Quadrant marked mesh patterns in words

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Abstract

In this paper, we study the distribution of quadrant marked mesh patterns in words over the alphabet of positive integers. Quadrant marked mesh patterns are based on how many elements lie in various quadrants of the graph of a permutation relative to the coordinate system centered at one of the points in the graph of the permutation.

Keywords: permutations, words, marked mesh patterns, distribution

Contents

1 Introduction

In this paper we study the distributions of quadrant marked mesh patterns in words. The notion of mesh patterns was introduced by Brändén and Claesson [4] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns (see [6] for a comprehensive introduction to the theory of permutation patterns). This notion was further studied in [3, 5, 13]. In particular, the

notion of a mesh pattern was extended to that of a *marked mesh pattern* by Úlfarsson in
[13]. The study of the distributions of quadrant marked mesh patterns in permutations
was initiated by Kitaev and Remmel [7]. Later Kitaev, Remmel, and Tiefenbruck studied
the distribution of quadrant marked mesh patterns in 132-avoiding permutations [9, 10]
and Kitaev and Remmel [8] studied the distribution of quadrant marked mesh patterns in
up-down and down-up permutations.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers and S_n denote the symmetric
group of permutations of $1, \ldots, n$. If $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, then we will consider the graph
of σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \ldots, n$. For example, the graph of the
permutation $\sigma = 471569283$ is pictured in Figure 1. Then if we draw a coordinate system
centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants
I, II, III, and IV of that coordinate system as pictured in Figure 1. For any $a, b, c, d \in \mathbb{N}$
and any $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that σ_i matches the quadrant marked mesh pattern
$MMP(a, b, c, d)$ in σ if in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i)
as its origin, there are $\geq a$ points in quadrant I, $\geq b$ points in quadrant II, $\geq c$ points in
quadrant III, and $\geq d$ points in quadrant IV. For example, if $\sigma = 471569283$, the point
$\sigma_4 = 5$ matches the quadrant marked mesh pattern $MMP(2, 1, 2, 1)$ since relative to the
coordinate system with origin $(4, 5)$, there are 3 points in $G(\sigma)$ in quadrant I, there is 1
point in $G(\sigma)$ in quadrant II, there are 2 points in $G(\sigma)$ in quadrant III, and there are 2
points in $G(\sigma)$ in quadrant IV. Note that if a coordinate in $MMP(a, b, c, d)$ is 0, then there
is no condition imposed on the points in the corresponding quadrant. In addition, we shall
consider patterns $MMP(a, b, c, d)$ where $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$. Here when one the parameters
a, b, c, or d in $MMP(a, b, c, d)$ is the empty set, then for σ_i to match $MMP(a, b, c, d)$
in $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, it must be the case that there are no points in $G(\sigma)$ relative to
the coordinate system with origin (i, σ_i) in the corresponding quadrant. For example, if
$\sigma = 471569283$, the point $\sigma_3 = 1$ matches the marked mesh pattern $MMP(4, 2, \emptyset, \emptyset)$ since
relative to the coordinate system with origin $(3, 1)$, there are 6 points in $G(\sigma)$ in quadrant
I, 2 points in $G(\sigma)$ in quadrant II, no points in $G(\sigma)$ in quadrant III, and no points in $G(\sigma)$
in quadrant IV. We let $\mathrm{mmp}^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches the
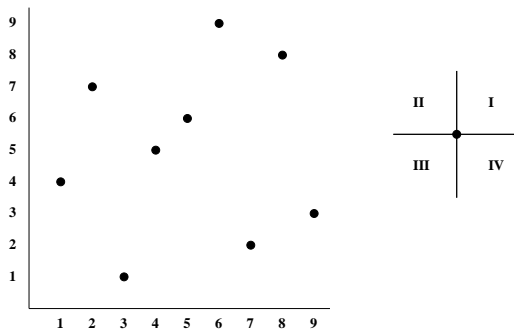marked mesh pattern $MMP(a, b, c, d)$ in σ.

Figure 1: The graph of $\sigma = 471569283$.

Note how the (two-dimensional) notation of Úlfarsson [13] for marked mesh patterns

corresponds to our (one-line) notation for quadrant marked mesh patterns. For example,

$$MMP(0,0,k,0) = \;\raisebox{-0.5em}{\includegraphics[]{}}\;, \quad MMP(k,0,0,0) = \;\raisebox{-0.5em}{\includegraphics[]{}}\;,$$

$$MMP(0,a,b,c) = \;\raisebox{-0.5em}{\includegraphics[]{}}\quad \text{and} \quad MMP(0,0,\emptyset,k) = \;\raisebox{-0.5em}{\includegraphics[]{}}\;.$$

In Section 2 we will consider $MMP(=k,0,0,0)$, another type of quadrant marked mesh patterns, which requires presence of exactly k elements in quadrant I. This type of patterns is expressed in the terminology of Úlfarsson [13] as follows:

$$MMP(=k,0,0,0) = \;\raisebox{-0.5em}{\includegraphics[]{}}\;.$$

Also, in Section **??** we define and study $MMP(k\leq\max,\emptyset,0,0)$, yet another type of quadrant marked mesh patterns, which is equivalent to the following pattern in the terminology of Úlfarsson [13]:

$$MMP(k\leq\max,\emptyset,0,0) = \;\raisebox{-1em}{\includegraphics[]{}}\;.$$

Given a sequence $\sigma = \sigma_1 \ldots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the i-th smallest integer that appears in σ by i. For example, if $\sigma = 2754$, then $\mathrm{red}(\sigma) = 1432$. Given a permutation $\tau = \tau_1 \ldots \tau_j \in S_j$, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ provided there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \ldots \sigma_{i_j}) = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. Let $S_n(\tau)$ denote the set of permutations in S_n which avoid τ. In the theory of permutation patterns, τ is called a *classical pattern*. See [6] for a comprehensive introduction to permutation patterns.

Kitaev and Remmel [7] were able to compute a number of generating functions for the distribution of quadrant marked mesh patterns in permutations. For example, if we let

$$P^{(k,0,0,0)}(t,x) = \sum_{n\geq k} \frac{t^{n-k}}{(n-k)!} \sum_{\sigma\in S_n} x^{\mathrm{mmp}^{(k,0,0,0)}(\sigma)},$$

then Kitaev and Remmel [7] proved that

$$P^{(k,0,0,0)}(t,x) = k! \left(\frac{1}{1-tx}\right)^{\frac{k}{x}+1}.$$

3

They also proved that

$$P^{(a,b,0,0)}(t,x) = \sum_{n \geq a+b} \frac{t^{n-a-b}}{(n-a-b)!} R_n^{(a,b,0,0)}(x) = (a+b)! \left(\frac{1}{1-tx}\right)^{\frac{a+b}{x}+1} \qquad (1)$$

In [8], Kitaev and Remmel studied the distribution of the statistics $\mathrm{mmp}^{(1,0,\emptyset,0)}$, $\mathrm{mmp}^{(0,1,0,\emptyset)}$, $\mathrm{mmp}^{(0,\emptyset,0,1)}$, and $\mathrm{mmp}^{(\emptyset,0,1,0)}$ in the set of up-down and down-up permutations. Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$. Then we say that σ is an *up-down permutation* if $Des(\sigma)$ is the set of all even numbers less than or equal to n and a *down-up permutation* if $Des(\sigma)$ is the set of all odd numbers less than or equal to n. That is, σ is an up-down permutation if

