Vassiliev Invariants, Chord Diagrams, and Jacobi Diagrams

By

John Dougherty X

Abstract:

The goal of this paper is to understand the topological meaning of Jacobi diagrams in relation to knot theory and finite type knot invariants, otherwise known as Vassiliev invariants. To do so, we will first set up the notions of: the Vassiliev invariant and its respective space, the space of chord diagrams, a weight system on chord diagrams, and Jacobi diagrams and their respective space. During this process we will prove some of the more significant results from the topics and briefly justify their motivation. Once we have done this, we will then consider how Jacobi diagrams relate to singular knots and introduce Habiro's clasper to provide a mapping between these two spaces.

1. The Vassiliev Knot Invariant and Classical Knot Polynomials:

Definition of Vassiliev Invariants:

Let $\mathcal{K}$ be the vector space over $\mathbb{C}$ freely spanned by oriented knots in S^3. Then a *singular knot* is an immersion of S^1 into S^3 whose singularities are transversal double points. Furthermore, we can think of a singular knot as an element in $\mathcal{K}$ by removing the singularity with the following relation:
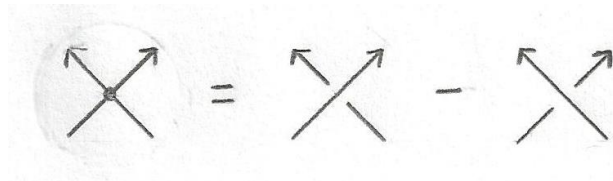

Example:
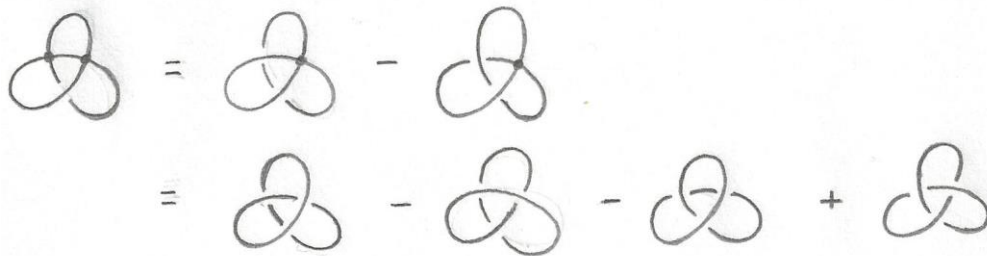
Next, let $\mathcal{K}_m$ denote the subspace of $\mathcal{K}$ spanned by singular knots with m double points. We then call a linear map $v : \mathcal{K} \to \mathbf{F}$ a *Vassiliev invariant* (or *finite type invariant*) *of degree m* if v vanished on $\mathcal{K}_{m+1}$. Furthermore, the space $\mathcal{V}$ of all Vassiliev invariants is *filtered* with $\mathcal{V}_m$ defined as the set of all degree m Vassiliev invariants.

The Polynomial Knots: Vassiliev invariants are important in understanding the structure of the polynomial invariants. To this note, we will show that the coefficients of some of the more famous polynomial invariants, including the Jones polynomial, are derived from finite type Vassiliev invariants.

Let L_1, L_+, L_-, and L_0 be local portions of a knot diagram as shown below, and let O be any diagram of the unknot. In doing so, we can now consider the polynomial knot invariants which we will define by their respective skein relations.

$$L_1 \qquad L_+ \qquad L_- \qquad L_0$$
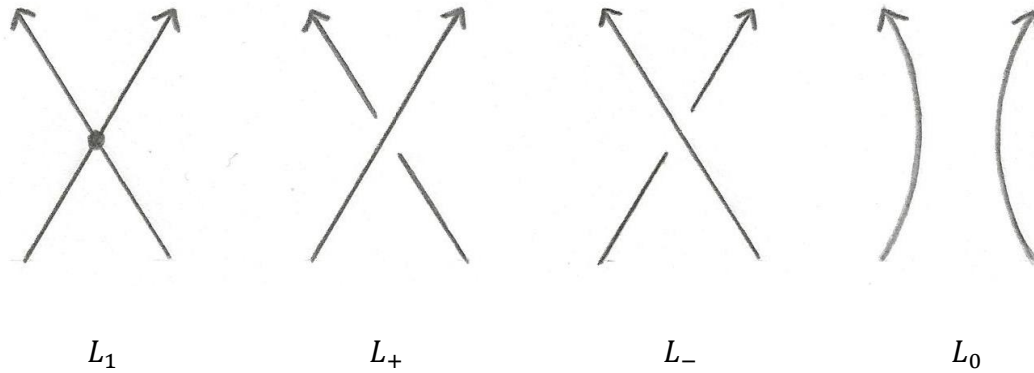
Conway Polynomial: The Alexander-Conway polynomial is a knot invariant which satisfies the two skein relations:

i) $\quad C(O) = 1$

ii) $\quad C(L_+) - C(L_-) = zC(L_0)$

HOMFLY Polynomial: The HOMFLY polynomial is a polynomial invariant of two variables satisfying the skein relations:

i) $\quad P(O) = 1$

ii) $\quad \alpha P(L_+) + \alpha^{-1}P(L_-) + zP(L_0) = 0$

Jones Polynomial: The Jones polynomial is another polynomial knot invariant which is a Laurent polynomial with integer coefficients in the variable $t^{+1/2}_{-}$. Typically it is defined using the writhe and the Kauffman Bracket, but is characterized by the skein relations:

i) $\quad V(O) = 1$

ii) $\quad \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)V(L_0) = t^{-1}V(L_+) - tVL_-)$

We note that both the Conway and Jones polynomials can be obtained from the HOMFLY polynomial after change of variables. Furthermore, the HOMFLY polynomial can be parameterized additional ways, and we will use one of these parameterizations when proving the below theorem.

