University of California, San Diego

Undergraduate Honors Thesis

# A quantum random walk model for the $(1+2)$ dimensional Dirac equation

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## Contents

# 1 Introduction

In this thesis we explore a quantum random walk on the hexagonal lattice as a discrete model of the (1+2) dimensional Dirac equation. It was shown by Meyer [16] that a quantum random walk on a one dimensional lattice models a discrete (1+1) dimensional Dirac equation. Additionally, it has the Dirac equation as its continuum limit. The key observation is that in one spatial dimension, the space of internal degrees of freedom present in a QRW is naturally isomorphic to the space of spinors of the (1+1) dimensional Dirac equation. Unfortunately there is no natural isomorphism when one goes to higher spatial dimensions. The main result of this thesis shows that there exists a natural embedding of the space of spinors of the (1+2) dimensional Dirac equation into the larger space of internal degrees of freedom present in a QRW.

**Theorem 1.1.** *Consider a QRW on the hexagonal lattice with $mn$ vertices and let $\Delta x$ be the lattice spacing. Then as $\Delta x \to 0$ there exists a $2mn$ dimensional subspace $\mathcal{H}' \subset \mathcal{H}$ that is invariant under the time evolution.*

Without going to the continuum limit, we show numerically that a quantum random walk on the hexagonal lattice potentially produces the discrete analog of a dispersion relation and plane waves, modeling a discrete (1+2) dimensional Dirac equation.

## 2 Background algebra: linear, multi-linear and associative

This thesis will rely heavily on basic linear, multilinear and associative algebra. This section will serve to give a refresher on basic definitions and well established theorems of which will be used freely. References for the linear and multilinear algebra include Goodman and Wallach [7], Halmos [10] and Horn and Johnson [11]; references for associative algebras include Atiyah, Bott and Shapiro [2], Goodman and Wallach [7], and Lawson and Michelson [15].

Recall that a **vector space** $V$ over a commutative field $k$ satisfies properties of an additive abelian group closed under distributive and associative multiplication of $k$. The set of linear functionals $f : V \to k$ form the vector *dual space* $V^*$.

**Definition 2.1.** Let $U$, $V$ and $W$ be vector spaces over a commutative field $k$. A map $\phi : U \to V$ is called *linear* if it respects scalar multiplication and group addition: for all $x, y \in U$ and $a \in k$, $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(ax) = a\phi(x)$. The set of all linear maps form a vector space over k denoted $\mathrm{Hom}(U, V)$ and in the case $U = V$ we denote it as $\mathrm{End}(V)$. End is short for *endomorphism*. If $\phi$ is also a bijection we call it an *isomorphism*. The set of isomorphism from $U$ to itself is denoted $\mathrm{Aut}(U)$, its elements referred to as automorphisms. A map $\phi : U \times V \to W$ is *bilinear* if it is linear in both coordinates.

**Examples 2.2.** Let $\phi : V \to W$ be a linear map of vector spaces.

1. The *kernel* $\mathrm{Ker}\phi = \{v \in V : \phi(v) = 0\}$.
2. The *image* $\mathrm{Im}\phi = \{w \in W : w = \phi(v) \text{ for some } v \in V\}$.
3. If $V \subset W$ then it is called a *subspace* of W.
4. Let $V \subset W$. Then the *quotient* $W/V = \{w + V : w \in W\}$ is a vector space. Here $w + V = \{w + v \in W : v \in V\}$ and is called a left coset.

Let $V$ and $W$ be *finite dimensional* vector spaces with respective bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$. If $\phi : V \to W$ is a linear transformation then there exists a unique matrix $A$ whose entries are given by $\phi(v_i) = \sum_{j=1}^{m} A_{ij} w_j$. The set of all $m \times n$ matrices over $k$ is denoted $M_{m\times n}(k)$. Thus we get a natural isomorphism $\mathrm{Hom}(V, W) \cong M_{m\times n}(k)$. We write $M_{m\times n}(k) = M_n$ for convenience if $m = n$ and $k$ is already established. Theorems from matrix theory will be very useful.

**Lemma 2.3.** *A matrix $U \in M_n(\mathbb{C})$ is* unitary *if $U^\dagger U = I$. If $\lambda$ is a eigenvalue of $U$ then $|\lambda|^2 = 1$.*

*Proof.* Let $x$ be any vector. Then $|Ux|^2 = (Ux)^\dagger Ux = x^\dagger U^\dagger U x = |x|^2$. If $x$ is an eigenvector than $Ux = \lambda x$ and $|Ux|^2 = |\lambda x|^2$. Thus $|\lambda x|^2 = |x|^2 \implies |\lambda|^2 = 1$. □

**Lemma 2.4.** *Let $A, B \in M_n$ be unitary and $X = \{x_1, \ldots, x_n\}$ be a set of eigenvectors for $A$. Then $AB = BA$ if and only if $B$ shares the eigenvectors $X$ of $A$.*

*Proof.* See Horn and Johnson [11] pg. 99. □

**Lemma 2.5.** *(Euler's Rotation Theorem) Every orthogonal matrix $R \in M_3(\mathbb{R})$ with $R^T R = I$ is unitarily equivalent to* $\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$*, for some $0 < \phi < 2\pi$. The parameter $\phi$ is the geometrical notion of the angle of rotation in $\mathbb{R}^3$ about a properly choosen axis.*

*Proof.* Since $R$ is orthogonal it is also unitary, thus its eigenvalues have unit norm. Let $\lambda = e^{i\phi}$ be an eigenvalue, with eigenvector $x$ and define the eigenpair $(\lambda, x)$. Then $Rx = \lambda x \implies \bar{R}\bar{x} = \bar{\lambda}\bar{x}$ $\implies R\bar{x} = \bar{\lambda}\bar{x}$, thus $(\bar{\lambda}, \bar{x})$ is also an eigenpair. Let $\gamma$ be the third eigenvalue. But since $\bar{\gamma}$ must also be an eigenvalue distinct from $\lambda$ this implies $\gamma = \bar{\gamma} = 1$. Diagonalize $R$ by $U = \begin{pmatrix} x & \bar{x} & \vec{n} \end{pmatrix}$ where $\vec{n}$ is the unit norm eigenvector with eigenvalue 1. Thus $U^\dagger R U = D := \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & e^{-i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Let $M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then check, $M^\dagger M = I$, $M^\dagger D M = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $(UM)^\dagger UM = I$. □

**Lemma 2.6.** *Let $A \in \mathcal{M}_n(\mathbb{C})$, $a \in \mathbb{C}$, and $b \in \mathbb{C}$. If $A_{ij} = a$, $\forall i = j$ and $A_{ij} = b$ otherwise then*

1. *The spectrum of $A$ is the set $eig(A) = \{a + (n-1)b, a - b\}$ where the algebraic multiplicity of $\lambda = a - b$ is $n - 1$: $am(a - b) = n - 1$,*
2. *$A$ has $n$ linearly independent eigenvectors:*

$$\begin{aligned} \lambda = a + (n-1)b \quad &\longrightarrow \quad v = (1, 1, \cdots, 1) && (2.1)\\ = a - b \quad &\longrightarrow \quad v_1 = (1, -1, 0, \cdots, 0) && (2.2)\\ &\qquad\quad v_2 = (1, 0, -1, 0, \cdots, 0) && (2.3)\\ &\qquad\quad \vdots && (2.4)\\ &\qquad\quad v_{n-1} = (1, 0, \cdots, -1) && (2.5) \end{aligned}$$

