset $B \in \mathbb{R}$, the set

$$\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\}$$

is in the σ-algebra of $\mathcal{F}$, where ω is a single event in Ω. The *expectation* of a random variable X can be computed as follows:

$$\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega) = \underbrace{\sum_{n=1}^{\infty} X(\omega)\mathbb{P}(\omega)}_{1} = \underbrace{\int_{\Omega} X(\omega)d\mathbb{P}(\omega)}_{2},$$

where,

1. for all $\omega \notin \Omega$ we assign probability 0,
2. is the Lebesgue integral over set Ω with respect to the probability measure $\mathbb{P}$.

On a random variable X whose expectation is defined, its variance, average squared distance from the mean, or second central moment is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}\mathbb{X})^2] = \mathbb{E}[X^2] - [\mathbb{E}X]^2.$$
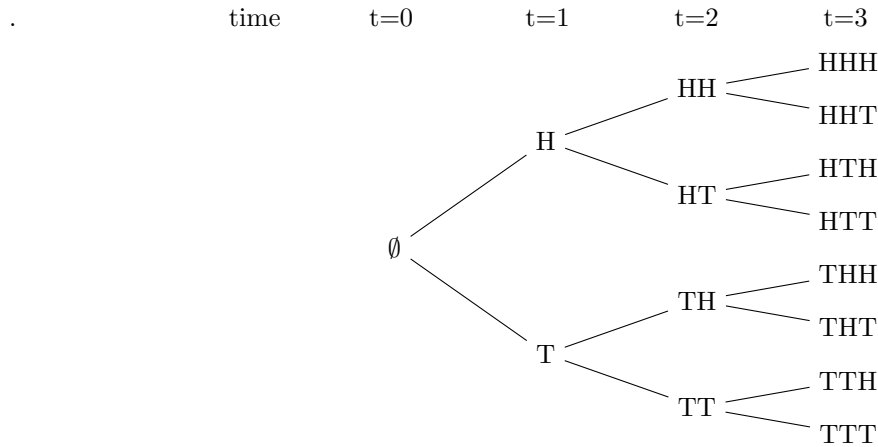
Given another random variable Y, the covariance of X and Y is

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = \mathbb{E}[XY] - \mathbb{E}X \cdot \mathbb{E}Y.$$

If the variances of X and Y are both nonzero, their correlation coefficient is defined as

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

Let us consider an outcome of 3 coin tosses, where one unit of time represents one flip. We can model this by the following tree

.

| time | t=0 | t=1 | t=2 | t=3 |

Then the σ-algebras $\mathcal{F}_t$ at time t follow

t=0: $\mathcal{F}_0 = \{\emptyset, \Omega\}$

t=1: $\mathcal{F}_1 = \{\emptyset, \Omega, H, T\}$

t=2: $\mathcal{F}_2 = \begin{cases} \emptyset, \Omega, H, T, HH, TT, HT, TH, HH^c, TT^c, HT^c, TH^c, \\ HH \cup TT, HH \cup TH, HT \cup TH, HT \cup TT \end{cases}$

Where $HH^c = HT \cup TH \cup TT$ is the complement of HH. This sequence of σ-algebras is known as a *filtration*. A filtration has the property that for any $s \leq t$ the sets in $\mathcal{F}_s$ are also in $\mathcal{F}_t$. It can also be thought of as the accumulation of information as time advances. Given what has already happened, a filtration will also tell us what can and cannot happen. For instance, if at $t = 1$ our coin lands on heads, we know that at time $t = 2$ the only feasible sets are $\{HH, HT\}$. In probability, information is measured in terms of σ-algebras.

Measurability

Given a σ-algebra $\mathcal{H}$, a random variable X is $\mathcal{H}$-measurable iff the set $\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\}$ is in $\mathcal{H}$ for every Borel set $B \in \mathbb{R}$. This means that the information in $\mathcal{H}$ is sufficient to determine the value of $X(\omega)$. Conversely, if the information contained within $\mathcal{H}$ is irrelevant to the calculation of X, we say X is *independent* of $\mathcal{H}$. Often times we encounter a random variable only partially dependent on this information, at which point we use the information in $\mathcal{H}$ to estimate the value of X, called the *conditional expectation* and denoted $\mathbb{E}[X|\mathcal{H}]$.