Theorem 1: *Each coefficient of the Conway polynomial is an invariant of finite type. Furthermore, after a suitable change of variables, each coefficient in the Taylor expansion of the Jones and HOMFLY is an invariant of finite type.*

> ***Proof:*** First we will show that the coefficients of the Conway polynomial are finite type invariants. Let K be a knot, and let $C(K)(z)$ be its corresponding Conway polynomial. Further, let C also denote the natural extension of the Conway polynomial to singular knots. Then, by definition of the Conway polynomial we have
>
> $$C(L_1) = C(L_+) - C(L_-) = zC(L_0).$$
>
> With this, we notice that if K has more than m double points, $C(K)$ is divisible by z^{m+1} and so the coefficient of z^m in $C(K)$ vanishes. Thus it follows that the mth coefficient of the Conway polynomial is a Vassiliev invariant of degree m.
>
> Now we consider the HOMFLY polynomial. As above, for the HOMFLY invariant $P(K)$, let P also denote the natural extension of the HOMFLY polynomial to singular knots. Next, we note that the HOMFLY polynomial has a standard parameterization in two parameters q and N satisfying the identity
>
> $$q^{N/2}P(L_+) - q^{-N/2}P(L_-) = \left(q^{1/2} - q^{-1/2}\right)P(L_0).$$
>
> Next we notice that with the change of variables $q = e^x$ and expanding in powers of x we can rewrite the above identity as
>
> $$P(L_+) - P(L_-) = x \cdot (some\ mess).$$
>
> In doing so, by the same argument as above, we have that the mth coefficient of HOMFLY polynomial is a degree m Vassiliev invariant. Furthermore, for the HOMFLY polynomial, when $N \equiv 2$ we yield the Jones polynomial and hence the mth coefficient of Jones polynomial is also a finite type invariant of degree m.
>
> $\blacksquare$

2. Chord Diagrams and a Weight System:

Definition: A *chord diagram* is an oriented circle with finitely many chords marked on it, regarded up to orientation preserving diffeomorphisms of the circle. We will denote the collection of all chord diagrams by D^c. Note that this collection is naturally graded by the number of chords in such a diagram. Let $G_m D^c$ denote the part of D^c which is the collection of all chord diagrams with precisely m chords.

> *Remark*: By convention we will always orient the circle in a chord diagram counterclockwise and will always use dashed lines for the chords.

Definition: An **F**-valued *weight system* of degree m is a function $W : G_m D^c \rightarrow \mathbf{F}$ with the following properties:

(1) If $D \in G_m D^c$ has an *isolated chord* (a chord that does not intersect any other chord in D), then $W(D) = 0$. This property is called *framing independence*.

(2) Whenever the four diagrams $S, E, W,$ and N differ only as shown below, their weights satisfy

$$W(S) - W(E) = -W(W) + W(N)$$

This property is called the $4T$ relation.
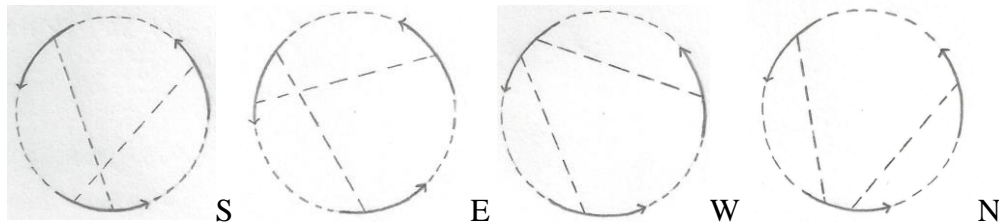
Let $\mathcal{W}$ denote the graded vector space of all weight systems.

Constructing a Weight System: Let v be a degree m knot invariant, and let $D \in G_m D^c$ a degree m chord diagram. Now an *embedding* of D in $\mathbf{R}^3$ will be an immersion $K_D : S^1 \rightarrow \mathbf{R}^3$ of the circle into $\mathbf{R}^3$ whose only singularities are self-intersections obtained by collapsing each chord of D. That is, for θ and $\tilde{\theta}$ the two ends of a chord in D, we have $K_D(\theta) = K_D(\tilde{\theta})$.

Next we notice that if K_D and $\overline{K}_D$ are different embeddings of D, we can obtain one from the other by a sequence of crossing changes. Furthermore, since the value of v is constant under a change of crossing, we can simply write

$$W(D) = W_m(v)(D) = v(K_D).$$

Theorem 2: *The above construction is in fact a mapping from v to $W_m(v)$. That is, for any given F-valued, degree m Vassiliev knot invariant v we have a degree m, F-valued weight system $W_m(v)$.*