*Proof.* A straight forward computation of $det(A - \lambda I)$ for each $\lambda \in eig(A)$ gives the result:

$$det(A - (a-b)I) = \begin{vmatrix} a-(a-b) & b & \cdots & b \\ b & a-(a-b) & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a-(a-b) \end{vmatrix} = \begin{vmatrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & b \end{vmatrix} = 0.$$

$$det(A - [a+(n-1)b]I) = \begin{vmatrix} a-(a+(n-1)b) & b & \cdots & b \\ b & a-(a+(n-1)b) & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a-(a+(n-1)b) \end{vmatrix}$$

$$= \begin{vmatrix} -(n-1)b & b & \cdots & b \\ b & -(n-1)b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & -(n-1)b \end{vmatrix}$$

$$= \begin{vmatrix} -(n-1)b & b & \cdots & b \\ b & -(n-1)b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)b-(n-1)b & (n-1)b-(n-1)b & \cdots & (n-1)b-(n-1)b \end{vmatrix} = 0.$$

Since $A$ is symmetric its eigenvectors must form an orthogonal set. One can check explicitly that the given set above satisfies the conditions. □

**Definition 2.7.** Let $U$ and $V$ be vector spaces over $k$. The *tensor product* of $U$ and $V$ is the vector space $U \otimes V$ such that given any vector space $W$ and bilinear map $\beta : U \times V \to W$, there exists a linear map $\tilde{\beta} : U \otimes V \to W$ which makes the following diagram commute:

$$\begin{array}{ccc} U \times V & \xrightarrow{\tau} & U \otimes V \\ & \searrow^{\beta} & \downarrow{\scriptstyle \tilde{\beta}} \\ & & W \end{array}$$

The above property is referred to as the *universal mapping property*. It follows that the tensor product is uniquely defined up to isomorphism, meaning for any vector space satisfying the universal mapping property there is a linear bijection between the two vector spaces.

**Construction:** Let $\{u_i \in U : i \in I \text{ and } \{v_j \in V : j \in J\}$ form bases for $U$ and $V$ for some indexing sets $I$ and $J$ with corresponding dual basis $\{u_i^*\}$ and $\{v_j^*\}$. Define $U \otimes V$ as the vector space with bases elements $\{u_i \otimes v_j \in U \otimes V : i \in I \text{ and } j \in J\}$ and the bilinear map $\tau$ by

$$\tau(u, v) = \sum_{i,j} u_i^*(u) v_j^*(v) u_i \otimes v_j.$$

Given $\beta : U \times V \to W$, define $\tilde{\beta}$ by its action on the basis vectors: $\tilde{\beta}(u_i \otimes v_j) = \beta(u_i, v_j)$. This gives $U \otimes V$ the universal structure required.

The vector product space has an inherent functorial characterization which is captured in the following proposition.

**Proposition 2.8.** *Given vector spaces $U$, $V$, $X$, and $Y$ and linear maps $f : U \to X$ and $g : V \to Y$, there is a unique linear map $f \otimes g : U \otimes V \to X \otimes Y$ such that $f \otimes g(u \otimes v) = f(u) \otimes g(v)$. Since $f, g \longmapsto f \otimes g$ is a bilinear map from $Hom(U, X) \times Hom(V, Y)$ to the vector space $Hom(U \otimes V, X \otimes Y)$. It extends to an injective linear map*

$$Hom(U, X) \otimes Hom(V, Y) \longrightarrow Hom(U \otimes V, X \otimes Y) \tag{2.7}$$

**Proposition 2.9.** *If $U$ and $V$ are of finite dimension then $dim(U)dim(V) = dim(U \otimes V)$.*

**Example 2.10.** (Kronecker Product) Let $U = X = \mathbb{C}^n$ and $V = Y = \mathbb{C}^m$, then $End(\mathbb{C}^n) = M_n(\mathbb{C})$ and $End(\mathbb{C}^m) = M_m(\mathbb{C})$. In this case the tensor product is the matrix Kronecker product, $End(\mathbb{C}^n) \otimes End(\mathbb{C}^m) \cong End(\mathbb{C}^n \otimes \mathbb{C}^m) \cong End(\mathbb{C}^{mn})$ defined for $U \in M_n(\mathbb{C})$ and $V \in M_m(\mathbb{C}^m)$

$$U \otimes V = \begin{pmatrix} u_{1,1}V & u_{1,2}V & \cdots \\ u_{2,1}V & u_{2,2}V & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} u_{1,1}v_{1,1} & u_{1,1}v_{1,2} & \cdots & u_{1,2}v_{1,1} & u_{1,2}v_{1,2} & \cdots \\ u_{1,1}v_{2,1} & u_{1,1}v_{2,2} & \cdots & u_{1,2}v_{2,1} & u_{1,2}v_{2,2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ u_{2,1}v_{1,1} & u_{2,1}v_{1,2} & \cdots & u_{2,2}v_{1,1} & u_{2,2}v_{1,2} & \cdots \\ u_{2,1}v_{2,1} & u_{2,1}v_{2,2} & \cdots & u_{2,2}v_{2,1} & u_{2,2}v_{2,2} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots \end{pmatrix}$$

**Definition 2.11.** Given a commutative field, $k$, an **associative algebra** $(A, \cdot)$ is the vector space $A$ over $k$ with a bilinear map $\cdot : A \times A \to A$ such that $x \cdot (y \cdot z) = (x \cdot y) \cdot z = xyz$. When it is clear that multiplication is in the algebra $\cdot$ will often be left out. $A$ is **unital** if there is a multiplicative identity $1 \in A$ such that $1x = x1 = x$. For convenience we denote an unital associative algebra over $k$ by $A$. Often times $k$ will be $\mathbb{C}$ or $\mathbb{R}$.

We would like to talk about maps which relate one algebra to another; interesting maps should preserve some algebraic structure. These relations are especially useful when two algebras share the same underlying vector space and will be used to define a notion of universality. This is a defining feature of the tensor algebra and Clifford algebra.

**Definition 2.12.** Let $A$ and $B$ be two associative algebras. An **algebra homomorphism** is a map $\phi : A \to B$ such that for all $x, y \in A$: *(i.)* $\phi(ax) = a\phi(x)$ for all $a \in k$, *(ii.)* $\phi(x+y) = \phi(x) + \phi(y)$, *(iii.)* $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ and *(iv.)* $\phi(1_A) = 1_B$. The first two conditions are *linearity*, the third says the map respects products and the last is a natural condition on the identity. If $\phi$ is an algebra homomorphism and injective then we call $\phi$ an **isomorphism** and say that $A$ is isomorphic to $B$ denoted by $A \cong B$.