Definition: Let $\mathcal{F}(t)$, $0 \leq t \leq T$ be a filtration of sub-σ-algebras of $\mathcal{F}$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A sequence of random variables $X(t)$ is an *adapted stochastic process* if for every $t \in [0, T]$, $X(t)$ is $\mathcal{F}_t$-measurable.

Consider a filtration as being the flow of public information and the value of a stock or portfolio as being an adapted stochastic process. We may estimate the value of the process based on the information we have at the time. This brings us to two very important types of adapted stochastic processes: the martingale and the Markov process.

Definition: Let $\mathcal{F}(t)$, $0 \leq t \leq T$ be a filtration of sub-σ-algebras of $\mathcal{F}$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. An adapted stochastic process $M(t)$ is a martingale if it satisfies:

$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s) \,\forall\, 0 \leq s \leq t \leq T,$$

where the expectation is constant over time. Given the information available at time s, the expected value of a martingale $M(t)$ at time $t \geq s$ is equal to the value of the martingale at time s. A martingale has no tendency to rise or fall. Extending the definition of martingale yields

1. A *submartingale* has no tendency to fall, may have tendency to rise, and $\forall\, 0 \leq s \leq t \leq T$ satisfies

$$\mathbb{E}[M(t)|\mathcal{F}(s)] \geq M(s).$$

2. A *supermartingale* has no tendency to rise, may have tendency to fall, and $\forall\, 0 \leq s \leq t \leq T$ satisfies

$$\mathbb{E}[M(t)|\mathcal{F}(s)] \leq M(s).$$

Definition: Let $\mathcal{F}(t)$, $0 \leq t \leq T$ be a filtration of sub-σ-algebras of $\mathcal{F}$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Given an adapted stochastic process $X(t)$, assume that for all $0 \leq s \leq t \leq T$ and every Borel-measurable function $f \geq 0$, there exists another Borel-measurable function g such that

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s)),$$

then $X(t)$ is a *Markov process*. A Markov process is often labeled as 'memoryless' in that having past information yields no greater benefit in predicting future values than having just the current value does. The estimate of $f(X(t))$ at time s depends only on the value $X(s)$ and is independent of path of the process before s.

Brownian Motion as a Scaled Random Walk

We now define another stochastic process at the heart of many models of finance. We obtain the definition of Brownian motion through the limit of scaled random walks. Consider a sequence of random variables X_j representing successive flips of a fair coin and defined by

$$X_j = \begin{cases} 1 & \text{if } \omega_i = H \\ -1 & \text{if } \omega_i = T, \end{cases}$$

where ω_i is the result of the coin on the ith flip. Then a random walk W_j is defined as

$$W_j = \sum_{i=1}^{j} X_i, \quad \forall j.$$

The value of X_j determined by an independent coin flip implies that $X_i \perp\!\!\!\perp X_j$, $\forall i \neq j$. The fair coin flip implies the random walk is symmetric in that the probability of increasing or decreasing are equal. The increments of a random walk are random variables defined by

$$M_{j_1} = (M_{j_1} - M_{j_0}),$$
$$\dots$$
$$M_{j_n} = (M_{j_n} - M_{j_{n-1}}),$$

and are independent of each other. It is clear that a symmetric random walk has increments satisfying $\mathbb{E}[M_{j_n} - M_{j_{n-1}}] = 0$. Moreover, the variance of each increment satisfies

$$\text{var}[M_{j_n} - M_{j_{n-1}}] = j_n - j_{n-1}.$$

This brings us to an important property.

