The Euler Characteristic of the Moduli Space of Stable Maps into a Grassmannian

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Abstract

The moduli space of stable maps has become a central object of study in algebraic geometry, and its cohomology is known to encapsulate important enumerative information. In an effort to further understanding of this cohomology, we describe a method which uses localization in order to calculate the Euler characteristic of the moduli space of stable maps into a Grassmannian.

Acknowledgements

The past few years have been, for me, an awakening of my passion for mathematics. To adequately recognize every individual who led me into and through this awakening would be entirely impossible, and what follows is a feeble attempt thereof.

First and foremost, I would like to thank my advisor, Dragos Oprea. His influence has extended beyond the mere fact that this thesis would have been impossible without his guidance. I have found his enthusiasm to be intensely contagious—I cannot recall a time that I did not leave his office excited and eager to get home to delve further into the topic of our discussion of the day. As a result, I have learned a great deal of mathematics over the course of the past year, both related and tangential to the contents of this thesis, and I have enjoyed every bit of it. I am indebted.

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Contents

1 Introduction

The moduli space of stable maps, constructed by Kontsevich in [6], has become a central object of study in algebraic geometry. The cohomology of this moduli space captures immensely useful algebro-geometric information which permits a remarkable number of previously impossible enumerative calculations [1]. In an effort to further understanding of this cohomology, we describe a method for calculating the Euler characteristic of the moduli space of stable maps into a Grassmannian. We approach this calculation using the following localization theorem, which follows from [7, Lemma 6].

Theorem 1 (Localization)**.** *Suppose X is a smooth Deligne-Mumford stack with a torus action of $\mathbf{C}^\times$ and fixed locus F. Then $\chi(X) = \chi(F)$.*

In fact, our applications of this theorem will be when the fixed locus F of the torus action is a discrete and finite set, in which case $\chi(F) = |F|$.

We will work exclusively over the complex numbers $\mathbf{C}$. We begin in section 2 by reviewing some elementary facts about the Grassmannian and the Plücker embedding. In section 3, we define the moduli space of stable maps by describing the functor it represents, and the set it parametrizes. Then in section 4, we produce a torus action on the moduli space and describe its fixed locus, thereby allowing us to apply localization and calculate the desired Euler characteristic. We conclude by discussing generalizations of the calculations presented.

2 The Grassmannian

The **Grassmannian $\mathbf{G} = \mathbf{G}(k, n)$**, as a set, consists of the k-dimensional subspaces of $\mathbf{C}^n$. In fact, it has the structure a smooth complex projective variety, and its embedding into projective space is called the **Plücker embedding**. Understanding this embedding will be useful, so we develop part of the theory here, following the exposition of [3, Lecture 6].

If a k-dimensional subspace $V \in \mathbf{G}$ is spanned by vectors $v_1, \ldots, v_k$, consider the multivector

$$v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbf{C}^n.$$

This multivector is determined uniquely up to scalars by V, for the choice of a different basis for V corresponds to multiplication by the determinant of the change of basis matrix. Thus, we acquire a well-defined map of sets

$$\psi : \mathbf{G} \to \mathbf{P}\left(\bigwedge^k \mathbf{C}^n\right).$$

In fact, it turns out that ψ is injective, and that its image can be carved out by homogeneous polynomials, thereby yielding the structure of a projective variety on $\mathbf{G}$.

We would like to have a description of precisely which elements of $\mathbf{P}(\bigwedge^k \mathbf{C}^n)$ are in $\mathbf{G}$. To obtain such a characterization, begin by noticing that the class $[\omega]$ of some $\omega \in \bigwedge^k \mathbf{C}^n$ is in the image of ψ if and only if ω is totally decomposable, so that

$$\omega = v_1 \wedge \cdots \wedge v_k$$

for some linearly independent $v_1, \dots, v_k \in \mathbf{C}^n$. Next, observe that, given $\omega \in \bigwedge^k \mathbf{C}^n$, a vector $v \in \mathbf{C}^n$ will divide ω (that is, ω can be written as $v \wedge \sigma$ for some $\sigma \in \bigwedge^{k-1} \mathbf{C}^n$) if and only if $\omega \wedge v = 0 \in \bigwedge^{k+1} \mathbf{C}^n$. Thus ω is totally decomposable if and only if the space of vectors dividing it is k-dimensional, which happens if and only if the linear map $v \mapsto \omega \wedge v$ has rank $n - k$. Thus, we arrive at the following characterization.