**Definition 2.13.** A subset $I \subset A$ is a **two sided ideal** if $I$ is a vector subspace over $k$ and for all $x \in I$ and $r \in A$, $r \cdot x$ and $x \cdot r$ are in $I$. If $S \subset A$ the two sided ideal generated by $S$ is the intersection of all two sided ideals containing $S$. Consider the ideal $I(S) = \{x \in A | x = \sum_i r_i a_i s_i, \text{ where } r_i, s_i \in A \text{ and } a_i \in S \text{ for finitely many non-zero } i's\}$. Letting $r_i = s_i = 1$ in the definition gives $S \subset I(S)$. Let $J$ be an ideal containing $S$, then it is necessarily closed under right and left multiplication of $A$. Thus by construction $I(S) \subseteq J$ and $I(S)$ is the ideal generated by $S$.

**Proposition 2.14.** *Let $I \subset A$ be an ideal. The map $\mu : A/I \times A/I \to A/I$ defined by*

$$\mu(a + I, b + I) = a \cdot b + I$$

*where $\cdot$ is the multiplication in $A$, is a well defined operation on $A/I$. It follows that $(A/I, \mu)$ is an associative algebra.*

**Example 2.15.** ($U \otimes V$ as an associative algebra)

Let $(U, f)$ and $(V, g)$ be associative algebras over $k$. The bilinear multiplication operations $f : U \times U \to U$ and $g : V \times V \to V$ extend to linear maps $\tilde{f} : U \otimes U \to U$ and $\tilde{g} : V \otimes V \to V$ by the universal mapping property 2.6. From Proposition 2.8 we have the linear map $\tilde{f} \otimes \tilde{g} : (U \otimes U) \otimes (V \otimes V) \to U \otimes V$, for $u_1 \otimes u_2 \in U \otimes U$ and $v_1 \otimes v_2 \in V \otimes V$ we have $(u_1 \otimes u_2) \otimes (v_1 \otimes v_2) \mapsto \tilde{f}(u_1 \otimes u_2) \otimes \tilde{g}(v_1 \otimes v_2) = f(u_1, u_2) \otimes g(v_1, v_2)$. With the natural isomorphism $(U \otimes U) \otimes (V \otimes V) \cong (U \otimes V) \otimes (U \otimes V)$, this extends to a map $\tilde{\mu} : (U \otimes V) \otimes (U \otimes V) \to U \otimes V$. Thus we define multiplication by $\mu : (U \otimes V) \times (U \otimes V) \to U \otimes V$ where

$$m(u_1 \otimes v_1, u_2 \otimes v_2) = (u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = f(u_1, u_2) \otimes g(v_1, v_2)$$

It is routine to check that $\mu$ is associative, thus $(U \otimes V, \mu)$ is a unital associative algebra with unit element $1_U \otimes 1_V$.

Given vector spaces $V_1, \ldots, V_p$, the process above may be iterated to produce the p-fold tensor product $V^{\otimes p} := V_1 \otimes V_2 \otimes \cdots \otimes V_p$ with the universal mapping property relative to p-multilinear maps.

**Proposition 2.16.** *Let $X$ be a vector space. Then there is an isomorphism $Hom(V^{\otimes p}, X) \cong L_p(V, X)$, where $L_p(V, X)$ is the space of p-multilinear maps.*

**Definition 2.17.** Let $V$ be a vector space over some commutative field $k$. The tensor algebra of $V$ is defined as

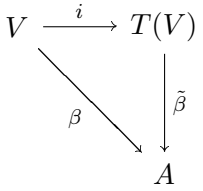
$$T(V) = \bigoplus_{k \geqslant 0} V^{\otimes k},$$

where $V^{\otimes i}$ are all tensors of rank $i$.

**Remark 2.18.** The tensor algebra has the following properties:

1. If $v \in T(V) \cap V^{\otimes i}$ for some $i$, then $v$ is called a *pure tensor* of rank $i$.
2. Every $v \in T(v)$ is a finite sum of pure tensors.
3. Define multiplication on $T(V)$ by multiplication on pure states $\mu : V^{\otimes k} \times V^{\otimes m} \to V^{\otimes k+m}$ where $\mu(x, y) = x \otimes y$. In general, this is highly non-commutative.
4. The inclusion map $i : V \to T(V)$ is injective.

The tensor algebra is the solution to the following universal mapping problem: Let $(A, k)$ be an associative algebra. If $\beta : V \to A$ is a homomorphism then there is a unique map $\tilde{\beta}$ which makes the following diagram commute:



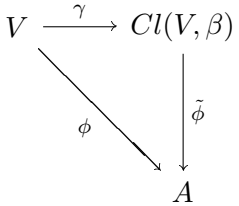

In fact for $x_j \in V$,

$$\tilde{\beta}(x_1 \otimes \cdots \otimes x_k) := \beta(x_1) \cdots \beta(x_k),$$

Let $V$ be a finite dimensional vector space over $\mathbb{C}$. A *bilinear form* on $V$ is a bilinear map $\beta : V \times V \to \mathbb{C}$. If $e_1, \ldots, e_n$ is a basis for $V$, then the components of the matrix associated with $\beta$ are given by $T_{ij} = \beta(e_i, e_j)$ and $\beta(x, y) = \langle x, Ty \rangle$. Then $\beta$ is symmetric if $\beta(x, y) = \beta(y, x)$ and nondegenerate if $\beta(x, y) = 0$ for all $y \in V$ implies $x = 0$.

**Definition 2.19.** A *Clifford algebra* for $(V, \beta)$ is a unital associative algebra $\mathrm{Cl}(V, \beta)$ with a linear map $\gamma : V \to \mathrm{Cl}(V, \beta)$ satisfying the following properties:

1. $\{\gamma(x), \gamma(y)\} := \gamma(x)\gamma(y) + \gamma(y)\gamma(x) = \beta(x, y)1$ where $1 \in \mathrm{Cl}(V, \beta)$ is the unit element.
2. $\gamma(V)$ generates $\mathrm{Cl}(V, \beta)$ as an algebra.
3. Given any unital associative algebra $A$ over $\mathbb{C}$ with a linear map $\phi : V \to A$ such that $\{\phi(x), \phi(y)\} = \beta(x, y)1$ where $1 \in A$, there exists a homomorphism $\phi : \mathrm{Cl}(V, \beta) \to A$ such that $\phi = \tilde{\phi} \circ \gamma$. It follows that $\mathrm{Cl}(V, \beta)$, unique up to isomorphism, is the solution to the following universal mapping problem:




Consider the two sided ideal $I(V, \beta) \subset T(V)$ generated by the set $S = \{z \in T(V) | z = x \otimes y + y \otimes x - \beta(x, y)1, \text{ for } x, y \in T(V)\}$, where $1 \in T(V)$ is the unit element. $I(V, \beta) = \{v \in T(V) | v = \sum_i a_i \otimes x_i \otimes b_i, \text{ for } x_i \in X, \ a_i, b_i \in T(V)\}$. Def Clifford algebra as

$$Cl(V, Q) = T(V)/I(V, \beta).$$

**Proposition 2.20.** *The Clifford Algebra has the following properties,*

1. *The inclusion map $i : V \to Cl(V, Q)$ is injective.*
2. *$dim(Cl(V, \beta)) = 2^n$, where $n = dim(V)$.*
3. *Consider the algebra homomorphism $\alpha : Cl(V, \beta) \to Cl(V, \beta)$ such that $\alpha\big(\gamma(v_1) \cdots \gamma(v_k)\big) = (-1)^k \gamma(v_1) \cdots \gamma(v_k)$. Then $\alpha^2(u) = u$ for all $u \in Cl(V, \beta)$. Let $Cl^+(V, \beta)$ be spanned by all even length products and $Cl^-(V, \beta)$ by spanned by the odd length products. There is a decomposition*
$$Cl(V, \beta) = Cl^+(V, \beta) \oplus Cl^-(V, \beta). \tag{2.8}$$
*This decomposition is often refereed to as a $\mathbb{Z}_2$-grading on $Cl(V, \beta)$.*