Proposition: The symmetric random walk is a martingale.
Proof: For any positive integers $s < t$, the conditional expectation satisfies:

$$\begin{aligned}
\mathbb{E}[M_t|\mathcal{F}_s] &= \mathbb{E}[(M_t - M_s) + M_s|\mathcal{F}_s] \\
&= \mathbb{E}[M_t - M_s|\mathcal{F}_s] + \mathbb{E}[M_s|\mathcal{F}_s] \\
&= \mathbb{E}[M_t - M_s] + M_s \\
&= 0 + M_s = M_s
\end{aligned}$$

$\square$

If we consider a random walk over n time steps, then as we let $n \to \infty$ while keeping the size of the interval the same, its distribution converges to the normal distribution, which is exactly that of a Brownian motion.

Definition: Define probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A continuous function $W(t)$ satisfying $W(0) = 0$ with independent, normally distributed increments is a Brownian motion if

$$\mathbb{E}[W_{j_n} - W_{j_{n-1}}] = 0, \quad \text{and}$$
$$\text{var}[W_{j_n} - W_{j_{n-1}}] = j_n - j_{n-1}.$$

Brownian motion satisfies the martingale property. This holds by the exact same logic as that of the symmetric random walk.

Proposition: A Brownian motion $W(t)$ is a Markov process.
Proof: To show that there exists a Borel-measurable function g s.t. $\mathbb{E}[f(W(t))|\mathcal{F}_s] = g(W(s))$ we first write

$$\mathbb{E}[f(W(t))|\mathcal{F}_s] = \mathbb{E}[f((W(t) - W(s)) + W(s))|\mathcal{F}_s].$$

Then, noting that the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$ and normally distributed with mean 0 and variance $t - s$, we replace $W(s)$ with dummy variable x and define

$$g(x) = \mathbb{E}f(W(t) - W(s) + x)$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(w + x) \exp\left\{-\frac{w^2}{2(t-s)}\right\} dw.$$

Replacing x with $W(s)$ we see that $g(W(s))$ holds.   □
   A remarkable property of Brownian motion is that it accumulates quadratic variation at a rate of one per unit time, which we write as

$$dW(t)dW(t) = dt.$$

A consequence of this is that for any interval of time T, partitioning the interval into infinite steps such that the length of each interval $\|T_i\| \to 0$ we find that the quadratic variation of Brownian motion within the interval is equal to the length of the interval itself. Moreover, Brownian motion satisfies:

$$dW(t) = dt, \quad dtdt = 0.$$

8.2   Ito calculus

Named after Kyoshi Ito [41], Ito calculus is a generalization of calculus through which the integrands and integrators become stochastic processes. Let $W(t)$ be a Brownian motion with filtration $\mathcal{F}_t$ and let $\Delta(t)$ be an adapted stochastic process. The Ito integral, defined by

$$I(t) = \int_0^t \Delta(u)dW(u),$$

has the following properties

1. **Continuity:** The paths of $I(t)$ are continuous
2. **Adaptivity:** $I(t)$ is $\mathcal{F}_t$-measurable for each t
3. **Martingale:** $I(t)$ is a martingale (constant expectation for all t)
4. **Ito isometry:** $\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)dW(u)$
5. **Quadratic variation:** $[I, I](t) = \mathbb{E}\int_0^t \Delta^2(u)dW(u)$

and is defined in differential notation as $dI(t) = \Delta(t)dW(t)$. The Ito integral is a stochastic adaptation of the Reimann-Stieltjes integral. Since $I(t)$ is a martingale and $I(0) = 0$ then $\mathbb{E}I(t) = 0 \; \forall \; t$. The variance of the Ito integral is given by Ito's isometry

$$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du.$$

Its quadratic variation up to time t is denoted by

$$[I, I](t) = \int_0^t \Delta^2(u) du$$

and is captured in differential notation as

$$dI(t)dI(t) = \Delta^2(t)dW(t)dW(t) = \Delta^2(t)dt.$$

An Ito process is of the form [67]

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du \quad \equiv \quad dX(t) = \Delta(t)dW(t) + \Theta(t)dt$$

where $\Delta(u), \Theta(u)$ are adapted stochastic processes as $X(0)$ is nonrandom. The accumulated quadratic variation over an interval $[0, t]$ is

$$[X, X](t) = \int_0^t \Delta^2(u) du,$$

equivalently,

$$dX(t)dX(t) = \Delta^2(t)dW(t)dW(t) + 2\Delta(t)\Theta(t)dW(t)dt + \Theta^2(t)dtdt = \Delta^2(t)dt.$$