Proposition 2. *The class $[\omega] \in \mathbf{P}(\bigwedge^k \mathbf{C}^n)$ is a point of $\mathbf{G}$ if and only if the linear map*

$$\alpha(\omega) : \mathbf{C}^n \to \bigwedge^{k+1} \mathbf{C}^n$$

given by $v \mapsto \omega \wedge v$ has rank $n - k$.

3 The Moduli Space of Stable Maps

We define a (genus zero) **nodal curve** to be a scheme with finitely many irreducible components, each isomorphic to a projective line $\mathbf{P}^1$, joined together at points called **nodes** to form a tree, so that there are no cycles. An isomorphism of nodal curves is an isomorphism of the underlying schemes: more explicitly, it is a bijection $\mu : C' \to C$ under which components are in one-to-one correspondence, and such that the restriction $\mu|_{C'_\lambda}$ to a component C'_λ of C' is an automorphism of $\mathbf{P}^1$, and is thus of the form

$$[z : w] \mapsto [\alpha z + \beta w : \gamma z + \delta w]$$

for some $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ with $\alpha\delta - \beta\gamma \neq 0$.

Let X be a smooth complex projective variety. An (unmarked) **stable map** (C, φ) of degree d into X is a map $\varphi : C \to X$ on a nodal curve C such that

(i) for each component C_λ of C, the restriction $\varphi|_{C_\lambda} : C_\lambda \to X$ is a morphism of degree d_λ such that $\sum d_\lambda = d$, and

(ii) if $d_\lambda = 0$, then C_λ has at least three nodes.

Let S be a scheme over $\mathbf{C}$. A **stable map over** S of degree d is a flat, proper map $C \to S$ and a map $\varphi : C \to X$ such that for every geometric point s of S, the restriction $\varphi_s : C_s \to X$ to the fiber over s defines a stable map (C_s, φ_s) of degree d. We define a functor

$$\overline{\mathrm{M}}(X, d) : (\mathbf{C}\text{-schemes})^{\mathrm{op}} \to (\text{sets})$$

which maps a scheme S over $\mathbf{C}$ to the isomorphism classes of stable maps of degree d over S.

Theorem 3. *Let X be a smooth complex projective variety. For every nonnegative integer d, the functor $\overline{\mathrm{M}}(X, d)$ is finely represented by a Deligne-Mumford stack [5]. Moreover, when $X = \mathbf{G}(k, n)$ is a Grassmannian, this stack is smooth.*

We will use the notation $\overline{\mathrm{M}}(X, d)$ also for the stack, called the **moduli space of stable maps**, which represents the functor $\overline{\mathrm{M}}(X, d)$. The points of this stack correspond to equivalence classes of stable maps, where (C, φ) and (C', φ') are equivalent if and only if there exists an isomorphism $\mu : C' \to C$ of nodal curves making the diagram

$$\begin{tikzcd} C' \arrow[r, "\varphi'"] \arrow[d, "\mu"'] & X \\ C \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commute. Observe that theorem 3 implies that when $X = \mathbf{G}(k,n)$, we can produce a torus action on $\overline{\mathrm{M}}(\mathbf{G}, d)$ in order to use localization to compute its Euler characteristic.

4 The Euler Characteristic

We now compute the Euler characteristic of the moduli space of stable maps $\overline{\mathrm{M}}(\mathbf{G}, d)$ into the Grassmannian $\mathbf{G} = \mathbf{G}(k,n)$ of k-planes in $\mathbf{C}^n$. We begin by defining a torus action on $\mathbf{C}^n$, which naturally induces a torus action on $\mathbf{G}$, which in turn induces a torus action on $\overline{\mathrm{M}}(\mathbf{G}, d)$. We then analyze fixed points of this action. In particular, we are able to completely describe and enumerate torus fixed stable maps of the form $\mathbf{P}^1 \to \mathbf{G}$. Finally, we describe a combinatorial method for piecing together this information in order to compute $\chi(\overline{\mathrm{M}}(\mathbf{G}, d))$.