**Definition 2.21.** Let $S$ be a vector space over $\mathbb{C}$ and let $\gamma : V \to End(S)$ be a linear map. Then the pair $(S, \gamma)$ is a *space of spinors* for $(V, \beta)$ if

1. $\{\gamma(x), \gamma(y)\} = \beta(x, y)1$ for all $x, y \in V$
2. Only the trivial subspace 0 and $S$ are invariant under $\gamma(V)$.

If $(S, \gamma)$ is a space of spinors for $(V, \beta)$, then the map $\gamma$ extends to a homomorphism

$$\tilde{\gamma} : \mathrm{Cl}(V, \beta) \to \mathrm{End}(S).$$

The homomorphism $\tilde{\gamma}$ along with property 2 above are referred to as an *irreducible representation* of $\mathrm{Cl}(V, \beta)$. Let $(S, \gamma)$ and $(S', \gamma')$ be spaces of spinors for $(V, \beta)$. If there is a linear bijection $T : S \to S'$ such that
$$T\gamma(v) - \gamma'(v)T = 0$$
for all $v \in V$, then we say that $(S, \gamma)$ and $(S', \gamma')$ are isomorphic.

**Theorem 2.22.** *Let $n = dim(V)$.*

1. *If $n$ is even, then up to isomorphism there is exactly one space of spinors and $dim(S) = 2^{\frac{n}{2}}$.*
2. *If $n$ is odd, then there are two isomorphism classes of spinors, $(S, \gamma)$ and $(S', \gamma')$, and $dim(S) = dim(S') = 2^{\lfloor \frac{n}{2} \rfloor}$.*

**Proposition 2.23.** *Let $n = dim(V)$.*

1. *Suppose $n$ is even. Let $(S, \gamma)$ be a space of spinors for $(V, \beta)$. Then $(End(S), \gamma)$ is a Clifford algebra for $(V, \beta)$.*
2. *Suppose $n$ is odd. Let $(S, \gamma_+)$ and $(S, \gamma_-)$ be the two non-isomorphic classes of spinors. Define $\gamma : V \to End(S) \oplus End(S)$ by $\gamma(v) = \gamma_+(v) \oplus \gamma_-(v)$. Then $(End(S) \oplus End(S), \gamma)$ is a Clifford algebra for $(V, \beta)$.*

The Clifford Algebra and its representations are the central mathematical structures of Dirac's relativistic wave theory.

# 3 Relativity and the Dirac Equation

The following is a short introduction to the Dirac equation. A full treatment of the subject is found in Grant [8].

With his conception of special relativity, Albert Einstein showed we really live in a 4-dimensional space time called *Minkowski spacetime*. Minkowski spacetime, $\mathcal{M}$ is the real subspace of a *Lorentzian Manifold* $(V,\ \beta)$, where $V = \mathbb{C}^{1+3}$ and $\beta\colon V \times V \to \mathbb{C}$ is a symmetric bilinear map: $\beta(v, w) = \langle v, gw\rangle$, we call $g$ the *Minkowski metric*. It can be shown that there exists an orthonormal basis of $V$ in which $g$ takes on the nice representation

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We write $(1 + 3)$ to identify the signature of the metric, in our case we take the convention above. The search for a relativistic wave equation to describe a spin-$\frac{1}{2}$ particle i.e., the free electron, came to fruition in the Dirac equation. Paul Dirac originally formulated the Dirac equation by considering the square root of the wave operator, $\Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$ in 1 time and 3 space dimensions. That is, the Dirac operator is a differential operator, $\not{\partial}\colon \mathbb{C}^\infty(V, S) \to \mathbb{C}^\infty(V, S)$ which satisfies

$$\not{\partial}^2 = \Box, \tag{3.1}$$

for some vector space $S$. Using Einstein summation convention, such an operator takes on the form $\not{\partial} = \gamma^\mu \partial_\mu$ where the $\gamma^\mu$ will be defined. Let's generalize to the $(1 + d)$ dimensional case. The Dirac equation is given as follows with $\mu \in \{0, \ldots, d\}$:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0, \tag{3.2}$$

where $m$ is the mass, $\psi(x) \in S$ and $\psi \in \mathbb{C}^\infty(V, S)$. By the definition of the equation, we need our $\gamma$ coefficients to act on $S$. Thus we take $\gamma^\mu \in End(S)$.

**Proposition 3.1.** *The $\gamma^\mu$ generate a basis of $End(S)$.*

Define $\gamma : V \to \mathrm{End}(S)$ by $\gamma(e_\mu) = \gamma^\mu$. Notice by (3.1) we get a defining anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{3.3}$$

Since not all $\gamma^\mu$ commute, by lemma 2.4 they cannot not be simultaneously diagonalized. Thus 0 and $S$ are the only invariant subspaces of S under $\gamma(V)$.

**Corollary 3.2.** *$(S, \gamma)$ is a space of spinors for $(V, g)$. Theorem 2.22 helps characterize Dirac's space of spinors for a $n$ dimensional Minkowski space: $S \cong \mathbb{C}^{2^{\frac{n}{2}}}$ or $S \cong \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$.*

**Example 3.3.** The (1+2) dimensional Dirac equation

Consider Minkowski spacetime of 1+2 dimensions, $(\mathcal{M}, \beta)$ where

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The Dirac equation is given by

$$(i\gamma^0\partial_0 + i\gamma^1\partial_1 + i\gamma^2\partial_2 - m)\psi(t, x_1, x_2) = 0 \tag{3.4}$$

where the $\psi(t, x_1, x_2) \in S = \mathbb{C}^2$, as given by Theorem 2.22 and a representation of the $\gamma$ matrices is given by, $\gamma^0 = -\sigma_x$, $\gamma^1 = \sigma_x\sigma_z$ and $\gamma^2 = \sigma_z$ for Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**Remark 3.4.** (Plane wave solutions) The Dirac equation (3.4) has solutions

$$\psi(x) = e^{-i(t\omega(|\mathbf{k}|)-\mathbf{x}\cdot\mathbf{k})}u(k) \tag{3.5}$$

where $k$ is momentum and $\omega$ is energy. The energy is given by

$$\omega(|\mathbf{k}|) = \pm\sqrt{m^2c^2 + \mathbf{k}^2} \tag{3.6}$$

which we refer to as the dispersion relation.