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \cdots$$

and σ is an down-up permutation if

$$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 > \cdots .$$

Let UD_n denote the set of all up-down permutations in S_n and DU_n denote the set of all down-up permutations in S_n. Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we define the reverse of σ, σ^r, to be $\sigma_n \sigma_{n-1} \ldots \sigma_2 \sigma_1$ and the complement of σ, σ^c, to be $(n+1-\sigma_1) \ldots (n+1-\sigma_n)$.
For $n \geq 1$, let

$$A_{2n}^{(a,b,c,d)}(x) = \sum_{\sigma \in UD_{2n}} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}, \qquad B_{2n-1}^{(a,b,c,d)}(x) = \sum_{\sigma \in UD_{2n-1}} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)},$$

$$C_{2n}^{(a,b,c,d)}(x) = \sum_{\sigma \in DU_{2n}} x^{\mathrm{mmp}^{(a,b,d,d)}(\sigma)}, \text{ and } D_{2n-1}^{(a,b,c,d)}(x) = \sum_{\sigma \in DU_{2n-1}} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}.$$

Then Kitaev and Remmel [8] proved the following simple proposition.

Proposition 1. For all $n \geq 1$,
(1) $A_{2n}^{(a,b,c,d)}(x) = C_{2n}^{(b,a,d,c)}(x) = C_{2n}^{(d,c,b,a)}(x) = A_{2n}^{(c,d,a,b)}(x)$,
(2) $C_{2n}^{(a,b,c,d)}(x) = A_{2n}^{(b,a,d,c)}(x) = A_{2n}^{(d,c,b,a)}(x) = C_{2n}^{(c,d,a,b)}(x)$,
(3) $B_{2n-1}^{(a,b,c,d)}(x) = B_{2n-1}^{(b,a,d,c)}(x) = D_{2n-1}^{(d,c,b,a)}(x) = D_{2n-1}^{(c,d,a,b)}(x)$, and
(4) $D_{2n-1}^{(a,b,c,d)}(x) = D_{2n-1}^{(b,a,d,c)}(x) = B_{2n-1}^{(d,c,b,a)}(x) = B_{2n-1}^{(c,d,a,b)}(x)$.

It follows from Proposition 1 that the study the distribution of the statistics $\mathrm{mmp}^{(1,0,0,0)}$, $\mathrm{mmp}^{(0,1,0,0)}$, $\mathrm{mmp}^{(0,0,1,0)}$, and $\mathrm{mmp}^{(0,0,0,1)}$ in the set of up-down and down-up permutations

can be reduced to the study of the following generating functions:

$$A^{(1,0,0,0)}(t,x) = 1 + \sum_{n \geq 1} A_{2n}(x) \frac{t^{2n}}{(2n)!}, \tag{2}$$

$$B^{(1,0,0,0)}(t,x) = \sum_{n \geq 1} B_{2n-1}(x) \frac{t^{2n-1}}{(2n-1)!}, \tag{3}$$

$$C^{(1,0,0,0)}(t,x) = 1 + \sum_{n \geq 1} C_{2n}(x) \frac{t^{2n}}{(2n)!}, \text{ and} \tag{4}$$

$$D^{(1,0,0,0)}(t,x) = \sum_{n \geq 1} D_{2n-1}(x) \frac{t^{2n-1}}{(2n-1)!}. \tag{5}$$

In the case when $x = 1$, these generating functions are well known. That is, for any (a,b,c,d), let $A_{2n}(1) = A_{2n}^{(a,b,c,d)}(1)$, $B_{2n+1}(1) = B_{2n+1}^{(a,b,c,d)}(1)$, $C_{2n}(1) = C_{2n}^{(a,b,c,d)}(1)$, and $D_{2n}(1) = D_{2n}^{(a,b,c,d)}(1)$. The operation of complementation shows that $A_{2n}(1) = C_{2n}(1)$ and $B_{2n_1}(1) = D_{2n-1}(1)$ for all $n \geq 1$ and André [1, 2] proved that

$$1 + \sum_{n \geq 0} A_{2n}(1) \frac{t^{2n}}{(2n)!} = \sec(t)$$

and

$$\sum_{n \geq 1} B_{2n-1}(1) \frac{t^{2n+1}}{(2n+1)!} = \tan(t).$$

In [8], we proved the following.

Theorem 1.

$$A^{(1,0,0,0)}(t,x) = \sec(xt)^{1/x}, \tag{6}$$

$$B^{(1,0,0,0)}(t,x) = \sec(xt)^{1/x} \int_0^t \sec(xz)^{-1/x} dz, \tag{7}$$

$$C^{(1,0,0,0)}(t,x) = 1 + \int_0^t \sec(xy)^{1+\frac{1}{x}} \int_0^y \sec(xz)^{1/x} dz \, dy, \text{ and} \tag{8}$$

$$D^{(1,0,0,0)}(t,x) = \int_0^t \sec(xz)^{1+\frac{1}{x}} dz. \tag{9}$$

The definition of quadrant marked mesh patterns can easily be extended to words over that alphabet of positive integers. That is, $\mathbb{P} = \{1, 2, \ldots\}$ denote the set of positive integers and for any $k \in \mathbb{P}$, let $[k] = \{1, \ldots, k\}$. Fix $k \geq 2$. Then given a word $w = w_1 \ldots w_n \in [k]$, we will consider the graph of w, $G(w)$, to be the set of points (i, w_i) for $i = 1, \ldots, n$. For example, the graph of the word $w = 134214532$ is pictured in Figure 2. Then if we draw a coordinate system centred at a point (i, w_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 2.

5

For any $a, b, c, d \in \mathbb{N}$ and any $w = w_1 \ldots w_n \in S_n$, we say that w_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$ in σ if in $G(w)$ relative to the coordinate system which has the point (i, w_i) as its origin, there are $\geq a$ points in quadrant I, $\geq b$ points in quadrant II, $\geq c$ points in quadrant III, and $\geq d$ points in quadrant IV. Here we do not consider points that lie on the x-axis of y-axis of the coordinate system which has the point (i, w_i) as its origin, to be part of any of quadrants I, II, III, of IV. Thus, for example, w_i matches the marked mesh pattern $MMP(1, 0, 0, 0)$ in w if and only if there is a $j > i$ such that $w_j > w_i$. For example, if $w = 134214532$, the point $w_4 = 2$ matches the quadrant marked mesh pattern $MMP(3, 2, 1, 1)$ since relative to the coordinate system with origin $(4, 2)$, there are 3 points in $G(w)$ in quadrant I, 2 points in $G(w)$ in quadrant II, 1 point in $G(w)$ in quadrant III, and 1 points in $G(w)$ in quadrant IV. As with permutations, if a coordinate in $MMP(a, b, c, d)$ is 0, then there is no condition imposed on the points in the corresponding quadrant. One can also consider patterns $MMP(a, b, c, d)$ where $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$. Here when one of the parameters a, b, c, or d in $MMP(a, b, c, d)$ is the empty set, then for w_i to match $MMP(a, b, c, d)$ in $w = w_1 \ldots w_n \in [k]^n$, it must be the case that there are no points in $G(w)$ relative to coordinate system with origin (i, w_i) in the corresponding quadrant. We let $\text{mmp}^{(a,b,c,d)}(w)$ denote the number of i such that w_i matches the marked mesh pattern $MMP(a, b, c, d)$ in w.
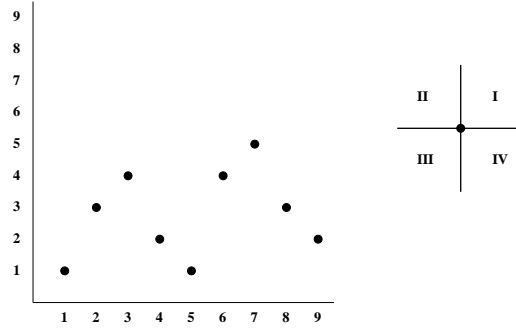

Figure 2: The graph of $w = 134214532$.