4.1 Torus Action

Fix once and for all a basis $\{e_1, \dots, e_n\}$ on $\mathbf{C}^n$, as well as n distinct integers $p_1, \dots, p_n$. We define a torus action of $\mathbf{C}^\times$ on $\mathbf{C}^n$ by

$$t \star \left(\sum_{i=1}^{n} a_i e_i \right) = \sum_{i=1}^{n} t^{p_i} a_i e_i,$$

where $t \in \mathbf{C}^\times$ and $a_i \in \mathbf{C}$. Equivalently, the action of $t \in \mathbf{C}^\times$ on a vector $v \in \mathbf{C}^n$ is given by left multiplication by the diagonal matrix

$$M_t = \begin{pmatrix} t^{p_1} & 0 & \cdots & 0 \\ 0 & t^{p_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{p_n} \end{pmatrix}, \tag{1}$$

so that $t \star v = M_t v$. Observe that M_t has full rank, so is invertible. If V is a k-dimensional subspace of $\mathbf{C}^n$ spanned by $v_1, \dots, v_k$, then invertibility of M_t implies that $t \star v_1, \dots, t \star v_k$ are linearly independent vectors spanning a k-dimensional subspace $t \star V$ of $\mathbf{C}^n$. In other words, the torus action on $\mathbf{C}^n$ induces an action on $\mathbf{G}$.

The eigenvalues of the matrix M_t are easily identified as its diagonal entries $t^{p_1}, \dots, t^{p_n}$, with corresponding eigenvectors $e_1, \dots, e_n$, respectively. Observe that two eigenvalues of M_t become equal if and only if $t^{p_i} = t^{p_j}$ for some distinct i and j, if and only if t is an $(p_i - p_j)$th root of unity. The collection of all $(p_i - p_j)$th roots of unity as i and j vary forms a finite set T of points of $\mathbf{C}^\times$.

Lemma 4. *If $V \in \mathbf{G}$ is such that $t \star V = V$ for some $t \notin T$, then V is spanned by $(e_i)_{i \in I}$ for some k-element subset $I \subseteq \{1, \ldots, n\}$.*

Proof. If $t \star V = V$, then V is an invariant subspace of the matrix M_t as defined in (1) above. Since $t \notin T$, the eigenvalues of the diagonal matrix M_t are all distinct, so, as in [2, Example 2.1.1], V must be the span of $(e_i)_{i \in I}$ for some k-element subset of $\{1, \ldots, n\}$. □

Corollary 5. *A point $V \in \mathbf{G}$ is fixed by the torus action if and only if V is spanned by $\{e_i\}_{i \in I}$ for some k-element subset $I \subseteq \{1, \ldots, n\}$.*

Proof. If $t \star V = V$ for all $t \in \mathbf{C}^\times$, then $t \star V = V$ for some particular $t \notin T$, and then lemma 4 implies that V is the span of $(e_i)_{i \in I}$. Conversely, any subspace V arising as the span of $(e_i)_{i \in I}$ for some $I \subseteq \{1, \ldots, n\}$ is a direct sum of the linear span of eigenvectors of M_t for all $t \in \mathbf{C}^\times$, so clearly V must be invariant. □

We can now use the torus action on $\mathbf{G}$ to naturally induce an action on $\overline{\mathrm{M}}(\mathbf{G}, d)$. If (C, φ) is a stable map, define

$$(t \star \varphi)(x) = t \star \varphi(x)$$

for all $x \in C$ and $t \in \mathbf{C}^\times$. It is clear that then $(C, t \star \varphi)$ is a stable map, so we have defined a torus action on the set of stable maps. Furthermore, it is also evident that this action factors through equivalence of stable maps, inducing a well-defined action on $\overline{\mathrm{M}}(\mathbf{G}, d)$ given by

$$t \star [C, \varphi] = [C, t \star \varphi].$$

Observe that $[C, \varphi]$ is fixed if and only if, for each component C_λ of C, the restriction $[C_\lambda, \varphi|_{C_\lambda}]$ is fixed. Therefore, we reduce the problem of studying torus fixed stable maps on arbitrary nodal curves reduces to studying torus fixed maps $\varphi : \mathbf{P}^1 \to \mathbf{G}$, where we call such a map fixed if it is fixed up to equivalence, so that the corresponding equivalence class $[\mathbf{P}^1, \varphi] \in \overline{\mathrm{M}}(\mathbf{G}, d)$ is torus fixed.