**Figure 3.5.** For the $m = 0$, massless $(1 + 2)$ dimensional Dirac equation, the dispersion relation is cone shaped.

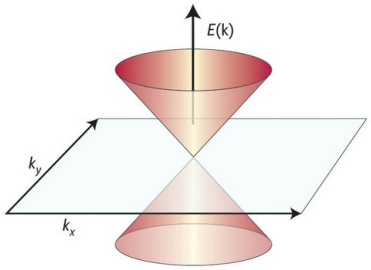



Image courtesy of Ryu, Murdy, Obuse, and Furusaki [20]

# 4 Random walks and the discrete heat equation

A simple random walk on a lattice describes a walker, who at each timestep, makes a random move of length one in one of the lattice directions. More formally, a random walk can be defined by considering a probability distribution, $p_n(x)$ on the lattice. The probability of the walker occupying a given vertex at the $n^{th}$ time step is thus given by $p_n(x)$. Let $\Pr(i|j)$ be the probability the walker moves to the $i^{th}$ vertex given he is previously at the $j^{th}$. By arranging $p_n(x)$ in a column vector $p_n$, the evolution of the random walk is naturally given by a Markov transition matrix $M$ where,

$$M_{ij} = \Pr(i|j) \quad \text{and} \quad Mp_n = p_{n+1}.$$

Random walks have nice behavior on the integer lattice $\mathbb{Z}^d$ and on finite integer lattices with proper boundary conditions. Our short account of random walks can be found in more detail in Lawler [14] or Doyle and Snell [5].

**Example 4.1.** The discrete heat equation on a finite 1 dimensional lattice

Consider the subset $A = \{1, 2, \ldots, N-1\} \subset \mathbb{Z}$ and $\partial A = \{0, N\}$. For each $x \in A$, let temperature be given by $p_n(x)$ at the $n^{th}$ time step. Let the heat be spread uniformly to its neighboring sites in the $n+1$ step. Then $p_{n+1}(x) = \frac{1}{2}(p_n(x-1) + p_n(x+1))$. Define linear operators $d_n$ and $\mathcal{L}$ by:

$$d_n[p_n(x)] = p_{n+1}(x) - p_n(x) \qquad \text{(disrete time differential)} \tag{4.1}$$

$$\mathcal{L}[p_n(x)] = \frac{1}{2}(p_n(x-1) + p_n(x+1) - 2p_n(x)) \qquad \text{(discrete Laplacian)} \tag{4.2}$$

Then $d_n[p_n(x)] = \mathcal{L}[p_n(x)]$; this is the discrete heat equation!

**Theorem 4.2.** *Given the discrete heat equation on $A$, initial condition $p_o(x) = f(x)$ and boundary conditions $p_n(x) = 0$ on $\partial A$, there exists a unique solution $p_n(x)$ to the discrete heat equation given by the finite Fourier Series:*

1. $p_n(x) = \sum_{j=1}^{N-1} c_j [\cos \frac{j\pi}{N}]^n \sin \frac{j\pi x}{N}$
2. $f(x) = \sum_{j=1}^{N-1} c_j \sin \frac{j\pi x}{N}$

*where* (2.) *defines the coefficients $c_j$.*

# 5 Dirac formalism

In this section we discuss notation and formalism standard in quantum mechanics. This notation was introduced by Dirac [4] with the goal that "from any given physical conditions, equations between the mathematical quantities may be inferred and vice versa" [4].

**Postulate 5.1.** *Associated to any isolated physical system is a* Hilbert space*, a vector space $\mathcal{H}$ over $\mathbb{C}$, and a bilinear form $\beta$ which gives rise to an inner product. The state of the system is completely described by a vector in $\mathcal{H}$, referred to as a* state vector*.*

**Definition 5.2.** Let $\mathcal{H}$ be a Hilbert space (possibly infinite dimensional) connected to a physical system. Then we refer the vectors in $\mathcal{H}$ as *ket vectors* and denote them by $|\cdot\rangle$. For a particular $\psi$ we write

$$|\psi\rangle. \tag{5.1}$$

Given an orthonormal basis $\{|v_i\rangle \in \mathcal{H}|\ i \in I\}$, there is a unique set of *scalars* $\{a_i \in \mathbb{C}|\ i \in I\}$ such that

$$|\psi\rangle = \sum_{i \in I} a_i |v_i\rangle \tag{5.2}$$

We call vectors in the dual Hilbert space *bra vectors* and denote them

$$\langle\psi|. \tag{5.3}$$

Consider the natural dual basis $\{\langle v_i| \in \mathcal{H}^*|\ i \in I\}$ defined by

$$\langle v_i| : \mathcal{H} \to \mathbb{C} \qquad \langle v_i|v_j\rangle := \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{5.4}$$

**Postulate 5.3.** *The evolution of a closed quantum system is described by a unitary transformation. Given a state $|\psi(t)\rangle$ there is a unitary operator $U$ such that*

$$U|\psi(t_1)\rangle = |\psi(t_2)\rangle. \tag{5.5}$$

**Definition 5.4.** Let $V$ and $W$ be Hilbert spaces. Then their tensor product $V \otimes W$ is a Hilbert space. Let $|v\rangle \in V$ and $|w\rangle \in W$ then we denote the tensor product in several equivalent ways:

1. $|v\rangle \otimes |w\rangle$
2. $|v\rangle|w\rangle$
3. $|v, w\rangle$

We adopt the last notation throughout this thesis.

# 6 Quantum random walks

The success of random walks in modeling a discrete heat equation motivates the search for a quantum analog that might implement a discrete Dirac equation. First let's point out a few properties of the Dirac equation:

1. The Dirac operator acts as a unitary operation on $S$, i.e., it preserves length.
2. The state $|\psi\rangle$ has the interpretation of a probability amplitude, i.e., $|\langle v_i|\psi\rangle|^2$ is the probability of observing the particle in the state $|v_i\rangle$.

The term **quantum random walk** (QRW) was first coined in 1993 by Aharonov and others [1]. In this paper we will discuss the discrete space and time quantum random walk in the setting of a quantum cellular automaton and one particle quantum lattice gas as described Meyer [16].

**Definition 6.1.** A *cellular automaton* (CA) consist of a lattice $L$ together with a vector field $\phi : \mathbb{N} \times L \to S$, where $S$ is the set of possible states. Naturally $\phi(t, x)$ is the *state* at the time $t$ and lattice position $x$. Let $E(t, x) \subset L$ be the set of lattice vertexes defining local neighborhoods. The evolution of the field is described by a local recurrence of the form

$$\phi(t + 1, x) = f(\phi(t, x)|x \in E(t, x)).$$

Let $\mathcal{H} = S$ be a discrete complex Hilbert space with basis $\{|x\rangle|x \in L\}$. If $\phi_t \in \mathcal{H}$ is a state vector we define a *quantum cellular automaton* (QCA) as a CA where $\phi(t, x)$ is the complex scalar coefficient of $|x\rangle$ and the evolution is local and unitary, $\phi_{t+1} = U\phi_t$ or equivalently

$$\phi_{t+1}(x) = \sum_{e \in E(t,x)} w(t, x + e)\phi_t(x + e),$$

where the $w(t, x + e)$ are constrained by the unitarity condition. If both $E(t, x)$ and $w(t, x + e)$ are independent of $t$ and $x$, the QCA is homogeneous.