Consider two Ito processes X, Y with differentials:

$$dX(t) = \Theta_1(t)dt + \sigma_{11}(t)dW_1(t) + \sigma_{12}dW_2(t),$$
$$dX(t) = \Theta_2(t)dt + \sigma_{21}(t)dW_1(t) + \sigma_{22}dW_2(t),$$

where $W_1(t), W_2(t)$ are independent Brownian motions. Multiplying the above yields

$$dX(t)dX(t) = (\sigma_{11}^2(t) + \sigma_{12}^2(t))dt,$$
$$dX(t)dY(t) = (\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t))dt,$$
$$dY(t)dY(t) = (\sigma_{21}^2(t) + \sigma_{22}^2(t))dt.$$

The multidimensional Ito formula shows that, given a function $f(t, x, y)$ with time t and dummy variables x, y, its differential satisfies

$$df(t, X(t), Y(t) = f_t(t, X(t), Y(t))dt + f_x(t, X(t), Y(t))dX(t) + f_y(t, X(t), Y(t))dY(t)$$
$$+ \frac{1}{2}f_{xx}(t, X(t), Y(t))dX(t)dX(t) + f_{xy}(t, X(t), Y(t))dX(t)dY(t) + \frac{1}{2}f_{yy}(t, X(t), Y(t))dY(t)dY(t),$$

which is just the Taylor expansion of f omitting the terms with differentials $dX(t)dt, dY(t)dt$ as these are 0. If we eliminate $Y(t)$ from the function we are left with the one-dimensional Ito formula

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))dX(t)dX(t).$$

If we define $f(t, x, y) = xy$ we obtain Ito's product rule

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t).$$

Using Ito's formula, we now derive the Black-Scholes PDE [67].

Black-Scholes PDE

Define the underlier $S(t)$ by a geometric Brownian motion

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t).$$

Define $c(t, S(t))$ to be the price of a European call with underlier $S(t)$ and strike K paying $(S(T) - K)^+$ at expiration. If we sell this call at time $t = 0$ for $X(0) = c(0, S(0))$ to obtain initial capital $X(0)$ we invest in a money market account paying constant interest r and given $\Delta(t)$ shares of $S(t)$ the portfolio differential becomes

$$dX(t) = \Delta dS(t) + r(X(t) - \Delta(t)S(t))dt.$$

Equating the differential of the discounted (risk-neutral) portfolio $e^{-rt}X(t)$ and the discounted call price $e^{-rt}c(t, S(t))$ we obtain the delta-hedging rule

$$\Delta(t) = c_x(t, S(t)),$$

and the Black-Scholes PDE

$$c_t(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t, x) = rc(t, x).$$

The solution to this PDE will be the call price of the option, $c(t, S(t))$. It must satisfy the boundary conditions

$$c(T, x) = (x - K)^+, \quad c(t, 0) = 0, \quad \text{and} \quad \lim_{x \to \infty} \left[c(t, x) - (x - e^{-r(T-t)}K) \right] = 0.$$

The solution is the Black-Scholes price of a European call defined as

$$c(t, x) = xN(d_+(T - t, x)) - Ke^{-r(T-t)}N(d_-(T - t, x)),$$

with

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau \right].$$

8.3   Infinitely Divisible Distributions

The most general type of process represented in this paper is the Lévy process. Named after French mathematician Paul Lévy, a Lévy process is a stochastic process with independent, stationary increments; it can be represented most simply as the continuous time version of the random walk [8].

Take the discrete random walk and construct a continuous time process whose unit interval is divided into n equal lengths which at time $t = 1$ has distribution D. The increments of this process are independent from the distribution $D^{(n)}$:

$$Y_t^{(n)} = \sum_{j=1}^{\lfloor tn \rfloor} C_j^{(n)}, \qquad \text{with} \qquad C_j^{(n)} \overset{i.i.d}{\sim} D^{(n)}.$$

If we let $n \to \infty$ then this process converges to a Brownian motion, which satisfies our results in the previous section. All distributions for which this construction holds are those for which D is *infinitely divisible*. These are Lévy processes.