If $w = w_1 \ldots w_n \in [k]^n$, then we let $|w| = n$ denote the length of w and, for any $1 \leq i \leq k$, $|w|_i$ denote the number of times i occurs in w. We let $w^r = w_n \ldots w_1$ denote the reverse of w and $w^{c,k} = (k + 1 - w_1) \ldots (k + 1 - w_n)$ denote the complement of w with respect to k.

The main goal of this paper is to study the generating functions

$$W_k^{(a,b,c,d)}(t, x, y_1, \ldots y_k) = 1 + \sum_{n \geq 0} t^n \sum_{w \in [k]^n} x^{\text{mmp}^{(a,b,c,d)}(w)} \prod_{i=1}^{k} y_i^{|w|_i}.$$

For any $a, b, c, d \in \{\emptyset\} \cup \mathbb{N}$, let

$$W_{n,k}^{(a,b,c,d)}(x, y_1, \ldots, y_k) = \sum_{w \in [k]^n} x^{\text{mmp}^{(a,b,c,d)}(w)} \prod_{i=1}^{k} y_i^{|w|_i}.$$

6

We note that operations of reverse and complement relative to k implies some obvious relationships among the polynomials $W_{n,k}^{(a,b,c,d)}(x_1, \ldots, x_k)$. That is, the map which sends $w \in [k]^n$ to w^r shows that

$$W_{n,k}^{(a,b,c,d)}(x, y_1, \ldots, y_k) = W_{n,k}^{(b,a,d,c)}(x, y_1, \ldots, y_k). \tag{10}$$

The map which sends $w \in [k]^n$ to $w^{c,k}$ shows that

$$W_{n,k}^{(a,b,c,d)}(x, y_1, \ldots, y_k) = W_{n,k}^{(d,c,b,a)}(x, y_k, \ldots, y_1). \tag{11}$$

Finally, the map which sends $w \in [k]^n$ to $(w^r)^{c,k}$ proves shows that

$$W_{n,k}^{(a,b,c,d)}(x, y_1, \ldots, y_k) = W_{n,k}^{(c,d,a,b)}(x.y_k, \ldots, y_1). \tag{12}$$

The main goal of this paper is to compute $W_k^{(a,b,c,d)}(t, x, y_1, \ldots y_k)$ for various special cases of k and (a, b, c, d). For example, the first result that we shall prove is

$$W_2^{(1,0,0,0)}(t, x, y_1, y_2) = \frac{(1 - txy_1)}{(1 - ty_1)(1 - txy_1 - ty_2)}. \tag{13}$$

Using Mathematica, the expansion of $W_2(t, x, y_1, 1)$ as a Taylor Series about $x = 0$ gives

$$\frac{1}{(1-t)(1-ty_1)} + \frac{t^2 xy_1}{(1-t)^2(1-ty_1)} +$$
$$\frac{t^3 x^2 y_1^2}{(1-t)^3(1-ty_1)} + \frac{t^4 x^3 y_1^3}{(1-t)^4(1-ty_1)} + \frac{t^5 x^4 y_1^4}{(1-t)^5(1-ty_1)} +$$
$$\frac{t^6 x^5 y_1^5}{(1-t)^6(1-ty_1)} + \frac{t^7 x^6 y_1^6}{(1-t)^7(1-ty_1)} + \frac{t^8 x^7 y_1^7}{(1-t)^8(1-ty_1)} + \cdots$$

This suggests that

$$W_2^{(1,0,0,0)}(t, x, y_1, 1)|_{x^s} = \frac{t^{s+1} y_1^s}{(1-t)^{s+1}(1-ty_1)}$$

and

$$W_2^{(1,0,0,0)}(t, x, y_1, 1)|_{x^0} = \frac{1}{(1-t)(1-ty_1)}.$$

This is true. In fact, we can prove the following theorem.

Theorem 2. For $s \geq 1$,

$$W_2^{(k,0,0,0)}(t, x, y_1, 1)|_{x^s} = \frac{t^{s+k} y_1^s}{(1-t)^{s+1}(1-ty_1)^k}$$

$$W_2^{(k,0,0,0)}(t,x,y_1,1)|_{x^0} = \frac{(1-t)(1-y_1t)^k - y_1t^{k+1}}{(1-t)(1-ty_1)^k(1-t-y_1t)}.$$

$$W_2^{(k,0,0,0)}(t,x,y_1,1) = \frac{((1-y_1t)^k(1-t-y_1xt) + y_1t^{k+1}(x-1)}{(1-ty_1)^k(1-t-y_1xt)(1-t-y_1t)}.$$

We shall also prove the following theorem

Theorem 3. For all $n \geq 2$,

$$W_n^{(1,0,0,0)}(t,x,y_1,\ldots,y_n) = W_{n-1}^{(1,0,0,0)}(t,x,y_1,\ldots,y_{n-1})\left(\frac{1 - tx(\sum_{i=1}^{n-1} y_i)}{1 - ty_n - tx(\sum_{i=1}^{n-1} y_i)}\right).$$

Clearly, $W_1^{(1,0,0,0)}(t,y_1) = \frac{1}{1-ty_1}$ so plugging this into the above, we get
$$W_2^{(1,0,0,0)}(t,x,y_1,y_2) = \frac{(1-txy_1)}{(1-ty_1)(1-txy_1-ty_2)}.$$
and
$$W_3^{(1,0,0,0)}(t,x,y_1,y_2,1) = \frac{(1-txy_1)(1-txy_1-txy_2)}{(1-ty_1)(1-txy_1y_2-ty_2)(1-t-txy_1-txy_2)}$$
.

Theorem 4.

$$W_2^{(1,1,0,0)}(t,x,y_1,y_2) = \frac{1 - y_1t(x+1) + y_1y_2t^2(1-x-xy_1)}{(1-y_1t)(1-y_1xt)(1-y_1xt-y_2t)}$$

2 $\quad W_{n,2}^{(k,0,0,0)}(x,y_1,y_2)$

Note that for any word $w \in [2]^n$, $|w| = |w|_1 + |w|_2$ so that we do not have to simultaneously keep track of $|w|$, $|w|_1$, and $|w|_2$ in our generating functions. Thus we let

$$W_n^{(k,0,0,0)}(x,y) = W_{n,2}^{(k,0,0,0)}(x,y,1)$$

and

$$W^{(k,0,0,0)}(t,x,y) = 1 + \sum_{n\geq 1} t^n W_n^{(k,0,0,0)}(x,y).$$

3 $\quad W_2^{(1,0,0,0)}(t,x,z)$ and $W_2^{(1,1,0,0)}$

In this section I will consider two specific examples that will aid in drawing some general conclusions.