It was noticed by Grössing and Zeilinger [9] that in the case of a one dimensional lattice, the above definition for a QCA was unsatisfactory. The following *no-go theorem* generalizes this idea for higher dimensions.

**Theorem 6.2.** *(Meyer [18]) For any dimension, the only homogeneous, scalar unitary CA (QCA) evolve by a constant translation with an overall phase multiplication.*

To overcome the difficulty presented above we modify the Hilbert space to account for an extra internal degree of freedom. This re-setup is motivated by the physical apparatus the one particle QLGA is designed to model. Meyer [17] describes it as follows: Consider a spin-$\frac{1}{2}$ particle, whose state is described by a 2-vector, moving on a lattice. At each vertex sits a nucleus of some sort. At the beginning of each timestep the particle is located at some vertex and is labelled with a velocity indicating along which edge incident to that vertex it will move during the advection half of the timestep. After moving along the designated edge to the the next vertex, a scattering occurs as the particle collides with the fixed nucleus according to some fixed rule which assigns new velocity labels. The evolution of the state is described by repeated advection and scattering.

**Definition 6.3.** Let $\mathcal{H}_p$ be the position space with an orthonormal basis given by $\{|x\rangle \mid x \in L\}$ and $\mathcal{H}_p$ be the internal velocity space where $\mathcal{H}_p \cong \mathbb{C}^d$ and $d = |E(t, x)|$. The value $d$ is well defined for the homogeneous case and $\{|\alpha\rangle| \ \alpha$ is an edge of $x\}$ forms an orthonormal basis for $\mathcal{H}_p$. Thus

we define our Hilbert space as $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_s$ with basis $\{|x, \alpha\rangle\}$. At each time the state of the QLGA is described by a state vector in $\mathcal{H}$:

$$\Psi(t) = \sum_{x,\alpha} \psi_\alpha(t, x)|x, \alpha\rangle \tag{6.1}$$

where $\psi_\alpha(t, x) \in \mathbb{C}$ and the norm of $\Psi(t)$ as measured by the inner product on $\mathcal{H}$ is:

$$\sum_{x,\alpha} \overline{\psi_\alpha(t, x)}\psi_\alpha(t, x) = 1. \tag{6.2}$$

Since the evolution is unitary, the inner product of the state is preserved. Thus we may interpret $\overline{\psi_\alpha(t, x)}\psi_\alpha(t, x)$ as the probability of the particle to be in the state $|x, \alpha\rangle$ at the time $t$. We enforce a *locality* condition. The vertex $y \in E(t, x)$ or equivalently $x = y + \beta$ if and only if

$$\langle x, \alpha|U|y, \beta\rangle \neq 0. \tag{6.3}$$

**Example 6.4.** A QRW on a one dimensional lattice [17]

Let $L$ be a one dimensional lattice. In the infinite case $L \cong \mathbb{Z}$ and in the finite case $L \cong \mathbb{Z}/n\mathbb{Z}$. This enforces a uniform lattice spacing and in the finite case periodic boundary conditions, $|a\rangle = |i \bmod n\rangle$. Let $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_s$ where a basis is given by $\{|x, 1\rangle,\ |x, -1\rangle \mid x \in L\}$. For $\Psi(t) = \sum_{x,\alpha} \psi_\alpha(t, x)|x, \alpha\rangle$ define the evolution by a unitary operation on the basis states by

$$U|x, \alpha\rangle = a|x + \alpha, \alpha\rangle + b|x + \alpha, -\alpha\rangle. \tag{6.4}$$

Let $A|x, \alpha\rangle = |x + \alpha, \alpha\rangle$ define the advection half of the time step and $S = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ define the scattering half of the time step. This leads to a natural decomposition of evolution as $U = (I_p \otimes S)A$, where $I_p$ is the identity on $\mathcal{H}_p$. Up to a phase multiplication of the form $e^{i\phi}$, $a = \cos\theta$ and $b = \sin\theta$ [16]. Letting $\psi(t, x) = (\psi_{-1}(t, x), \psi_1(t, x))$ we may also write our state vector as

$$\Psi(t) = \sum_x \psi(t, x)|x\rangle. \tag{6.5}$$

We can rewrite the action of $U$ as

$$U\psi(t, x) = \psi(t + 1, x) = w_{-1}\psi(t, x - 1) + w_1\psi(t, x + 1), \tag{6.6}$$

where

$$w_{-1} = \begin{pmatrix} 0 & i\sin\theta \\ 0 & \cos\theta \end{pmatrix} \quad w_1 = \begin{pmatrix} \cos\theta & 0 \\ i\sin\theta & 0 \end{pmatrix}. \tag{6.7}$$

**Theorem 6.5.** *Let $\Delta x$ and $\Delta t$ be the lattice and time interval length respectively, $\theta$ be the parameter in the scattering operation and $m$ be the mass. If one makes the association $m = \tan\theta$ then the $\Delta x = \Delta t \to 0$ limit of the discrete time evolution QRW on a one dimensional lattice is the (1+1) dimensional Dirac equation.*

The proof, presented by Meyer [16], was motivated by the sum-over-histories approach to quantum mechanics introduced by Feynman [6] for the Dirac particle in one dimension. The $(1 + 1)$ dimensional Dirac equation describes the dynamics of a free spin-$\frac{1}{2}$ particle. Theorem 2.22 shows that in $(1 + 1)$ dimensions for the space of spinors $S$, $\deg(S) = 2$ and $S \cong \mathbb{C}^2$. We were thus free to associate the internal degree of freedom for the QRW on a one dimensional lattice with the space of spinors in our Dirac equation

$$S \cong \mathcal{H}_s \cong \mathbb{C}^2. \tag{6.8}$$

This association really is the key to the many nice results in this section.

**Remark 6.6.** (Plane waves and a dispersion relation) Without taking the continuum limit, the quantum random walk still reproduces the quantum mechanical phenomena of plane waves obeying a dispersion relation. Let $L \cong \mathbb{Z}/N\mathbb{Z}$ and consider the translation operator $T$ defined by $T\psi(t,x) = \psi(t,x+1)$. The eigenvalues of $T$ are $e^{ik}$ for $k = \frac{2\pi n}{N}$ and $n \in \{0,\dots,N-1\}$. The corresponding eigenvectors $\Psi^{(k)}(t) = \sum_x \psi(t,x)|x\rangle$ satisfy:

$$\psi^{(k)}(x+1) = e^{ik}\psi^{(k)}(x). \tag{6.9}$$

The value $k$ has a natural interpretation as the *wave number* or *momentum*. Since our QRW is homogeneous by definition $UT = TU$ and by Lemma 2.4

$$U\Psi^{(k)}(t) = e^{-i\omega_k}\Psi^{(k)}(t). \tag{6.10}$$

The value $\omega_k \in \mathbb{R}$ takes the interpretation as the *frequency* or *energy*. The eigenvectors $\Psi^{(k)}$ are the discrete analogues of plane waves since they evolve by phase multiplication. From equations 6.6 and 6.7 we have

$$\begin{aligned} e^{-i\omega_k}\psi^{(k)}(x) &= w_{-1}\psi^{(k)}(x-1) + w_1\psi^{(k)}(x+1) \\ &= (e^{-ik}w_{-1} + e^{ik}w_1)\psi^{(k)}(x) \\ &= \begin{pmatrix} e^{ik}\cos\theta & e^{-ik}i\sin\theta \\ e^{ik}i\sin\theta & e^{-ik}\cos\theta \end{pmatrix} \\ &=: D(k)\psi^{(k)}(x). \end{aligned} \tag{6.11}$$