Definition: a cadlag stochastic process $Y = \{Y_t\}_{t \geq 0}$ with $Y_0 = 0$ is a Lévy process iff it has independent and strictly stationary increments.

The cadlag criteria literally means 'continu á droite, limites á gauche' and translates to right continuous $(\lim_{s \downarrow t} Y_s = Y_t)$ with left limits $(Y_{t-} = \lim_{s \uparrow t} Y_s)$. The stationarity assumption ensures that the distribution of any increment $Y_{t+s} - Y_t$ may change with s but does not depend on t. Combined with the assumption that all increments are independent from each other implies that the Lévy process is fully governed by the cumulant function of Y_t

$$C_{Y_t}\{\theta\} = \log\left[\mathbb{E}\exp\{i\theta Y_t\}\right] = t\log\left[\mathbb{E}\exp\{i\theta Y_1\}\right] = tC_{Y_t}\{\theta\}.$$

This shows that the distribution of Y_t depends only on the value of the process at time $t = 1$.

Lévy-Khintchine Representation

The Lévy-Khintchine representation for Lévy processes with increments on the entire real line required the following developments. The Lévy measure must be allowed to have support on the real line but exclude the possibility that the measure has an atom at zero. We imagine that an independent Brownian motion is added to the process. With $a \in \mathbb{R}, \sigma \geq 0$ and the Levy measure ν satisfying:

$$\int_{\mathbb{R}} \min\{1, y^2\}\nu(dy) < \infty, \qquad \text{and } \nu \text{ has no atom at zero.} \tag{8.3.1}$$

Then, the Lévy-Khintchine representation for Lévy process Y in terms of its cumulant function is:

$$C_{Y_1}\{\zeta\} = ai\zeta - \frac{1}{2}\sigma^2\zeta^2 + \int_{\mathbb{R}} \left(e^{iy\zeta} - 1 - iy\zeta\mathbb{I}_{y\in[-1,1]}\right)\nu(dy).$$

Lévy processes can be fully represented by a characteristic triplet (a, σ^2, ν) with drift a, variance of the diffusion component (Brownian motion) σ^2 and Lévy measure ν satisfying (8.3.1). Processes such that $\nu = 0$ do not have jumps and are instead a Brownian motion with drift. Processes such that $\sigma = 0$ are of pure jump variety such as variance gamma. All Lévy processes can be decomposed into the sum of a drift a, a Brownian motion W_t, and an independent pure jump process L_t:

$$Y_t = at + \sigma W_t + L_t$$

Lévy-Ito decomposition

Further more, any Lévy process can be decomposed into a Brownian motion with drift, a compensated jump process with jump sizes less than ϵ, and a jump process with jump sizes greater than or equal to ϵ. [1]

Definition: Levy-Ito representation. Let L be a Levy process with Levy measure ν then L has the following representation:

$$L_t = at + \sigma W_t + \int_0^t \int_{\mathbb{R}} y\{N(dy, ds) - \mathbb{I}_{y\in[-1,1]}\}\nu(dy)ds$$

where W_t is a Brownian motion and N is a Poisson basis on $\mathbb{R}^> \times \mathbb{R}$ independent of W_t and with measure $\nu(dy)dt$. We may define the average number of jumps per unit time $\lambda = \int \nu(dx) < \infty$ then X_t is of finite activity and can be used to describe the distribution of jump sizes:

$$\mu(dx) = \frac{\nu(dx)}{\lambda}.$$

Then, X_t is a compound Poisson process. The truncation of small jumps can be omitted from the Lévy-Khintchine representation which becomes

$$\mathbb{E}e^{izX_t} = \exp\left\{t\left[-\frac{z^2\sigma^2}{2} + i\gamma_0 z + \int_{-\infty}^{\infty}(e^{izx} - 1)\nu(dx)\right]\right\}.$$