> **Proof:** First, we will show that W is framing independent. Let D be a chord diagram with an isolated chord; further, when embedding D into $\mathbf{R}^3$, let the ends of this isolated chord intersect at the point p. Then K_D can be chosen to have a double isthmus at p. Next, we can remove this singularity by applying the definition of a Vassiliev invariant which yields:
>
> $$W(D) = v(K_D) = v(K_D^+) - v(K_D^-)$$
>
> Where K_D^+ and K_D^- denotes the positive and negative crossings respectively. Now we notice that K_D^+ and K_D^- are ambient isotopic and so $v(K_D^+) - v(K_D^-) = 0$, hence $W(D) = 0$ as desired.
>
> Now we will show that W also satisfies the $4T$ relation. Consider the four chord diagrams $S, E, W,$ and N from the $4T$ relation. Each of these chord diagrams locally corresponds to two singular points in their respective knot diagrams, shown below.
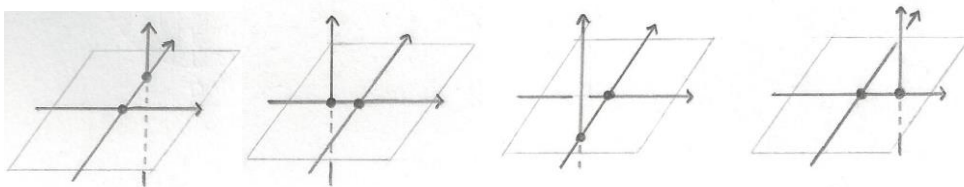
> Now we can consider removing the vertical singularity using the definition of the Vassiliev invariant v . In doing so, we see that:

$$N^* = v\left(\text{[figure]}\right) = v\left(\text{[figure]}\right) - v\left(\text{[figure]}\right)$$

$$W^* = v\left(\text{[figure]}\right) = v\left(\text{[figure]}\right) - v\left(\text{[figure]}\right)$$

$$S^* = v\left(\text{[figure]}\right) = v\left(\text{[figure]}\right) - v\left(\text{[figure]}\right)$$

$$E^* = v\left(\text{[figure]}\right) = v\left(\text{[figure]}\right) - v\left(\text{[figure]}\right)$$

Now, we want to show that the Vassiliev invariant v on the corresponding singular knots in $\mathbf{R}^3$ satisfy the $4T$ relation. But this follows from the equation
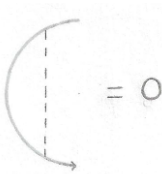
$$N^* - W^* + S^* - E^* = 0.$$

Hence we have that W is in fact a weight system.

∎

Remark: In this proof we showed that an isolated chord of a chord diagram does not results in the weight being 0, and we call this property the framing invariant property, or *FI* relation for short. So we have that this weight system satisfies the 4T and the FI relations.

The 4T and FI relations:

(1) The 4T relation:

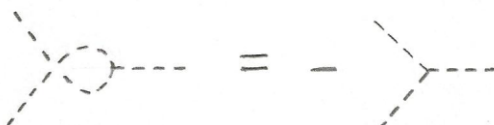(2) The Framing Independence relation:

3. Jacobi Diagrams and Lie Algebras:

Definition: Let X be a compact oriented 1-manifold with boundary. A *Jacobi diagram* on X is the manifold X together with a uni-trivalent graph such that the univalent vertices of the graph are distinct points on X and the trivalent vertices are vertex-oriented.

When drawing a Jacobi diagram, we draw X with solid lines and the uni-trivalent graphs with dashed lines. Furthermore, we will draw each trivalent vertex such that it is vertex-oriented in the counterclockwise direction. We define the *degree* of a Jacobi diagram to be half the number of vertices of the uni-trivalent graph of the Jacobi diagram.

Now, let $\mathcal{A}(X)$ denote the quotient space spanned by Jacobi diagrams on X subject to the three relations: *AS, IHX* and *STU*, shown below. We then call $\mathcal{A}(X)$ the *space of a Jacobi diagrams on X*. Furthermore, we denote the vector subspace of $\mathcal{A}(X)$ spanned by the Jacobi diagrams of degree d as $\mathcal{A}(X)^d$.

The AS, IHX and STU relations:

(1) The AS relation:

(2) The STU relation:

(3) The IHX relation:
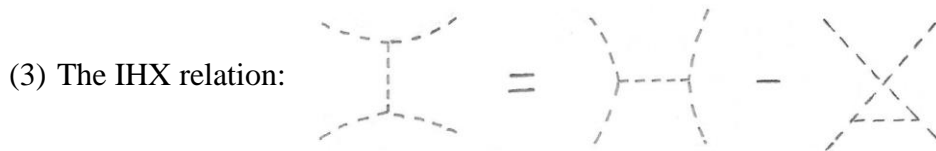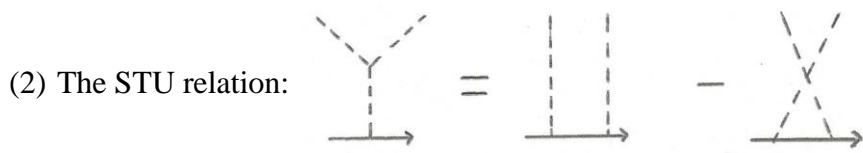
Remark: We note that when X is S^1, a Jacobi diagram whose uni-trivalent graph has no trivalent vertices is in fact also a chord diagram. Furthermore, by the *STU* relation, we see that the *4T* relation holds in the space of Jacobi diagrams on S^1. We call a Jacobi diagram on S^1 with connected uni-trivalent graph *primitive*.