4.2 Fixed Maps

The degree 0 maps $\mathbf{P}^1 \to \mathbf{G}$ are precisely the constant maps, and a constant map is fixed under the torus action if and only if its image is a fixed point of $\mathbf{G}$. Corollary 5 completely classifies fixed points of $\mathbf{G}$, and we see that there are precisely

$$N_{\mathbf{G}} = \binom{n}{k}$$

torus fixed constant maps. In fact, the structure of torus fixed maps of higher degree is rigidly constrained as well.

Proposition 6. *If $\varphi : \mathbf{P}^1 \to \mathbf{G}$ is a torus fixed map of positive degree d, its image is a rational curve through two fixed points of $\mathbf{G}$.*

Proof. It is clear that the scheme-theoretic image $\varphi(\mathbf{P}^1)$ is at most one-dimensional, since $\mathbf{P}^1$ is one-dimensional, and cannot be zero-dimensional since φ is not constant. By Hurwitz's theorem [4, Corollary 2.4], the curve $\varphi(\mathbf{P}^1)$ has genus at most the genus of $\mathbf{P}^1$, which has genus 0. Thus, $\varphi(\mathbf{P}^1)$ is a rational curve.

Observe that φ being torus fixed implies that the torus action on $\mathbf{G}$ restricts to a torus action on $\varphi(\mathbf{P}^1)$. Moreover, $\chi(\varphi(\mathbf{P}^1)) = 2$ since $\varphi(\mathbf{P}^1)$ is rational, and localization now implies that $\varphi(\mathbf{P}^1)$ has two fixed points. $\square$

Proposition 7. *If $\varphi : \mathbf{P}^1 \to \mathbf{G}$ is a torus fixed map of positive degree d, it factors as a composition*

$$\mathbf{P}^1 \to \mathbf{P}^1 \to \mathbf{G}$$

where the first map is $[z : w] \mapsto [z^d : w^d]$, up to a linear change of coordinates, and the second map is a torus fixed line in $\mathbf{G}$.

Proof. By proposition 6, $\varphi(\mathbf{P}^1)$ is a rational curve in $\mathbf{G}$ through two fixed points V and W in $\mathbf{G}$. Thus, $\varphi(\mathbf{P}^1)$ can be embedded in the projective line in $\mathbf{G}$ which passes through V and W, and this line must be torus invariant since φ is torus fixed. In other words, it suffices to classify torus fixed maps $\psi : \mathbf{P}^1 \to \mathbf{P}^1$, where the action on the codomain arises as a restriction of the action on $\mathbf{G}$, and thus must be of the form $t \star [z : w] = [z : t^q w]$ for some integer q.

Since ψ is torus fixed, its branch points must be fixed points of the torus action on the codomain. It is evident that only $0 = [0 : 1]$ and $\infty = [1 : 0]$ are fixed points of this action, so all ramified points of ψ must lie over 0 and ∞. Since $\deg(\psi) = d$, we know that

$$\sum_{p \text{ over } 0} e_p = \sum_{p \text{ over } \infty} e_p = d.$$

Also, by Hurwitz's theorem,

$$2d - 2 = \sum_{p \text{ over } 0,\infty} (e_p - 1),$$

so it is clear that we must have exactly one point over each of 0 and ∞. In other words, if

$$\psi([z : w]) = [a_0 z^d + a_1 z^{d-1} w + \cdots + a_d w^d : b_0 z^d + \cdots + b_d w^d],$$

both polynomials $a_0 z^d + \cdots + a_d w^d$ and $b_0 z^d + \cdots + b_d w^d$ have exactly one root. Thus ψ must be of the form $\psi([z : w]) = [z^d : w^d]$, possibly after a linear change of coordinates. $\square$

In other words, proposition 7 states that, for every degree d, each torus fixed line in $\mathbf{G}$ corresponds uniquely to a torus fixed degree d map $\mathbf{P}^1 \to \mathbf{G}$. Thus, we have reduced the problem of counting the number of torus fixed maps to the problem of counting the number of fixed lines in $\mathbf{G}$.