Theorem 5. $W_2^{(1,0,0,0)}(t,x,y_1,1) = \frac{1-txy_1}{(1-y_1t)(1-t-xy_1t)}$

Proof. By definition,

$$W_n^{(k,0,0,0)}(t,x,z) = \sum_{w\in\{1...n\}^*} t^{|w|} y_1^{|w|_1} x^{mmp^{(k,0,0,0)}} = 1 + \sum_{n\geq 1} t^n \sum_{k=0}^{n} W_2^{(k,0,0,0)}(n,k,x)y_1^k$$

So for this case our equation becomes

$$W_2^{(1,0,0,0)}(t,x,y_1) = \sum_{w\in\{1,2\}^*} t^{|w|} y_1^{|w|_1} x^{mmp^{(1,0,0,0)}(w)}$$

$$= 1 + \sum_{n\geq 1} t^n \sum_{k=0}^{n} h(n,k,x)y_1^k$$

where

$$h(n,k,x) = \sum_{w\in R(1^k 2^{n-k})} x^{mmp^{(1,0,0,0)}(w)}$$

,

which is all possible rearrangements of k ones and $n-k$ twos. The sum breaks into the following parts:

$$W_2^{(1,1,0,0)}(t,x,y_1) = 1 + \sum_{n\geq 1} t^n h(n,0,x) \quad (1)$$

$$+ \sum_{n\geq 1} t^n h(n,n,x)z^k \quad (2)$$

$$+ \sum_{n\geq 2} t^n \sum_{k=1}^{n-1} h(n,k,x)z^k \quad (3)$$

where (1) is words of all twos, (2) is words of all ones, and (3) is words with at least one of each, and its length is greater than 2.

For (1), since $h(n,0,x) = 1$ we get

$$\sum_{n\geq 1} t^n = t + t^2 + t^3 \ldots = \frac{t}{1-t}$$

Similarly, for (2), we have that $h(n,n,x) = 1$ so

$$\sum_{n\geq 1} t^n h(n,n,x)y_1^n = \sum_{n\geq 1} (y_1 t)^n = y_1 t + (y_1 t)^2 + (y_1 t)^3 \ldots = \frac{y_1 t}{1 - y_1 t}$$.

Finally, for (3) we need to use recursion: If a word starts with 1, then we get $xh(n-1,k-1,x)$ since the first one matches the pattern. If the word starts with 2 then we get $h(n-1,k,x)$ so for $n\geq 2$, $h(n,k,x) = xh(n-1,\ k-1,\ x) + h(n-1,\ k,\ x)$ (3) becomes

$$\sum_{n\geq 2} t^n \sum_{k=1}^{n-1} h(n,k,x)y_1^k = \sum_{n\geq 2} t^n \sum_{k=1}^{n-1} (xh(n-1,k-1,x) + h(n-1,k,x))y_1^k$$ which we split up into:

9

$$\sum_{n \geq 2} t^n \sum_{k=1}^{n-1} x h(n-1, k-1, x) y_1^k \quad (A)$$

$$+ \sum_{n \geq 2} t^n \sum_{k=1}^{n-1} h(n-1, k, x) y_1^k \quad (B)$$

We solve each one separately:

$$
\begin{aligned}
A &= \sum_{n \geq 2} t^n \sum_{k=1}^{n-1} x h(n-1, k-1, x) y_1^k \\
&= t x y_1 \sum_{n \geq 2} t^{n-1} \sum_{k=1}^{n-1} h(n-1, k-1, x) y_1^{k-1} \\
&= t x y_1 \sum_{n \geq 2} t^{n-1} [(\sum_{k=1}^{n-1} h(n-1, k-1, x) y_1^{k-1}) + h(n-1, n-1, x) y_1^{n-1}] \\
&- t x y_1 \sum_{n \geq 2} t^{n-1} f(n-1, n-1, x) y_1^{n-1} \\
&= t x y_1 (W_2(t, x, y_1) - 1) - t x y_1 (\frac{y_1 t}{1 - y_1 t})
\end{aligned}
$$

Similarly, for (B) we get that

$$
\begin{aligned}
B &= \sum_{n \geq 2} t^n \sum_{k=1}^{n-1} h(n-1, k, x) \\
&= t \sum_{n \geq 2} t^{n-1} \sum_{k=1}^{n-1} h(n-1, k, x) y_1^k + h(n-1, 0, x) \\
&- t \sum_{n \geq 2} t^{n-1} h(n-1, 0, x) \\
&= t(W_2(t, x, y_1) - 1) - t \frac{t}{1 - t}
\end{aligned}
$$

Combining each of these four parts, we get

10

$$W_2^{(1,0,0,0)}(t,x,y_1) \;=\; 1 + \frac{t}{1-t} + \frac{y_1 t}{1-y_1 t} + txy_1 W_2^{(1,0,0,0)}(t,x,y_1) - txy_1$$
$$\phantom{W_2^{(1,0,0,0)}(t,x,y_1) \;=\;} - \frac{t^2 xy_1^2}{1-y_1 t} + t(W_2^{(1,0,0,0)}(t,x,y_1)) - t - \frac{t^2}{1-t}$$

From which it follows that
$$(1 - t - txy_1)W_2^{(1,0,0,0)}(t,x,y_1) = 1 + \tfrac{t}{1-t} + \tfrac{y_1 t}{1-y_1 t} - txy_1 - \tfrac{t^2 xy_1^2}{1-y_1 t} - t - \tfrac{t^2}{1-t}$$
And finally, solving for $W_2^{(1,0,0,0)}(t,x,y_1)$ and simplifying with Mathematica, we get
$$W_2^{(1,0,0,0)}(t,x,y_1) = \tfrac{1-txy_1}{(1-y_1 t)(1-t-txy_1)}$$

$\square$

This gives us the generating function for words with letters that match $mmp(1,0,0,0)$. Using Mathematica to expand this generating function about $x = 0$ we obtain the following:

$$\frac{1}{(1-t)(1-ty_1)} + \frac{t^2 xy_1}{(1-t)^2(1-ty_1)} +$$
$$\frac{t^3 x^2 y_1^2}{(1-t)^3(1-ty_1)} + \frac{t^4 x^3 y_1^3}{(1-t)^4(1-ty_1)} + \frac{t^5 x^4 y_1^4}{(1-t)^5(1-ty_1)} +$$
$$\frac{t^6 x^5 y_1^5}{(1-t)^6(1-ty_1)} + \frac{t^7 x^6 y_1^6}{(1-t)^7(1-ty_1)} + \frac{t^8 x^7 y_1^7}{(1-t)^8(1-ty_1)} + \cdots$$

from which we can prove the following about the coefficient of x for various s.

Theorem 6. For $s \geq 1$,

$$W_2^{(1,0,0,0)}(t,x,y_1,1)|_{x^s} = \sum_{w \in \{1,2\}^*,\, mmp^{(1,0,0,0)}(w)=s} t^{|w|} y_1^{|w|_1} = \frac{t^{s+1} y_1^s}{(1-t)^{s+1}(1-ty_1)}$$

$$W_2^{(1,0,0,0)}(t,x,y_1,1)|_{x^0} = \sum_{w \in \{1,2\}^*,\, mmp^{(1,0,0,0)}(w)=0} t^{|w|} y_1^{|w|_1} = \frac{1}{(1-t)(1-ty_1)}.$$

Proof. To count the number of words with exactly s ones that match the pattern there are two cases to be examined.