Lie Algebraic Connection:

Definition: A Lie algebra $\mathfrak{g}$ is a vector space over a field F with a binary operation, the Lie bracket, $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following axioms for all $x, y, z \in \mathfrak{g}$ and $a, b \in F$:

  i.   Bilinearity:
$$[ax + by, z] = a[x, z] + b[y, z], [z, ax + by] = a[z, x] + b[z, y]$$

  ii. Alternating on $\mathfrak{g}$:
$$[x, x] = 0$$

  iii. The Jacobi Identity:
$$\big[x, [y, z]\big] + \big[z, [x, y]\big] + \big[y, [z, x]\big] = 0$$

Remark: We notice that the bilinearity and alternating properties of the Lie bracket imply that it is antisymmetric. That is, $[x, y] = -[y, x]$. Furthermore, the dimension of a Lie algebra $\mathfrak{g}$ is its dimension as a vector space over F.

Now let $\mathfrak{g}$ be a finite dimensional Lie algebra over F let t be a metric on $\mathfrak{g}$, and let R be a finite dimensional representation of $\mathfrak{g}$. We can now consider the tensors $(\mathfrak{g}, t, R)$.

The Lie Algebra: First we note that the bilinear form t induces an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$. Furthermore, letting $\tilde{f}$ of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ represent the Lie bracket tensor, we have that $\tilde{f}$

corresponds to some tensor $f \in \mathbf{g}^* \otimes \mathbf{g}^* \otimes \mathbf{g}^*$ by the induced isomorphism from t. Now we notice that f is antisymmetric and will denote it by the anticlockwise oriented graph:

$$\leftrightarrow \qquad f \in \mathbf{g}^* \otimes \mathbf{g}^* \otimes \mathbf{g}^*$$
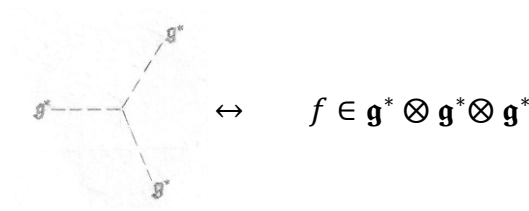
The Bilinear Form: The bilinear form t is a tensor in $\mathbf{g}^* \otimes \mathbf{g}^*$ with its inverse $t^{-1} \in \mathbf{g} \otimes \mathbf{g}$, and we will represent these tensors with the graphs:

 $\leftrightarrow \quad t \in \mathbf{g}^* \otimes \mathbf{g}^* \quad ; \quad$ $\leftrightarrow \quad t^{-1} \in \mathbf{g} \otimes \mathbf{g}$
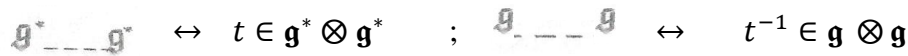
The Representation: A representation R is a vector space together with a distinguished tensor $r \in \mathbf{g}^* \otimes R \otimes R^*$ which we will represent with the graph:

$$\leftrightarrow \qquad r \in \mathbf{g}^* \otimes R \otimes R^*$$
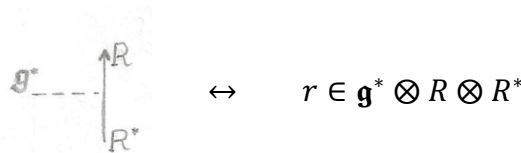
Now let D be a diagram composed from the above parts. For such a diagram we can construct a tensor $\mathcal{T}(D)$ defined below. We note that the tensor product is taken over univalent vertices v of D.

$$\mathcal{T}(D) = \mathcal{T}_{\mathbf{g},R}(D) \in \otimes \begin{pmatrix} the\ vector\ space \\ marked\ near\ v \end{pmatrix}$$

By separating D into a union of its components and consider what you have as a tensor in some higher tensor product of the spaces involved, and contract the obvious pairs of spaces with their duals.

4. The Space of Chord Diagrams and Jacobi Diagrams:

**Theorem:** *We have that $\mathcal{A}(S^1) \cong Span\{D^c\}/the\ 4T\ relation.$*

Proof: By the *STU* relation we have that the *4T* relation holds in $\mathcal{A}(S^1)$ as seen below. This then shows that the $Span\{D^c\}/the \ 4T \ relation \subseteq \mathcal{A}(S^1)$, and thus implying the surjectivity of the linear mapping $\varphi: \mathcal{A}(S^1) \to Span\{D^c\}/the \ 4T \ relation$. We now use induction to prove that φ is also injective. To do so, we first notice that the *STU* relation expresses a Jacobi diagram with k trivalent vertices as the difference of two diagrams with $k - 1$ trivalent vertices. It is then follows that by repeating this process we can inductively construct a mapping from $\mathcal{A}(S^1)$ to $Span\{D^c\}/the \ 4T \ relation$.
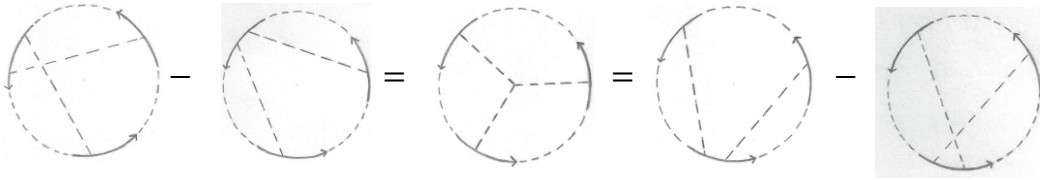
For induction, consider the base case when there are no trivalent crossings in our Jacobi diagram on S^1. Then our Jacobi diagram is also a chord diagram and thus φ is clearly injective. Now for our inductive hypothesis, suppose that φ is injective for all Jacobi diagrams on S^1 with up to $k - 1$ trivalent crossings. We then consider a Jacobi diagram on S^1 with k trivalent crossings, call it D. Choose any trivalent crossing of D and write it as the difference of two diagrams with $k - 1$ trivalent crossings using the *STU* relation. We then have, by or inductive hypothesis, that φ is also injective for D assuming that this process is consistent. Thus we will now show the consistency of this mapping.