We turn our attention to exponential Lévy models which have the form:

$$S_t = e^{rt+X_t}$$

where there exists a measure $\mathbb{Q}$ under which S_t is a martingale. Here r denotes the interest rate and X_t has characteristic triplet (σ, γ, ν) upon which the constraint

$$\varphi(1) = 0 \iff \gamma = \gamma(\sigma, \nu) = -\frac{\sigma^2}{2} - \int (e^y - 1 - y\mathbb{I}_{|y|\leq 1})\nu(dy)$$

---

[1] The alternate form for the Lévy-Ito decomposition is:

$$L_t = at + \sigma W_t + \int_{\{|y|<\epsilon\}} y\left(N_t(dy) - t\nu(dy)\right) + \int_{\{|y|\geq\epsilon\}} y\left(N_t(dy)\right)$$

Where $N_t(dy) = \int_0^t N(dy, ds)$

is imposed since the process $e^{-rt}S_t = e^{X_t}$ is a martingale. A profound consequence is that the entire Lévy process simply corresponds to different parameterizations of its Lévy measure. The Lévy measures for some processes used in this paper are now shown. The measure for VGSA is not available in closed form.

Variance gamma:

$$d\nu(x) = k(x)dx,$$

$$k(x) = \frac{e^{-\lambda_p x}}{\nu x}\mathbb{I}_{x>0} + \frac{e^{-\lambda_n |x|}}{\nu |x|}\mathbb{I}_{x<0},$$

$$\lambda_p = \left(\frac{\theta^2}{\sigma^2} + \frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}} - \frac{\theta}{\sigma^2},$$

$$\lambda_n = \left(\frac{\theta^2}{\sigma^2} + \frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}} + \frac{\theta}{\sigma^2}.$$

CGMY:

$$k(x) = C\frac{e^{-Gx}}{x^{1+Y}}\mathbb{I}_{x>0} + \frac{e^{-M|x|}}{|x|^{1+Y}}\mathbb{I}_{x<0},$$

where it is easy to see that CGMY can be reduced to variance gamma.

8.4   The Characteristic Function

The characteristic function of a random variable X is the Fourier transform of the probability density function $p(x)$ and defined as:

$$\varphi(\nu) = \mathbb{E}e^{i\nu X} = \int_{-\infty}^{\infty} e^{i\nu x}p(x)dx$$

$$= \int_{-\infty}^{\infty} p(x)dx + i\nu \int_{-\infty}^{\infty} xp(x)dx + \frac{1}{2}(i\nu)^2 \int_{-\infty}^{\infty} x^2 p(x)dx$$

$$= \sum_{k=0}^{\infty} \frac{(i\nu)^k}{k!}\mathbb{E}[X^n]$$

where $\mathbb{E}[X^n] = i^{-n}\frac{d}{d\nu^n}\varphi(\nu)\big|_{\nu=0}$ is the n-th moment of random variable X. Fully identified by its characteristic function, a random variable's cumulative distribution can be written as

$$F(x) = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\nu x}\varphi(-\nu) - e^{-i\nu x}\varphi(\nu)}{i\nu}d\nu$$

and its density function can be found by Fourier inversion

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\nu x}\varphi(\nu)d\nu = \frac{1}{\pi}\int_{0}^{\infty} e^{-i\nu x}\varphi(\nu)d\nu.$$

For example, given the normal probability density function with mean μ and variance σ^2,

$$p_N(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

its characteristic function can be determined:

$$\varphi_N(\nu) = \int_{-\infty}^{\infty} e^{i\nu x}p_N(x)dx = e^{i\mu\nu}e^{-\frac{1}{2}\sigma^2\nu^2}.$$

Similarly, given the gamma probability density function with shape k and scale θ,

$$p_\Gamma(x) = \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\theta^k\Gamma(k)},$$

where $\Gamma(k)$ is the gamma function, its characteristic function is

$$\varphi_\Gamma(\nu) = (1 - i\nu\theta)^{-k}.$$

Since the characteristic function completely determines all information about the probability distribution of a stochastic process, we use it as an alternate analytical route for deriving numerical solutions.

9   References

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