Case 1. $s = 0$ counts all words with no instances of a 1 followed by a 2:
For the word of all ones, $\{1\}^* = 1 \ldots 1 = \frac{1}{1-y_1 t}$.
For the word of all twos, $2\{2\}^*$ (We need at least one 2, so we do not count the empty word twice) $= 2 \ldots 2 = \frac{t}{1-t}$

The last case looks like:

$2\underline{s}211\underline{s}$

The first block of twos gives $\frac{1}{1-t}$, the mandatory 2 followed by a 1, gives $y_1 t^2$, and the last block of ones gives $\frac{1}{1-y_1 t}$, hence, for $s = 0$ the coefficient of x^0 is

$$\frac{1}{1-zt} + \frac{t}{1-t} + \frac{zt^2}{(1-t)(1-zt)}$$

$$= \sum_{w \in \{1,2\}^*, \text{mmp}^{(1,0,0,0)}(w)=0} t^{|w|} y_1^{|w|_1} = \frac{1}{(1-t)(1-ty_1)}$$

Case 2. If $s \geq 1$ then the word must look like

$2\underline{s}\ 1\ 2\underline{s}\ 1\ 2\underline{s}...$ again, the block of twos gives $(\frac{1}{1-t})^{s+1}$ the mandatory 2, and the s 1s give $y_1^s t^{s+1}$ which means that my coefficient of x^s is:

$$\frac{y_1^s t^{s+1}}{(1-t)^{s+1}}$$

$\square$

The next example counts $mmp(1, 1, 0, 0)$, that is finding a generating function for words of length n with $w_i \in \{1, 2\}^*$ that counts the 1's with at least one 2 to the right of it and one 2 to the left of it.

Theorem 7. $W_2^{(1,1,0,0)}(t, x, y, z) = \sum_{w \in \{1,2\}^*} t^{|w|} y_1^{|w|_1} x^{mmp(1,1,0,0)|w|} = \dfrac{1 - t(x+y)z + t^2 z(yx + xyz)}{(1 - tz)(1 - t - txz)(1 - tyz)}$

Proof. Consider

$$W_2^{(1,1,0,0)}(t, x, y, z) = \sum_{w \in \{1,2\}^*} t^{|w|} z^{|w|_1} x^{mmp(1,1,0,0)} y^{mmp(1,\emptyset,0,0)}$$

$$= 1 + \sum_{n \geq 1} t^n \sum_{k=0}^{n} g(n, k, x, y) z^k$$

where

$$g(n, k, x, y) = \sum_{w \in R(1^k, 2^{n-k})} x^{mmp(1,1,0,0)} y^{mmp(1,\emptyset,0,0)}$$

Let us first consider our special cases:

1. If the word is all ones, then $w \in \{1\}^*$, then we get that $W_2^{(1,1,0,0)}(t, x, y, z) = \frac{1}{1-zt}$

2. If the word is all twos, then $w \in 2\{2\}^*$, so $W_2^{(1,1,0,0)}(t, x, y, z) = \frac{t}{1-t}$

3. If the word has one 2 followed by only ones, then: $w \in 2\{1\}^*$ and $W_2^{(1,1,0,0)}(t, x, y, z) = \frac{zt^2}{1-zt}$

4. If the word has exactly one 2 and it is not the first entry, the word looks like $1\underline{s}121\underline{s}1$ so $W_2^{(1,1,0,0)}(t, x, y, z) = \frac{yzt}{1-yzt} \frac{1}{1-zt}$

And lastly, we consider the case where we have at least two 2s, and for that expression we use a recursion again. We must count the letters that match $mmp(1, \emptyset, 0, 0)$ in order to be able to use recursion, and again, we get two cases.

12

If a word begins with 1, then $g(n,k,x,y) = yg(n-1,k-1,x,y)$ because the 1 matches $y^{mmp(1,\emptyset,0,0)}$ since there are at least two 2s. If the word starts with a 2, then $g(n,k,x,y) = g(n-1,k,x,x)$, because now every subsequent 1 matches the pattern for y. So,
$$g(n,k,x,y) = y * g(n-1,k-1,x,y) + g(n,k,x,x).$$
And

$$
\begin{aligned}
W_2^{(1,1,0,0)}(t,x,y,z) &= 1 + \sum_{n\geq 1} z^n g(n,n,x,y)t^n + \sum_{n\geq 1} g(n,0,x,y)t^n \\
&\quad + \sum_{n\geq 2} t^n \sum_{k=1}^{n-1} z^k g(n,k,x,y)t^n \qquad (*)
\end{aligned}
$$

where $(*) = \displaystyle\sum_{n\geq 2} t^n \sum_{k=1}^{n-1} [yg(n-1,k-1,x,y) + g(n-1,k,x,x)]z^k$

Let

$$
\begin{aligned}
A &= \sum_{n\geq 2} t^n \sum_{k=1}^{n-1} y * g(n-1,k-1,x,y)z^k \\
&= \sum_{n\geq 2} t^n zy \sum_{k=1}^{n-1} g(n-1,k-1,x,y)z^{k-1} \\
&= tyz \sum_{n\geq 2} t^{n-1} \sum_{k=1}^{n-1} g(n-1,k-1,x,y)z^{k-1} \\
&= tyz \sum_{n\geq 1} t^n \sum_{k=1}^{n} g(n,k-1,x,y)z^{k-1} \\
&= tyz \sum_{n\geq 1} t^n \sum_{k=0}^{n-1} g(n,k,x,y)z^k
\end{aligned}
$$

This last sum can be evaluated using a recursion. To get the expression in terms of $W_2^{(1,1,0,0)}(t,x,y,z)$, add the nth term and then subtract it:
$$tyz \sum_{n\geq 1} t^n \sum_{k=0}^{n-1} g(n,k,x,y)z^k + g(n,n,x,y)z^n - tyz \sum_{n\geq 1} t^n g(n,n,x,y)z^n$$
Since $g(n,n,x,y) = 1$ we get
$$tyz \sum_{n\geq 1} t^n z^n = \frac{tz}{1-tz}$$
and the first part is just
$$tyz * W_2^{(1,1,0,0)}(t,x,y,z) - 1$$

so $A = tyz(W_2^{(1,1,0,0)}(t,x,y,z) - 1) - tyz\frac{tz}{1-tz}$

If the word starts with 2, and there are at least two 2s, then the word looks like $2\ldots 12\ldots 12\ldots$ In this case, any 1 that matches y now also matches x, because of the 2 at the beginning of the word. Thus, y can be replaced in to obtain the expression $g(n-1,k,x,x)$ Let

$$
\begin{aligned}
B &= \sum_{n\geq 2} t^n \sum_{k=1}^{n-1} g(n-1,k,x,x)z^k \\
&= t\sum_{n\geq 2} t^{n-1} \sum_{k=1}^{n-1} g(n-1,k,x,x)z^k \\
&= t\sum_{n\geq 1} t^n \sum_{k=1}^{n} g(n,k,x,y)z^k \\
&= tyz\sum_{n\geq 1} t^n \sum_{k=1}^{n} g(n,k,x,x)z^k
\end{aligned}
$$

Again, this expression can be simplified using recursion to get an expression in terms of $W_2^{(1,1,0,0)}(t,x,y,z)$. this can be done by adding the "0th" term to the sum and then subtracting it:

$$
t\sum_n \geq 1 t^n \sum_{k=1}^{n} g(n,k,x,x)z^k + g(n,0,x,x) - t\sum_{n\geq 1} t^n g(n,0,x,x)
$$

However, since $g(n,0,x,x) = 1$, the second sum becomes

$$
\frac{t}{1-t}
$$
.