If $D \in \mathcal{A}(S^1)$ has only one trivalent vertex, then the consistency is clear, for it is in fact the 4T relation. So suppose that φ is consistent for Jacobi diagrams on S^1 with $k - 1$ trivalent crossings and now consider $D \in \mathcal{A}(S^1)$ where D has exactly k trivalent crossings. Next, suppose that the *STU* relation was used to express D as the difference of two diagrams with $k - 1$ trivalent crossings in two different ways. There are a few ways this could be done.

First, suppose this is done by using it to remove two edges which connects the circle to a trivalent vertex, call them i and j. If i and j are not connected to the same trivalent vertex, then the two figures agree as seen below.
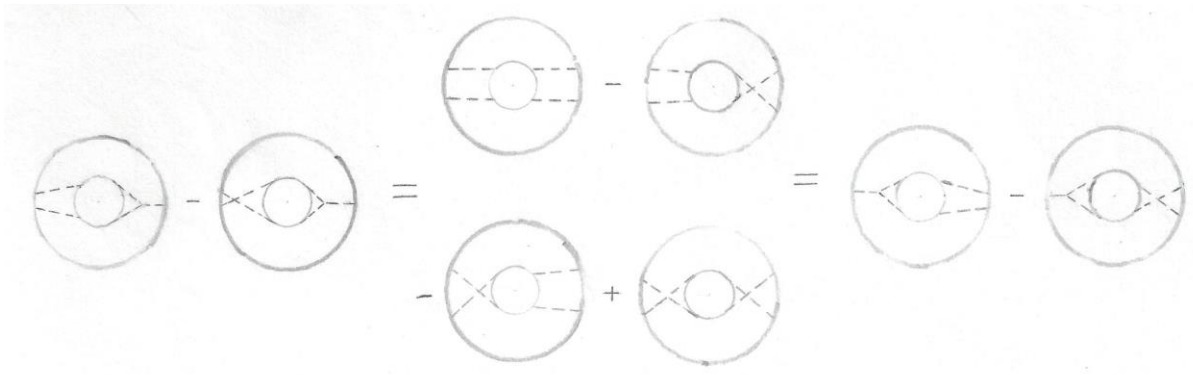
Next, suppose that i and j are in fact connected to the same trivalent vertex, and further suppose that there exists a third arc l such that l connects the circle to a different trivalent vertex. Then simply use transitivity to show the consistency of φ.

Finally, we have the case where the arc l does not exist. That is the case where i and j are connected to the same trivalent vertex and there are no other arcs connecting a trivalent vertex to S^1. In this case, D looks as below.
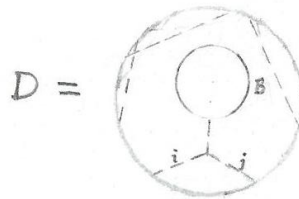
In this case, regardless of how the STU relation is used to reduce D, the resulting chord diagrams are equivalent mod the $4T$ relation. Consider the below calculation.
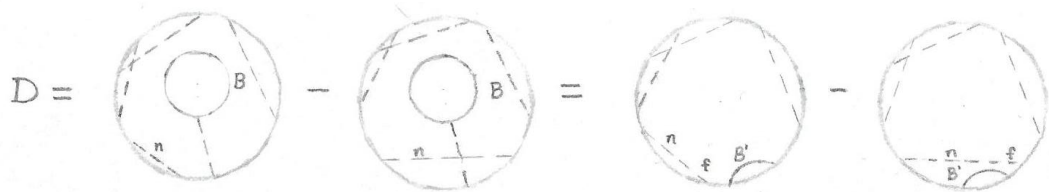
We have that the two diagrams on the left hand side are equivalent by the $4T$ relation. That is, we can obtain the first diagram from the second passing the end f of the chord marked n over the blob B'. This is possible by the $4T$ relation and thus $D = 0$.

∎

5. Claspers:

We have now shown that the space of Jacobi diagrams on S^1 is isomorphic to the space of chord diagrams mod the 4T relation. We also have that there is a clear mapping from chord diagrams of degree d to the space of singular knots with d double points modulo singular knots with $d + 1$ double points. That is, we obtain a knot with d double points from a degree d chord diagram D by collapsing each chord of D to a point. In doing so, we in turn constructed a weight system of a Vassiliev invariant. However, we do not have an obvious mapping from $\mathcal{A}(S^1)^{(d)}$ to the space of singular knots with d double points modulo singular knots with $d + 1$ double points. To do this, we introduce the notion of the clasper as discovered by Habiro.