4.3 Fixed Lines

Let $V_1, \ldots, V_{N_\mathbf{G}}$ be an enumeration of the $N_\mathbf{G}$ fixed points of $\mathbf{G}$. Proposition 6 implies that any fixed line must pass through two distinct fixed points V_j, and the choice of these two V_j completely determines the line. This means that there can be at most

$$\binom{N_\mathbf{G}}{2}$$

fixed lines in $\mathbf{G}$. However, this overcounts the fixed lines because the line through two arbitrary V_j might not exist in $\mathbf{G}$ (even though it exists in the ambient projective space into which $\mathbf{G}$

embeds). To determine which of these lines exist in $\mathbf{G}$, we embed $\mathbf{G}$ into $\mathbf{P}(\bigwedge^k \mathbf{C}^n)$ via the Plücker embedding. Then V_j corresponds to the class $[\varepsilon_j]$ of the multivector

$$\varepsilon_j = e_{j_1} \wedge \cdots \wedge e_{j_k},$$

where $I_j = \{j_1, \ldots, j_k\}$ is the k-element subset of $\{1, \ldots, n\}$ such that V_j is the span of $\{e_{j_1}, \ldots, e_{j_k}\}$. The line $\varphi_{jj'} : \mathbf{P}^1 \to \mathbf{P}(\bigwedge^k \mathbf{C}^n)$ through V_j and $V_{j'}$ is given by

$$\varphi_{jj'}([z : w]) = [z\varepsilon_j + w\varepsilon_{j'}].$$

Then, by proposition 2, the line $\varphi_{jj'}$ lies completely in $\mathbf{G}$ if and only if the linear map

$$\alpha(z\varepsilon_j + w\varepsilon_{j'}) : \mathbf{C}^n \to \bigwedge^{k+1} \mathbf{C}^n$$

has rank $n - k$ for all $[z : w] \in \mathbf{P}^1$. Under this map,

$$e_i \mapsto z\varepsilon_j \wedge e_i + w\varepsilon_{j'} \wedge e_i.$$

Observe that $\varepsilon_j \wedge e_i = 0$ whenever $i \in I_j$. Letting multivectors of the form $\varepsilon_j \wedge e_i$ form a basis for the codomain, the matrix of this map can be described as follows. For each $i \in \{1, \ldots, n\}$, the ith column of the matrix is

(i) all zeores, if $i \in I_j \cap I_{j'}$,

(ii) all zeores except a single z in the position corresponding to $\varepsilon_j \wedge e_i$, if $i \in I_j \backslash I_{j'}$,

(iii) all zeores except a single w in the position corresponding to $\varepsilon_{j'} \wedge e_i$, if $i \in I_{j'} \backslash I_j$, and

(iv) all zeores except a z in the position corresponding to $\varepsilon_j \wedge e_i$ and a w in the position corresponding to $\varepsilon_{j'} \wedge e_i$ if $i \notin I_j \cup I_{j'}$.

Columns of type (i) contribute nothing to the rank. This leaves $n - |I_j \cap I_{j'}|$ columns of type (ii), (iii) and (iv), each of which can contribute up to 1 to the rank. In fact, it is clear that each of these columns do contribute 1 to the rank, *unless* there are distinct $i, i' \in \{1, \ldots, n\}$ such that

$$I_j \cup \{i\} = I_{j'} \cup \{i'\}. \tag{2}$$

In this case, the ith and i'th columns will be redundant, so we will have overcounted the rank by exactly 1. In other words, the rank of the matrix will be precisely

$$n - |I_j \cap I_{j'}| - \delta_{jj'}$$

where $\delta_{jj'}$ is 1 if condition (2) is satisfied, and 0 otherwise. The line $\varphi_{jj'}$ is contained entirely in $\mathbf{G}$ if and only if this rank is $n - k$, or, equivalently, if and only if

$$|I_j \cap I_{j'}| + \delta_{jj'} = k. \tag{3}$$

Observe that, since I_j and $I_{j'}$ are distinct, $|I_j \cap I_{j'}| < |I_j| = k$. This means that condition (3) is satisfied if and only if $|I_j \cap I_{j'}| = k - 1$ and $\delta_{jj'} = 1$. But $\delta_{jj'} = 1$ alone already forces