Let $I$ be the identity in $M_2(\mathbb{C})$, we get that the eigenvalues $e^{-i\omega_k}$ must satisfy

$$\begin{aligned} 0 &= \det(D(k) - e^{-i\omega_k}I) \\ &= \det\begin{pmatrix} e^{ik}\cos\theta - e^{-i\omega_k} & e^{-ik}i\sin\theta \\ e^{ik}i\sin\theta & e^{-ik}\cos\theta - e^{-i\omega_k} \end{pmatrix}\psi^{(k)}(x) \\ &= (e^{ik}\cos\theta - e^{-i\omega_k})(e^{-ik}\cos\theta - e^{-i\omega_k}) + \sin^2\theta \\ &= e^{-2i\omega_k} - (e^{-ik-i\omega_k} + e^{ik-i\omega_k})\cos\theta + \cos^2\theta + \sin^2\theta \\ &= 2(\cos\omega_k - \cos k\cos\theta)(\cos\omega_k - i\sin\omega_k). \end{aligned} \tag{6.12}$$

The characteristic equation for $D(k)$ produces the *dispersion relation*:

$$\cos\omega = \cos\theta\cos k. \tag{6.13}$$

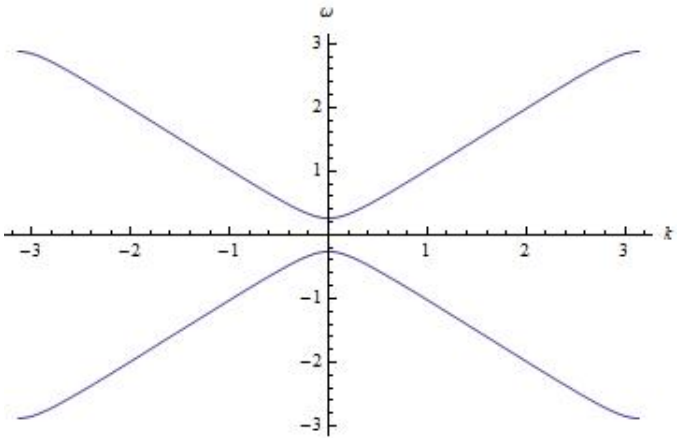



**Figure 6.7.** The dispersion relation for $\theta = \pi/12$.

# 7 Quantum random walk on the honeycomb lattice

We seek to generalize the example above to a two dimensional lattice. For the one dimensional lattice, the key was to associate the internal degree of freedom, $\mathcal{H}_p \cong \mathbb{C}^2$, with the space of spinors for the Dirac equation, $S \cong \mathbb{C}^2$. For any non-trivial planar lattice however, $E(x,t) \geqslant 3$ while for the space of spinors for the $(2+1)$ dimensional Dirac equation, $S \cong \mathbb{C}^2$. To overcome this apparent difficulty, we look for a two dimensional subspace of $\mathcal{H}_p$, say spanned by $\{|v_1\rangle, |v_2\rangle\}$, such that the subspace $\mathcal{H}' = \text{span}\{|x, v_1\rangle, |x, v_2\rangle|$ for all $x \in L\} \subset \mathcal{H}$ is invariant under time evolution of a QRW. To simplify life we work on a honeycomb lattice $L$ where $E(x,t) = 3$ for all $x \in L$ and $t \in \mathbb{N}$. Thus it follows that $\mathcal{H}_p \cong \mathbb{C}^3$.

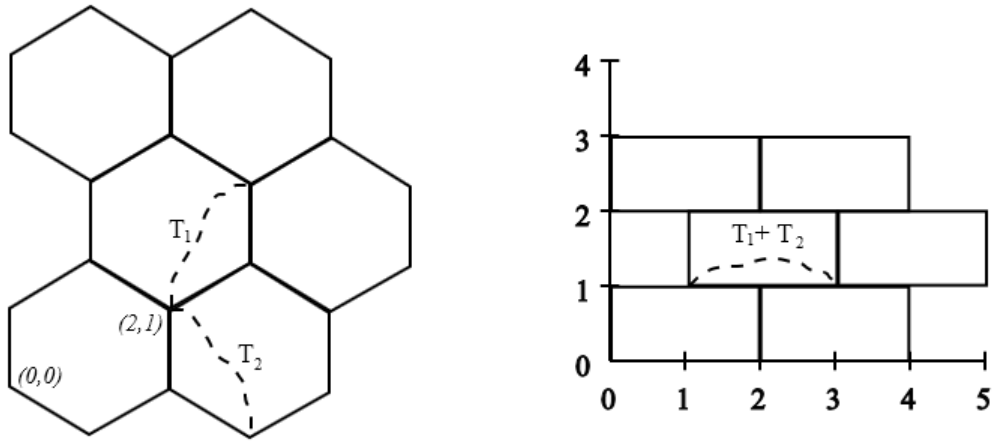



**Figure 7.1.**
Two isomorphic representations of the hexagonal lattice; the left we call the honeycomb and the right we call the brick wall.

**Definition 7.2.** Consider a regular hexagon with side length 1. Tile the plane with these regular hexagons. The *hexagonal lattice* is the set of vetices and edges formed by the tiling of the plane. Notice that $|E(x)| = 3$ for $x \in L$. Let $A$ be the set of vertices $(x,y) \in L$ such that $(z,w) \in L$ is a neighbor if and only if $(z,w) = (x, y+1)$, $(z,w) = (x + \frac{\sqrt{3}}{2}, y - \frac{1}{2})$ or $(z,w) = (x - \frac{\sqrt{3}}{2}, y - \frac{1}{2})$ and $B$ be the remaining vertices. Then $A \cup B$ forms a bi-partitioning of $L$. This bi-partitioning is seen more vividly if we think of the hexagonal lattice in the brick wall perspective. Here the vertex set is $\mathbb{Z}^2$. A vertex $(x,y) \in L$ is *even* if $x + y = 2n$ and odd otherwise. Define the neighborhoods by

$$E((x,y)) = \begin{cases} \{(x-1,y),(x+1,y),(x,y+1)\} & \text{if } (x,y) \text{ even} \\ \{(x+1,y),(x-1,y),(x,y-1)\} & \text{if } (x,y) \text{ odd} \end{cases}$$

This forms a natural bi-partition of $L$ by even and odd sites. Notice that the neighborhood of a vertex is composed completely of the opposite parity. To distinguish the first lattice from the brick wall, we call it the *honeycomb*, see the figure 7.1. In practice we will work with the brick wall picture but have the honeycomb in mind. Define a finite periodic hexagonal lattice by $L \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for $m \geqslant 2$ and $n \geqslant 4$. Set $m = 2k$ and $n = 2l$ to ensure to even vertices are strictly adjacent to odd vertices.