**Definition:** Habiro's clasper is defined as follows**:**
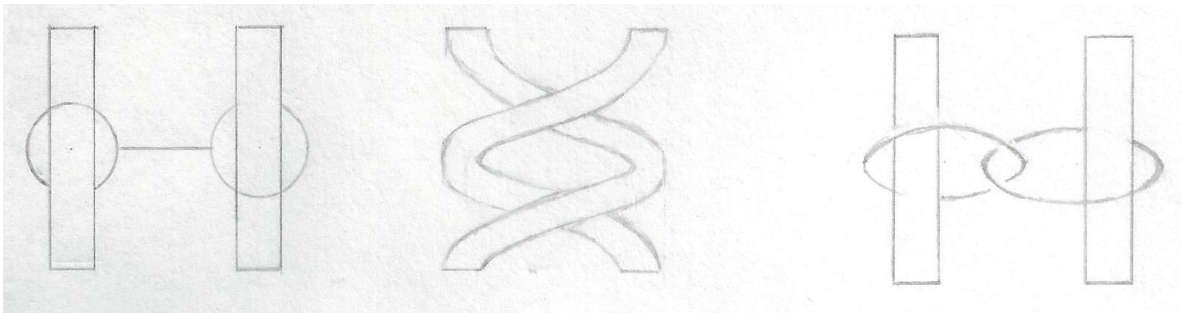
where the band indicates part of a link or clasper, and the first picture denotes the other two pictures. That is, the middle picture is obtained from the right picture by blackboard framed surgery along the Hopf link. The framed embedded graph in the left picture is called a *clasper*, and each loop at the end of a clasper is called a *leaf* of the clasper.

We have the following notation when dealing with claspers:
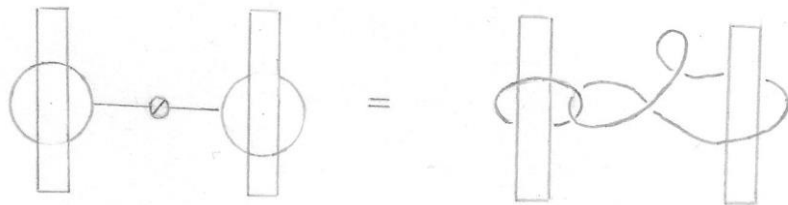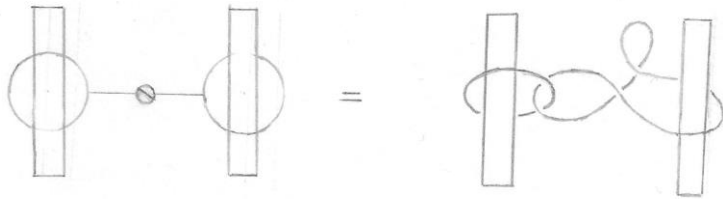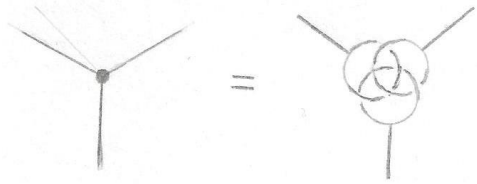
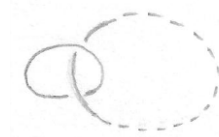*Relations*: Claspers satisfy the following relations:

(A.1)

(A.2)

(A.3)

(A.4)
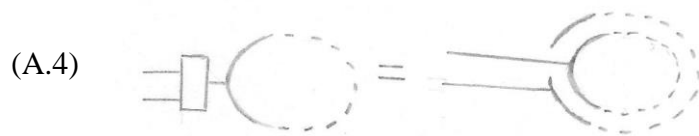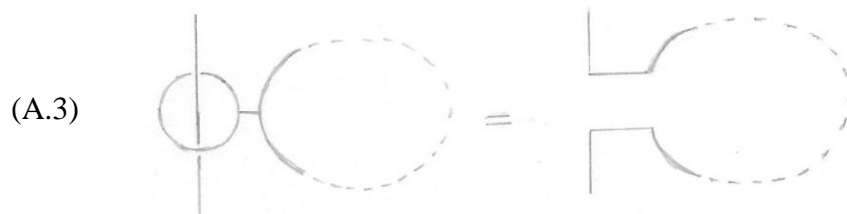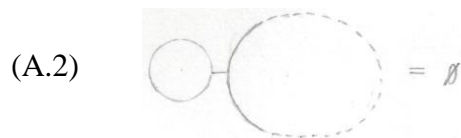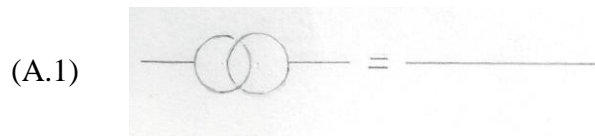
(A.5)

where the dotted lines indicate strands which are possibly knotted and linked in some manner.

Proof: We have that (A.1) is obtained by

where the equalities are obtained by surgery along the links. We get (A.2) by noting that the 3-space obtained by the surgery along the link

is homeomorphic to the initial manifold. (A.3) is obtained by

$$LHS \quad = \quad \text{} \quad = \quad \text{} \quad = \quad RHS$$

where the middle equality is obtained by the handle slide of the vertical strand over the component of the dotted line. (A.4) follows from (A.3). And finally (A.5) is obtained by applying (A.2) and (A.1) to the definition. ∎

6. Claspers and Knots:

Definition: Let $\mathbb{K}$ denote the set of knots and note that $\mathbb{K}$ is a commutative semigroup with respect to connect sum of knots. We call a connected graph without cycles a *tree*. We define a

tree clasper as a union of claspers obtained from a connected uni-trivalent tree in a three dimensional space by replacing univalent vertices with leaves and trivalent vertices with sets of Borromean rings. A *disc*-leaf is a leaf bounding a disk intersecting at precisely one point. A *tree clasper on a knot K* is a tree clasper such that at least one of the leaves of the clasper is a disc-leaf and each of the other leaves bounds a disk intersecting parts on K and edges of the clasper. This is graphically shown as:
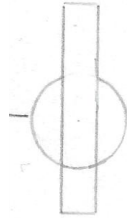

where the band implies some nonempty bundle of strands of K and other edges of the clasper.