The first sum looks like the original expression with a x where there was a y:

i.e. $W_2^{(1,1,0,0)}(t,x,x,z) - 1$

Thus $B = t(W_2^{(1,1,0,0)}(t,x,x,z) - 1) - \frac{t^2}{1-t}$

$L(t,x,y,z)$ can now be found by combining the four special cases, with the expressions for A and B

$$
\begin{aligned}
W_2^{(1,1,0,0)}(t,x,y,z) &= \frac{1}{1-zt} + \frac{t}{1-t} \\
&+ tyz(W_2^{(1,1,0,0)}(t,x,y,z) - 1) - \frac{t^2yz^2}{1-tz} \\
&+ t(W_2^{(1,1,0,0)}(t,x,x,z) - 1) - \frac{t^2}{1-t}
\end{aligned}
$$

This expression has $W_2^{(1,1,0,0)}(t,x,x,z)$, so to solve for this in terms of $W_2^{(1,1,0,0)}(t,x,y,z)$

set $y = x$ in the expression for $W_2^{(1,1,0,0)}(t, x, y, z)$:

$$W_2^{(1,1,0,0)}(t, x, x, z) = \frac{1}{1 - zt} + \frac{t}{1 - t}$$
$$+ \quad txz(W_2^{(1,1,0,0)}(t, x, x, z) - 1) - \frac{t^2 xz^2}{1 - tz}$$
$$+ \quad t(W_2^{(1,1,0,0)}(t, x, x, z) - 1) - \frac{t^2}{1 - t}$$

$$\Rightarrow (1 - t - txz)W_2^{(1,1,0,0)}(t, x, x, z) = \frac{1}{1 - zt} + \frac{t}{1 - t}$$
$$- \quad txz - \frac{t^2 xz^2}{1 - tz}$$
$$- \quad t - \frac{t^2}{1 - t}$$

$$\Rightarrow W_2^{(1,1,0,0)}(t, x, x, z) = \frac{1}{1 - t - txz}[\frac{1}{1 - zt} + \frac{t}{1 - t} - txz - \frac{t^2 xz^2}{1 - tz} - t - \frac{t^2}{1 - t}]$$
$$= \frac{1 - txz}{(1 - tz)(1 - t - txz)}$$

(Using Mathematica to simplify.)

To get the final expression for $W_2^{(1,1,0,0)}(t,x,y,z)$ plug this expression in for $W_2^{(1,1,0,0)}(t,x,x,z)$

$$W_2^{(1,1,0,0)}(t,x,y,z) = \frac{1}{1-zt} + \frac{t}{1-t}$$
$$+ \quad tyz(W_2^{(1,1,0,0)}(t,x,y,z) - 1) - \frac{t^2yz^2}{1-tz}$$
$$+ \quad t(\frac{1-txz}{(1-tz)(1-t-txz)} - 1) - \frac{t^2}{1-t}$$

$$\Rightarrow (1-tyz)W_2^{(1,1,0,0)}(t,x,y,z)$$

$$= \frac{1}{1-zt} + \frac{t}{1-t} - tyz - \frac{t^2yz^2}{1-tz}$$
$$+ \quad t(\frac{1-txz}{(1-tz)(1-t-txz)} - 1) - \frac{t^2}{1-t}$$

$$\Rightarrow W_2^{(1,1,0,0)}(t,x,y,z)$$

$$= \frac{1}{1-tyz}[\frac{1}{1-zt} + \frac{t}{1-t} - tyz - \frac{t^2yz^2}{1-tz}$$
$$+ \quad t(\frac{1-txz}{(1-tz)(1-t-txz)} - 1) - \frac{t^2}{1-t}]$$
$$= \frac{1 - t(x+y)z + t^2z(y-x-xyz)}{(1-tz)(1-t-txz)(1-tyz)}$$

Again using Mathematica to simplify. However, recall that the expression $W_2^{(1,1,0,0)}(t,x,y,z)$ gave us the generating function that counted $mmp(1,1,0,0)$ and $mmp(1,\emptyset,0,0)$. So in order to get the generating function for just $mmp(1,1,0,0)$ simply let $y=1$ in our polynomial expression. $\square$

Again, we would like to observe the general case for words with exactly s 1s that match the pattern.

Case 3. $s \geq 1$
Then the word must look like:
$\underline{1s}\ \underline{2s}\ 2\ 1\ \underline{2s}\ 1\ \underline{2s}\ 1\ 2\ \underline{2s}\ \underline{1s}$ The numerator becomes $z^s t^{s+2}$; the first block of 1s gives, $\frac{1}{1-yzt}$; the last block of ones, gives $\frac{1}{1-zt}$; the middle blocks of 2s, gives $\frac{1}{(1-t)^s+1}$. Thus, for $s \geq 1$
$$\frac{z^s t^{s+2}}{(1-t)^{s+1}(1-zt)(1-yzt)}$$

Case 4. If s $= 0$, then there are several ways in which a word can be arranged such that it does not match the pattern at all.
(1) $\{1\}^* = 1 \ldots 1$ gives $\frac{1}{1-zt}$
(2) 22^* which gives $\frac{t}{1-t}$
(3) There is at least one 1 and one 2. Then two situations arise. In the first situation we have $1\underline{1s}2\underline{2s}\underline{1s}$ (A) and the second situation looks like $2\underline{2s}1\underline{1s}$ (B). for A, our block looks like 1 1s 2 2s 1s so the generating function is:

16

Similarly for B we get:
$$\frac{yzt^2}{(1-yzt)(1-t)(1-zt)}$$

Combining these expressions and using Mathematica to simplify:
$$\frac{zt^2}{(1-t)(1-zt)}.$$

$$\sum_{w\in\{1,2\}^* mmp^{(1,1,0,0)}|w|=0} t^{|w|}z^{|w|_1}$$

$$= \frac{1}{1-zt} + \frac{t}{1-t} + \frac{yzt^2}{(1-yzt)(1-t)(1-zt)} + \frac{zt^2}{(1-t)(1-zt)}$$

$$= \frac{(1-yzt)(1-t) + t(1-yzt)(1-zt) + yzt^2 + zt^2(1-yzt)}{(1-yzt)(1-t)(1-zt)}$$

4 The generating functions $W_n^{(1,0,0,0)}(t, x, y_1, \ldots, y_n)$.

In this section, we will consider the distribution of the quadrant marked mesh pattern $MMP(1, 0, 0, 0)$ over words in $[n]^*$ where $n \geq 2$. The key idea is consider $W_n^{(1,0,0,0)}(t, x, y_1, \ldots, y_n)|_{y_n^s}$ which is is the sum of $t^{|w|}x^{\mathrm{mmp}^{(1,0,0,0)}(w)} \prod_{i=1}^{n} y_i^{|w|_i}$ over all words $w \in [n]^*$ with exactly s occurrences of the letter n. Clearly if $s = 0$, then

$$W_n^{(1,0,0,0)}(t, x, y_1, \ldots, y_n)|_{y_n^0} = W_{n-1}^{(1,0,0,0)}(t, x, y_1, \ldots, y_{n-1}). \tag{14}$$