$|I_j \cap I_{j'}| = k-1$. In other words, $\varphi_{jj'}$ is a line in $\mathbf{G}$ if and only if $\delta_{jj'} = 1$, if and only if condition (2) is satisfied. Moreover, every such $\varphi_{jj'}$ is indeed torus fixed, since

$$t \star \varphi_{jj'}([z:w]) = t \star (z\varepsilon_j + w\varepsilon_{j'}) = z(t^{p_{j_1}+\cdots+p_{j_k}})\varepsilon_j + w(t^{p_{j'_1}+\cdots+p_{j'_k}})\varepsilon_{j'} = \varphi_{jj'}\mu_t([z:w])$$

where $\mu_t : \mathbf{P}^1 \to \mathbf{P}^1$ is the automorphism

$$[z:w] \mapsto [(t^{p_{j_1}+\cdots+p_{j_k}})z : (t^{p_{j'_1}+\cdots+p_{j'_k}})w].$$

Therefore, counting torus fixed lines is equivalent to counting the number ways of choosing two distinct k-element subsets of $\{1,\ldots,n\}$ such that condition (2) is satisfied. There are $\binom{n}{k-1}$ ways of selecting $k-1$ elements for the intersection $I_j \cap I_{j'}$, and then we must choose two additional elements from the remaining $n-k+1$ elements, one element for each of the two sets. Thus, there are

$$L_{\mathbf{G}} = \binom{n}{k-1}\binom{n-k+1}{2} \tag{4}$$

ways of choosing k-element subsets I_j and $I_{j'}$ of $\{1,\ldots,n\}$ such that condition (2) are satisfied, and exactly the same number of torus fixed lines in $\mathbf{G}$.

4.4 Intersections

Proposition 8. *If $[C,\varphi] \in \overline{\mathrm{M}}(\mathbf{G},d)$ is torus fixed and $x \in C$ is a node, then $\varphi(x)$ is a torus fixed point of $\mathbf{G}$.*

Proof. Let C_1 and C_2 be components of C joined by x. If either of $\varphi|_{C_1}$ or $\varphi|_{C_2}$ are of degree 0, then it is must be a constant map onto a fixed point of $\mathbf{G}$, so in particular $\varphi(x)$ must be a fixed point of $\mathbf{G}$. So suppose both $\varphi|_{C_1}$ and $\varphi|_{C_2}$ are of positive degree. Then $\varphi(x) \in \mathbf{G}$ is a point on both the curves $\varphi(C_1)$ and $\varphi(C_2)$. These curves are torus invariant, so

$$t \star \varphi(x) \in \varphi(C_1) \cap \varphi(C_2)$$

for every $t \in \mathbf{C}^\times$. If $\varphi(C_1)$ and $\varphi(C_2)$ are distinct, their intersection is a discrete set, so in fact $\varphi(x)$ must be torus fixed. If $\varphi(C_1)$ and $\varphi(C_2)$ are the same curve, we change coordinates to get $\varphi(x)$ to be a torus fixed point of $\mathbf{G}$. $\square$

We now count the number of torus fixed lines passing through a given torus fixed point. Equivalently, we can count the number of distinct k-element subsets of $\{1,\ldots,n\}$ which can be obtained by replacing exactly 1 element from a given k-element subset of $\{1,\ldots,n\}$, and it is apparent that there are

$$D_{\mathbf{G}} = \binom{k}{k-1}(n-k) = k(n-k)$$

ways to do this. In fact, this point of view allows us to recompute the number $L_{\mathbf{G}}$ of torus fixed lines in $\mathbf{G}$ as

$$L_{\mathbf{G}} = \frac{1}{2}\left(\sum_{j=1}^{N_{\mathbf{G}}} D_{\mathbf{G}}\right) = \frac{1}{2}N_{\mathbf{G}}D_{\mathbf{G}},$$

where the sum is over the $N_{\mathbf{G}}$ fixed points of $\mathbf{G}$.