We say that a tree clasper on a knot K has *degree d* if it has $d - 1$ trivalent vertices. We define the C_d-*equivalence* to be the equivalence relation of two knots K and $\widetilde{K}$ where $\widetilde{K}$ is obtained by surgery along tree claspers on K of degree d. Furthermore, we write $\mathbb{K}_d$ to mean the set of knots which are C_d-equivalence to the unknot. The set $\mathbb{K}_d$ forms a commutative semigroup as a sub-semigroup of $\mathbb{K}$.

> *Remark:* Surgery along a tree clasper on a knot in S^3 results in another knot in S^3 by the structure of the tree claspers, use of (A.2), and the fact that Borromean rings completely unlink when one of its components unlinks.

For the following lemma's, relations and theorems, the pictures indicate local diagrams of tree claspers except for a ball where they differ as indicated. Furthermore, we denote the unknot as 0. It should also be noted that the proofs of these lemmas can be found in detain in Habiro's paper circa 2000.
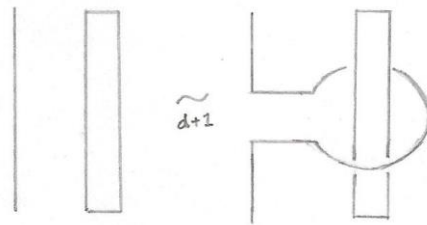
Lemma 1: We have that
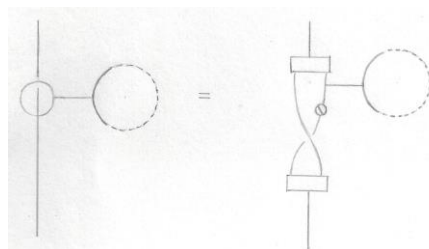
*Lemma 2:* A box moves beyond a trivalent vertex as
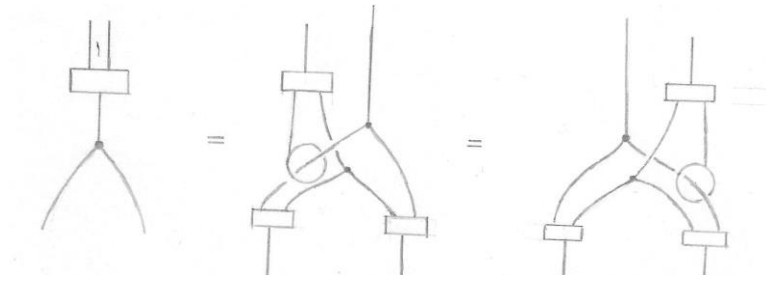
Lemma 3: Let K be a knot, and let $\widetilde{K}$ be the knot obtained from K by surgery along a degree d tree clasper T on K. Then K can be obtained from $\widetilde{K}$ by surgery along a degree d tree clasper $\widetilde{T}$ on $\widetilde{K}$.

Lemma 4: We have the following relation between two knots obtained from another knot via surgery along two degree d tree claspers which differ as below

Lemma 5: We have the following relation between two knots obtained from a third knot by surgery on collections of tree claspers
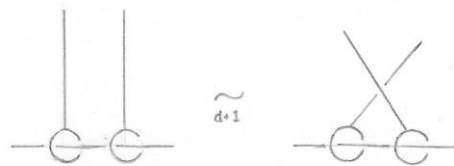
Here we note that the left leaf in the left picture, and so the right leaf in the right picture, is a leaf of a degree d tree clasper, and the other leaves belong to some other tree clasper.

Lemma 6: We have the following relation between the trivial knot and a knot obtained by connect-summing together a pair of knots, each which is obtained from the trivial knot by surgery along a tree clasper:
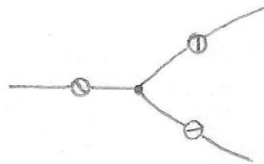
Lemma 7: We have the following relation among the three knots, each of which is obtained from the trivial knot by surgery along a degree d tree clasper.

The AS Relation: We have the following *AS* relation, which relates the connect sum of two knots, each obtained from the trivial knot by surgery along a tree clasper of degree d, to the trivial knot.
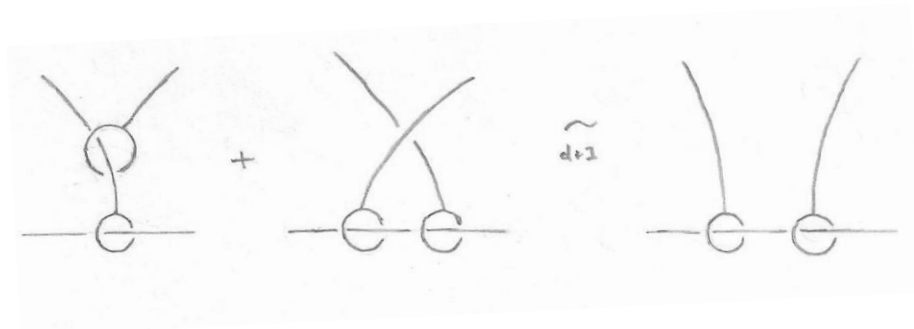