Now if $s \geq 1$ and $|w|_n = s$, then we can write $w = u_1 n u_2 n \ldots u_s n u_{s+1}$ where $u_1, u_2, \ldots, u_s, u_{s+1}$ are words in $[n-1]^*$. Clearly, every letter in each of $u_1, \ldots, u_s$ will match the pattern $MMP(1, 0, 0, 0)$ due to the presence of the an occurrence of n to the right of that letter in w. Thus the contribution to $W_n^{(1,0,0,0)}(t, x, y_1, \ldots, y_n)|_{y_n^s}$ as we sum over all possibilities of each u_i is just

$$\frac{1}{1 - tx(y_1 + \cdots y_{n-1})}.$$

For letters in u_{s+1}, the letters to the left of u_{s+1} in w have no effect on whether a letter in u_{s+1} matches $MMP(1, 0, 0, 0)$ in w. Thus Thus the contribution to $W_n^{(1,0,0,0)}(t, x, y_1, \ldots, y_n)|_{y_n^s}$ as we sum over all possibilities of u_{s+1} is just $W_{n-1}^{(1,0,0,0)}(t, x, y_1, \ldots, y_{n-1})$. Thus for $s \geq 1$.

$$W_n^{(1,0,0,0)}(t, x, y_1, \ldots, y_n)|_{y_n^s} = \left(\frac{y_n t}{1 - tx(y_1 + \cdots + y_{n-1})}\right)^s W_{n-1}^{(1,0,0,0)}(t, x, y_1, \ldots, y_{n-1}). \tag{15}$$

It follows that

$$
\begin{aligned}
W_n^{(1,0,0,0)}(t,x,y_1,\ldots,y_n) &= \sum_{s\geq 0} W_n^{(1,0,0,0)}(t,x,y_1,\ldots,y_n)|_{y_n^s} \\
&= \sum_{s\geq 0}\left(\frac{y_n t}{1-tx(y_1+\cdots+y_{n-1})}\right)^s W_{n-1}^{(1,0,0,0)}(t,x,y_1,\ldots,y_{n-1}) \\
&= W_{n-1}^{(1,0,0,0)}(t,x,y_1,\ldots,y_{n-1})\sum_{s\geq 0}\left(\frac{y_n t}{1-tx(y_1+\cdots+y_{n-1})}\right)^s \\
&= W_{n-1}^{(1,0,0,0)}(t,x,y_1,\ldots,y_{n-1})\frac{1}{1-\left(\frac{y_n t}{1-tx(y_1+\cdots+y_{n-1})}\right)} \\
&= W_{n-1}^{(1,0,0,0)}(t,x,y_1,\ldots,y_{n-1})\frac{1-tx(y_1+\cdots y_{n-1})}{1-y_n t-tx(y_1+\cdots y_{n-1})}.
\end{aligned}
$$

Thus we have proved the following Theorem.

Theorem 8. For all $n\geq 2$,

$$
W_n^{(1,0,0,0)}(t,x,y_1,\ldots,y_n) = W_{n-1}^{(1,0,0,0)}(t,x,y_1,\ldots,y_{n-1})\frac{1-tx(y_1+\cdots+y_{n-1})}{1-y_n t-tx(y_1+\cdots+y_{n-1})}. \tag{16}
$$

It is easy to see that

$$
W_1^{(1,0,0,0)}(t,x,y_1) = \frac{1}{(1-y_1 t)}.
$$

Thus

$$
W_2(t,x,y_1,y_2) = \frac{1}{(1-y_1 t)}\frac{(1-y_1 xt)}{(1-y_1 xt-y_2 t)},
$$

$$
W_3(t,x,y_1,y_3) = \frac{1}{(1-y_1 t)}\frac{(1-y_1 xt)}{(1-y_1 xt-y_2 t)}\frac{(1-(y_1+y_2)xt)}{(1-(y_1+y_2)xt-y_3 t)},
$$

$$
W_3(t,x,y_1,y_3) = \frac{1}{(1-y_1 t)}\frac{(1-y_1 xt)}{(1-y_1 xt-y_2 t)}\frac{(1-(y_1+y_2)xt)}{(1-(y_1+y_2)xt-y_3 t)}\frac{(1-(y_1+y_2+y_3)xt)}{(1-(y_1+y_2+y_3)xt-y_4 t)}.
$$

etc..

5 The generating functions $W_3^{(k,0,\ell,0)}(t,x,y_1,y_2,y_3)$

In this section, we will study the generating function $W_3^{(k,0,\ell,0)}(t,x,y_1,y_2,1)$ for $k,\ell\geq 1$. This time, our first task is to to understand the generating functions

$$
U(s,k,l,t,y_1,y_2) = W_3^{(k,0,\ell,0)}(t,x,y_1,y_2,1)|_{x^s}
$$

for $s\geq 1$. Thus

$$
U(s,k,l,t,y_1,y_2) = \sum_{w\in[3]^*,\,\mathrm{mmp}^{(k,0,\ell,0)}(w)=s} t^{|w|}y_1^{|w|_1}y_2^{|w|_2}. \tag{17}
$$

18

Now fix $s \geq 1$. If $w \in [3]^*$ and $\mathrm{mmp}^{(k,0,\ell,0)}(w) = s$, then the only letters that can match the quadrant marked mesh pattern $MMP(k, 0, \ell, 0)$ in w are 2's. Thus we must be able to write w as $w = u_1 2 u_2 2 \ldots u_s 2 u_{s+1}$ where the 2's following $u_1, u_2, \ldots u_s$, respectively, are the elements of w which match the quadrant marked mesh pattern $MMP(k, 0, \ell, 0)$ in w. Thus there must be at least ℓ 1's in u_1 and k 3's in u_{s+1}. Then we can write $u_1 = v_1 1 v_2 1 \ldots v_\ell 1 v_{\ell+1}$ where the 1s following $v_1, \ldots, v_\ell$, respectively are the leftmost ℓ 1s that appear in u_1. Similarly, we $u_{s+1} = w_1 3 w_2 3 \ldots w_k 3 v_{k+1}$ where the 3s following $w_1, \ldots, w_k$, respectively are the rightmost ℓ 3s that appear in u_{s+1}. For example, Figure ?? pictures such a factorization when $\ell = 3$, $k = 2$, and $s = 2$. Thus we have written

$$w = v_1 1 v_2 1 \ldots v_\ell 1 v_{\ell+1} 2 u_2 2 \ldots u_s 2 w_1 3 w_2 3 \ldots w_k 3 w_{k+1}.$$

$\mathrm{mmp}^{(2,0,3,0)}(w) = 2$ First observe that since $\mathrm{mmp}^{(k,0,\ell,0)}(w) = s$, there can be no 2s in $v_{\ell+1}, u_2, \ldots, u_s, w_1$ since any 2s in these words would automatically match the quadrant marked mesh pattern $MMP(k, 0, \ell, 0)$ in w. Thus each of these word can be arbitrary words in $\{1, 3\}^*$. Thus as we allow $v_{\ell+1}, u_2, \ldots, u_s, w_1$ to vary over all possible words in $\{1, 3\}^*$, we would get a contribution of $\left(\frac{1}{(1-y_1 t - t)} \right)^s$ to $U(s, k, l, t, y_1, y_2)$.