4.5 Calculations

We now proceed with calculations of $\chi(\overline{\mathrm{M}}(\mathbf{G},d))$ for some small values of d. To fix some terminology, define a **labeling** of a nodal curve C to be an assignment of nonnegative integers d_λ to each component C_λ of C. We will say that the labeling **sums** to d if $\sum d_\lambda = d$, and that the labeling is **valid** if C_λ has at least three nodes whenever $d_\lambda = 0$, so that in a valid labeling which sums to d, d_λ is the degree of the restriction to C_λ of some stable map of degree d on C. Also, we will use the following constants, computed earlier.

$$
\begin{aligned}
N_{\mathbf{G}} &= \binom{n}{k}. \\
D_{\mathbf{G}} &= k(n-k). \\
L_{\mathbf{G}} &= \frac{1}{2}N_{\mathbf{G}}D_{\mathbf{G}} = \frac{1}{2}\binom{n}{k}k(n-k) = \binom{n}{k-1}\binom{n-k+1}{2}.
\end{aligned}
$$

These represent the number of torus fixed points on $\mathbf{G}$, the number of torus fixed lines in $\mathbf{G}$ through a given torus fixed point, and the total number of torus fixed lines in $\mathbf{G}$, respectively.

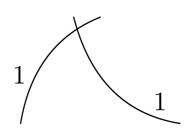
Calculation 9. When $d = 1$, the only possible validly labeled nodal curve has just one component, with label 1. In other words, the torus fixed stable maps of degree 1 are precisely the torus fixed lines in $\mathbf{G}$. Thus, by localization,

$$\chi(\overline{\mathrm{M}}(\mathbf{G},1)) = L_{\mathbf{G}}.$$

Calculation 10. Let $d = 2$. This time, there are two validly labeled nodal curves C.

(i) C has just one component, labeled 2.

(ii) C has two components C_1 and C_2, both labeled 1.



For case (i), by proposition 7, there are as many torus fixed degree 2 maps $\mathbf{P}^1 \to \mathbf{G}$ as there are torus fixed lines, for which there are $L_{\mathbf{G}}$ options. Now consider case (ii). We can choose $L_{\mathbf{G}}$ lines on the first component, and then we have 2 options for which of the fixed points of that line to assign to the node of C, and then $D_{\mathbf{G}}$ options for lines through the fixed point of $\mathbf{G}$ corresponding to that node. This double-counts precisely when the line on both components is the same (since in that case a coordinate change on both components would identify two choices of values for the node of C), and there are $L_{\mathbf{G}}$ ways to do this. So, there are

$$2L_{\mathbf{G}}D_{\mathbf{G}} - L_{\mathbf{G}} = L_{\mathbf{G}}(2D_{\mathbf{G}} - 1)$$

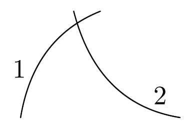
torus fixed stable maps on a curve with two components. Putting these counts together, we arrive at

$$\chi(\overline{\mathrm{M}}(\mathbf{G},2)) = L_{\mathbf{G}} + L_{\mathbf{G}}(2D_{\mathbf{G}} - 1) = 2L_{\mathbf{G}}D_{\mathbf{G}}.$$

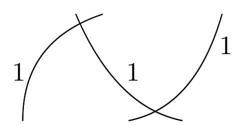
Calculation 11. Now, let $d = 3$. There are four cases to consider.

(i) C has just one component, labeled 4.

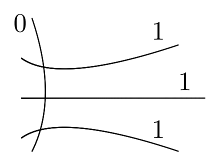
(ii) C has two components C_1 and C_2, labeled 1 and 2, respectively.



(iii) C has three components C_1, C_2, and C_3, each labeled 1.



(iv) C has four components C_1, C_2, C_3 and C_4, with C_1 labeled 0, and the other three components joined to C_1 and labeled 1.



The first two cases are largely analogous to their $d = 2$ counterparts. By proposition 7, there are precisely $L_{\mathbf{G}}$ options to consider for case (i). For case (ii), there are $L_{\mathbf{G}}$ options for a degree 2 map on C_2, then two choices for which of the fixed points of that curve to assign to the node of C, and then $D_{\mathbf{G}}$ options for a line on C_1 through that fixed point, which overcounts when the map on both components pass through the same two fixed points; so, there are

$$2L_{\mathbf{G}}D_{\mathbf{G}} - L_{\mathbf{G}} = L_{\mathbf{G}}(2D_{\mathbf{G}} - 1)$$

torus fixed maps in case (ii).