      Proof: The tree clasper on the far left hand side is equal to
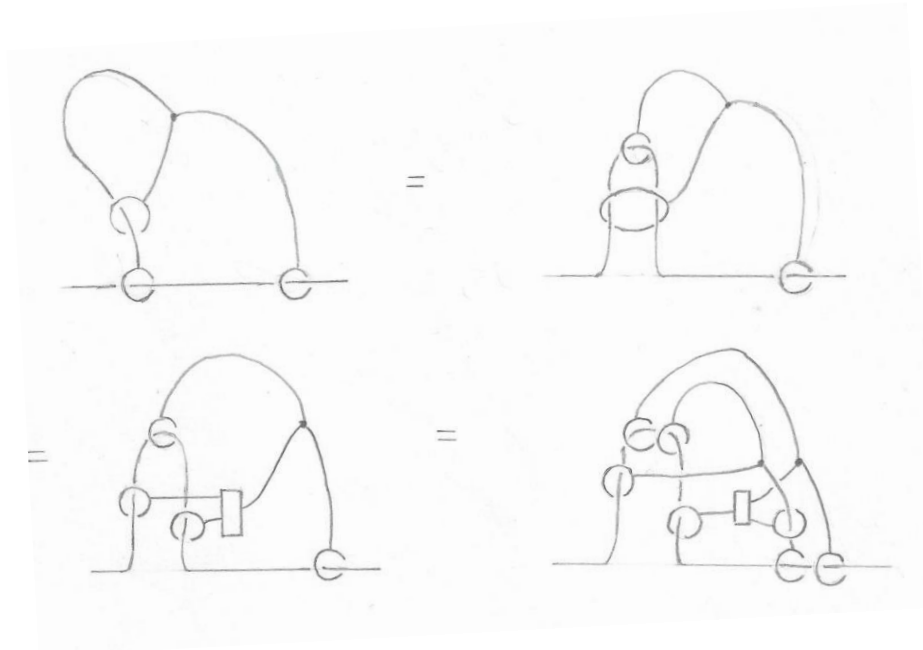

by a half twist of a neighborhood of the trivalent vertex. Hence we can apply Lemma 6 to it three times and yield the desired formula.
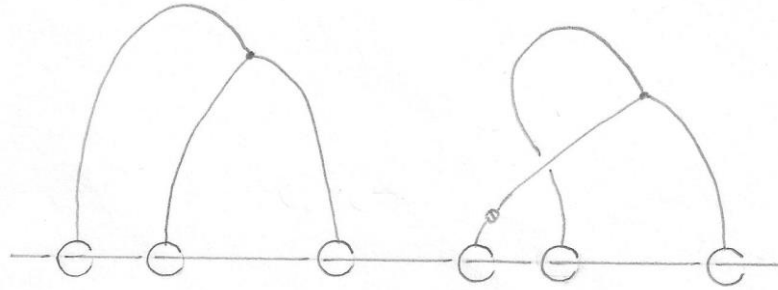
∎

The STU Relation: We have the following *STU* relation, which relates three knots, each obtained from the trivial knot by surgery along a degree d tree clasper.

***Proof:*** We consider the simple case below for simplicity. We have then that

where the three equalities are obtained by isotopy, the definition of the box, and lemma 2 respectively. Moreover, the last picture is equal to the following picture by lemmas 4 and 5. Thus, by lemma 6 we have the desired formula.

■

The IHX Relation: We have the following IHX relation, which relates three knots, each obtained from the trivial knot by surgery along a degree d tree clasper.
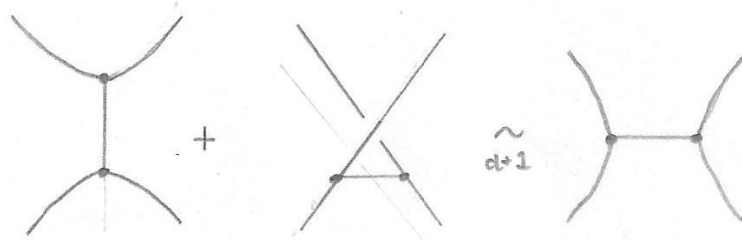
Theorem: *For each positive integer d, we have the following surjective homomorphism,*

$$\varphi : span \left\{ \begin{matrix} primitive\ Jacobi\ diagrams \\ on\ S^1\ of\ degree\ d \end{matrix} \right\} / AS, IHX, STU \;\to\; \mathbb{K}_d / C_{d+1}\text{-}equivalence$$

Which takes primitive Jacobi diagrams to a knot obtained from the unknot by surgery along a tree clasper on the unknot. We obtain a tree clasper from a Jacobi diagram by replacing trivalent and univalent vertices as follows:

I.

II.
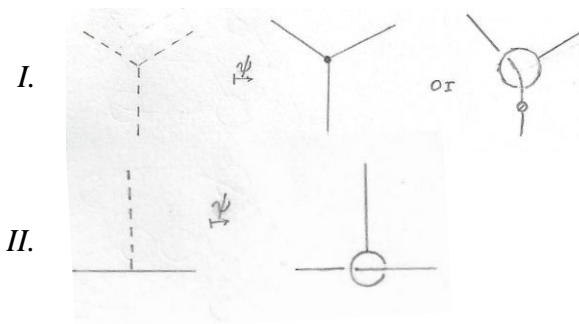
where we choose the image of each trivalent vertex in such a way that the resulting image is a tree clasper.

          Proof: This follows from the above lemmas.

Thus we now have representations of both Chord diagrams and Jacobi diagrams as knots.

References:

We would like to note that most of the information here, with some modification, comes from either Dror Bar-Natan in his paper on Vassiliev invariants circa 1995 or from Tomotada Ohtsuki in his book "Quantum Invariants: a Study of Knots, 3-Mannifolds, and Their Sets."