Next we observe that there can be no 1's in $v_1, \ldots, v_\ell$ by the fact that the 1's following $v_1, \ldots, v_\ell$ where chosen to the leftmost ℓ 1s in u_1. We claim that each v_i can be an arbitrary word in $\{2, 3\}^*$ since each 2 that occurs in v_i will have at most $i - 1$ 1s to its left and hence cannot match the quadrant marked mesh pattern $MMP(k, 0, \ell, 0)$ in w. It follows that as we allow $v_1, \ldots, v_\ell$ to vary over all possible words in $\{2, 3\}^*$, we would get a contribution of $\left(\frac{1}{(1-y_2 t - t)} \right)^\ell$ to $U(s, k, l, t, y_1, y_2)$.

Finally, observe that there can be no 3's in $w_2, \ldots, w_{k+1}$ by the fact that the 3's following $w_1, \ldots, w_k$ where chosen to the rightmost k 3s in u_{s+1}. We claim that each w_i where $i \geq 2$ can be an arbitrary word in $\{1, 2\}^*$ since each 2 that occurs in such w_i will have at most $k - i$ 3s to its right and hence cannot match the quadrant marked mesh pattern $MMP(k, 0, \ell, 0)$ in w. It follows that as we allow $w_2, \ldots, w_{k+1}$ to vary over all possible words in $\{1, 2\}^*$, we would get a contribution of $\left(\frac{1}{(1-y_1 t - y_2 t)} \right)^k$ to $U(s, k, l, t, y_1, y_2)$.

It follows that for all $s \geq 1$,

$$U(s, k, l, t, y_1, y_2) = \frac{y_1^\ell y_2^s t^{k+\ell+s}}{(1 - y_1 t - y_2 t)^k (1 - y_2 t - t)^\ell (1 - y_1 t - t)^{s+1}}. \tag{18}$$

Next let

$$V_3(k, l, t, x, y_1, y_2) = \sum_{s \geq 1} x^s U(s, k, l, t, y_1, y_2).$$

Thus

$$V_3(k, l, t, x, y_1, y_2) = \sum_{w \in [3]^*, \mathrm{mmp}^{(k,0,\ell,0)}(w) \neq 0} x^{\mathrm{mmp}^{(k,0,\ell,0)}(w)} t^{|w|} y_1^{|w|_1} y_2^{|w|_2} \tag{19}$$

19

We see that

$$V_3(k,l,t,x,y_1,y_2) \;=\; \sum_{s\geq 1} x^s U(s,k,l,t,y_1,y_2) \tag{20}$$

$$= \sum_{s\geq 1} x^s \frac{y_1^\ell y_2^s t^{k+\ell+s}}{(1-y_1t-y_2t)^k(1-y_2t-t)^\ell(1-y_1t-t)^{s+1}}$$

$$= \frac{y_1^\ell t^{k+\ell}}{(1-y_1t-y_2t)^k(1-y_2t-t)^\ell(1-y_1t-t)} \times$$

$$\sum_{s\geq 1}\left(\frac{xy_2t}{(1-y_1t-t)}\right)^s$$

$$= \frac{y_1^\ell t^{k+\ell}}{(1-y_1t-y_2t)^k(1-y_2t-t)^\ell(1-y_1t-t)} \times$$

$$\frac{xy_2t}{1-y_1t-t}\frac{1}{1-\frac{xy_2t}{1-y_1t-t}}$$

$$= \frac{xy_2y_1^\ell t^{k+\ell+1}}{(1-y_1t-y_2t)^k(1-y_2t-t)^\ell(1-y_1t-t)(1-t-y_1t-xy_2t)}.$$

It follows that

$$W_3^{(k,0,\ell,0)}(t,x,y_1,y_2,1)|_{x^0} \;=\; \frac{1}{(1-y_1t-y_2t-t)} - V(k,l,t,1,y_1,y_2) \tag{21}$$

$$= \frac{(1-y_1t-y_2t)^k(1-y_2t-t)^\ell(1-y_1t-t)-y_2y_1^\ell t^{k+\ell+1}}{(1-y_1t-y_2t)^k(1-y_2t-t)^\ell(1-y_1t-t)(1-t-y_1t-y_2t)}.$$

Hence

$$W_3^{(k,0,\ell,0)}(t,x,y_1,y_2,1) = W_3^{(k,0,\ell,0)}(t,x,y_1,y_2,1)|_{x^0} + V_3(k,l,t,x,y_1,y_2). \tag{22}$$

Combining (20) and (21), one can easily show that the following theorem holds.

Theorem 9. For all $k,\ell \geq 1$,

$$W_3^{(k,0,\ell,0)}(t,x,y_1,y_2,1) = \tag{23}$$

$$\frac{A_3^{(k,0,\ell,0)}(t,x,y_1,y_2,1)}{(1-y_1t-t)(1-t-y_1t-y_2t)(1-t-y_1t-y_2xt)(1-y_1t-y_2t)^k(1-y_2t-t)^\ell}$$

where

$$A_3^{(k,0,\ell,0)}(t,x,y_1,y_2,1) \;=\; (1-y_1t-t)(1-t-y_1t-y_2xt)(1-y_1t-y_2t)^k(1-y_2t-t)^\ell -$$
$$y_2y_1^\ell t^{k+\ell+1}(1-t-y_1t-y_2xt) + xy_2y_1^\ell t^{k+\ell+1}(1-t-y_1t-y_2t).$$

References

[1] D. André, Développements de sec x et de tang x, *C. R. Acad. Sci. Paris*, **88** (1879), 965–967.

[2] D. André, Mémoire sur les permutations alternées, *J. Math. Pur. Appl.*, **7** (1881), 167–184.

[3] S. Avgustinovich, S. Kitaev and A. Valyuzhenich, Avoidance of boxed mesh patterns on permutations, preprint.

[4] P. Brändén and A. Claesson, Mesh patterns and the expansion of permutation statistics as sums of permutation patterns, *Elect. J. Comb.* **18(2)** (2011), #P5, 14pp.

[5] Í. Hilmarsson, I. Jónsdóttir, S. Sigurdardottir and S. Vidarsdóttir, Wilf classification of mesh patterns of short length, in preparation.

[6] S. Kitaev, Patterns in permutations and words, Springer-Verlag, 2011.

[7] S. Kitaev and J. Remmel, Quadrant marked mesh patterns, Journal of Integer Sequences, **12** Issue 4 (2012), Article 12.4.7.

[8] S. Kitaev and J. Remmel, Quadrant marked mesh patterns in alternating permutations, preprint.

[9] S. Kitaev, J. Remmel and M. Tiefenbruck, Quadrant marked mesh patterns in 132-avoiding permutations I, preprint.

[10] S. Kitaev, J. Remmel and M. Tiefenbruck, Quadrant marked mesh patterns in 132-avoiding permutations II, in preparation.

[11] N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at `http://www.research.att.com/~njas/sequences/`.

[12] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.

[13] H. Úlfarsson, A unification of permutation patterns related to Schubert varieties, arXiv:1002.4361 (2011).