For case (iii), let x be the node joining C_1 and C_2, and let y be the node joining C_2 and C_3. Since $\varphi|_{C_2}$ is degree 1, so, in particular injective, it must be that $\varphi(x)$ and $\varphi(y)$ are distinct fixed points of $\mathbf{G}$. There are $L_{\mathbf{G}}$ options for a line on C_1, then 2 options for a choice of fixed point $\varphi(x)$, and then $D_{\mathbf{G}}$ options for a line on C_2. The choice of $\varphi(x)$ and a line on C_2 determines $\varphi(y)$, and then there are $D_{\mathbf{G}}$ options for a line on C_3. However, this double-counts precisely when the line on all three components are the same, so we arrive at a total of

$$2L_{\mathbf{G}}D_{\mathbf{G}}^2 - L_{\mathbf{G}} = L_{\mathbf{G}}(2D_{\mathbf{G}}^2 - 1)$$

torus fixed maps in case (iii). Finally, for case (iv), there are $N_{\mathbf{G}}$ options for the constant map on C_1, and then there are

$$D_{\mathbf{G}} + \binom{D_{\mathbf{G}}}{2} + \binom{D_{\mathbf{G}}}{3}$$

options for the maps on the other components. Summing over the $N_{\mathbf{G}}$ options for the constant map on C_1, we arrive at

$$\begin{aligned} N_{\mathbf{G}} &\left(D_{\mathbf{G}} + \frac{1}{2} D_{\mathbf{G}}(D_{\mathbf{G}} - 1) + \frac{1}{6} D_{\mathbf{G}}(D_{\mathbf{G}} - 1)(D_{\mathbf{G}} - 2) \right) \\ &= 2L_{\mathbf{G}} + L_{\mathbf{G}}(D_{\mathbf{G}} - 1) + \frac{1}{3} L_{\mathbf{G}}(D_{\mathbf{G}} - 1)(D_{\mathbf{G}} - 2) = \frac{1}{3} L_{\mathbf{G}}(D_{\mathbf{G}}^2 + 5) \end{aligned}$$

torus fixed maps in case (iv). Thus,

$$\begin{aligned} \chi(\overline{\mathrm{M}}(\mathbf{G}, 3)) &= L_{\mathbf{G}} + L_{\mathbf{G}}(2D_{\mathbf{G}} - 1) + L_{\mathbf{G}}(2D_{\mathbf{G}}^2 - 1) + \frac{1}{3} L_{\mathbf{G}}(D_{\mathbf{G}}^2 + 5) \\ &= \frac{1}{3} L_{\mathbf{G}}(7D_{\mathbf{G}}^2 + 6D_{\mathbf{G}} + 2). \end{aligned}$$

5 Conclusion

Much of the work presented here has a routine generalization to flag varieties, which, given a fixed sequence of integers

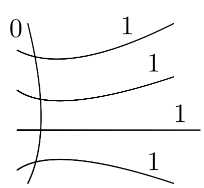
$$0 = k_0 < k_1 < k_2 < \cdots < k_l = n,$$

parametrize strictly ascending chains of vector subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_l = \mathbf{C}^n$$

of $\mathbf{C}^n$, where $\dim(V_i) = k_i$ (so the Grassmannian is a flag variety for which $l = 1$).

On the other hand, in performing the calculations presented in section 4.5 for higher degrees, we run into two obstacles. The first is that the fixed locus of $\overline{\mathrm{M}}(\mathbf{G}, d)$ starts to contain continuous families when $d \geqslant 4$. For instance, one of the validly labeled nodal curves that must be considered when $d = 4$ is the following curve C.



One can show that the fixed maps of the form $[C, \varphi]$ (with C as depicted above) form a locus isomorphic to $\mathbf{P}^1$, so the overall fixed locus of $\overline{\mathrm{M}}(\mathbf{G}, 4)$ is certainly not finite. However, since $\chi(\mathbf{P}^1)$ is known, localization would still allow us to calculate the Euler characteristic in much the same way as we did in section 4.5. The second, more difficult obstacle is the enumeration of the possible validly labeled nodal curves. This is a combinatorial problem, and the number of possibilities grows very quickly. While a tedious enumeration could be performed for any particular d, it is difficult to understand the growth of the number of possibilities systematically in a way that would permit some understanding of $\chi(\overline{\mathrm{M}}(\mathbf{G}, d))$ for arbitrary d.

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