Let $T$ be a finite periodic hexagonal lattice. Let $\mathcal{H}_p$ be the position space with a basis $\{|(x,y)\rangle|\ (x,y) \in L\}$. Let $\mathcal{H}_s$ be the velocity space. Recall that $E((x,y)) = 3$ for all $(x,y) \in L$, thus $\dim(\mathcal{H}_p) = 3$. At even sites a particle will have as a basis $\{|1\rangle, |2\rangle, |3\rangle\} := \{|\swarrow\rangle, |\searrow\rangle, |\uparrow\rangle\}$ and at odd sites $\{|1\rangle, |2\rangle, |3\rangle\} := \{|\nearrow\rangle, |\nwarrow\rangle, |\downarrow\rangle\}$. The numbering scheme will be convenient when we define the time evolution. It follows that the set $\{|(x,y),\alpha\rangle|\ (x,y) \in L$ and $\alpha \in \{1,2,3\}\}$ forms

a basis for our Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_s$. By proposition 2.9 $\dim(\mathcal{H}) = 3mn$. We define a state vector as

$$\Psi(t) = \sum_{(x,y),\alpha} \psi_\alpha(t,x,y)|(x,y),\alpha\rangle, \tag{7.1}$$

noting that $\alpha$ depends on the parity of the vertex.

Define maps $\phi_{even},\ \phi_{odd} : \mathcal{H}_s \to L$ by

$$\left\{ \begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \end{array} \right\} = \left\{ \begin{array}{c} |\swarrow\rangle \\ |\searrow\rangle \\ |\uparrow\rangle \end{array} \right\} \longmapsto \left\{ \begin{array}{c} (-1,0) \\ (1,0) \\ (0,1) \end{array} \right\}; \qquad \left\{ \begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \end{array} \right\} = \left\{ \begin{array}{c} |\nearrow\rangle \\ |\nwarrow\rangle \\ |\downarrow\rangle \end{array} \right\} \longmapsto \left\{ \begin{array}{c} (1,0) \\ (-1,0) \\ (0,1) \end{array} \right\}.$$

We define the advection half of our time evolution, motivated by the one dimensional case, as

$$A|(x,y),\alpha\rangle = \left\{ \begin{array}{l} |(x,y) + \phi_{even}(\alpha),\alpha\rangle \text{ if } (x,y) \text{ even} \\ |(x,y) + \phi_{odd}(\alpha),\alpha\rangle \text{ if } (x,y) \text{ odd} \end{array} \right. \tag{7.2}$$

On the honeycomb lattice, the scattering step of our quantum random walk is the operator $I_p \otimes S \colon \mathcal{H}_p \otimes \mathcal{H}_s \to \mathcal{H}_p \otimes \mathcal{H}_s$. The local scattering operator $S \in U(3)$ can be parametrized as

$$S = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$$

where $a = e^{i\alpha}\cos(\theta)$ and $b = Be^{i\beta}\sin(\theta)$ . We may factor out $e^{i\alpha}$ from $S$, accounting for an unobservable phase. Given that $S$ is a unitary operator, the following conditions must hold:

$$|a|^2 + 2|b|^2 = 1 \tag{7.3}$$

$$\bar{a}b + a\bar{b} + |b|^2 = 0 \tag{7.4}$$

From (7.3) we find that our normalizing constant $B = \frac{1}{\sqrt{2}}$ and from (7.4) we find that

$$\begin{aligned} 0 &= \cos\theta(\frac{1}{\sqrt{2}}e^{i\beta}\sin\theta) + \cos\theta(\frac{1}{\sqrt{2}}e^{-i\beta}\sin\theta) + \frac{1}{2}\sin^2\theta \\ &= \frac{1}{\sqrt{2}}\sin\theta\cos\theta(e^{i\beta} + e^{-i\beta}) + \frac{1}{2}\sin^2\theta \\ &= \frac{1}{\sqrt{2}}\sin\theta\cos\theta(2\cos\beta) + \frac{1}{2}\sin^2\theta \end{aligned} \tag{7.5}$$

$$\implies \qquad \beta(\theta) = \cos^{-1}(-\frac{\sqrt{2}}{4}\tan(\theta)). \tag{7.6}$$

Thus the scattering matrix, unique up to multiplication by a phase, is

$$S = \begin{pmatrix} \cos\theta & e^{i\beta(\theta)}\sin\theta & e^{i\beta(\theta)}\sin\theta \\ e^{i\beta(\theta)}\sin\theta & \cos\theta & e^{i\beta(\theta)}\sin\theta \\ e^{i\beta(\theta)}\sin\theta & e^{i\beta(\theta)}\sin\theta & \cos\theta \end{pmatrix}. \tag{7.7}$$

Lemma 2.6 allows us to quickly retrieve the eigenvectors for $S$. We find an orthogonal set of eigenvectors for $S$ to be

$$\{v_1, v_2, v_3\} := \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}. \tag{7.8}$$

Let $\psi(t,x,y) = (\psi_1(t,x,y), \psi_2(t,x,y), \psi_3(t,x,y))$. Rewrite state vectors as

$$\Psi(t) = \sum_{(x,y)} \psi(t,x,y)|(x,y)\rangle \tag{7.9}$$

Thus we may define $U\psi(t,x,y) = \psi(t+1,x,y)$ by

$$\psi(t+1,x,y) = \begin{cases} w_1\psi(t,x-1,y) + w_2\psi(t,x+1,y) + w_3\psi(t,x,y+1) & \text{if } (x,y) \text{ even} \\ w_1\psi(t,x+1,y) + w_2\psi(t,x-1,y) + w_3\psi(t,x,y-1) & \text{if } (x,y) \text{ odd} \end{cases}, \tag{7.10}$$

where

$$w_1 = \begin{pmatrix} \cos\theta & 0 & 0 \\ e^{i\beta(\theta)}\sin\theta & 0 & 0 \\ e^{i\beta(\theta)}\sin\theta & 0 & 0 \end{pmatrix}; \quad w_2 = \begin{pmatrix} 0 & e^{i\beta(\theta)}\sin\theta & 0 \\ 0 & \cos\theta & 0 \\ 0 & e^{i\beta(\theta)}\sin\theta & 0 \end{pmatrix}; \quad w_3 = \begin{pmatrix} 0 & 0 & e^{i\beta(\theta)}\sin\theta \\ 0 & 0 & e^{i\beta(\theta)}\sin\theta \\ 0 & 0 & \cos\theta \end{pmatrix}.$$

By working on the honeycomb lattice, which has the fewest edges per vertex, we were able to describe the time evolution with essentially one free parameter. If we were on a lattice where $|E(x,y)| > 3$, say a square or triangular lattice, the increased degrees of freedom would correspond to more parameters in the time evolution.

Let $(x,y) \in Ł$ be some point on the lattice. The velocity $|v\rangle \in \mathcal{H}_s$ bundled at $P$, after advection and scattering, is completely determined by the velocities of its three neighboring lattice sites.

**Question**: *If the velocities at each neighboring site of $(x,y)$, $E(x,y)$, are in some subspace of $\mathcal{H}_s \cong \mathbb{C}^3$ spanned by any pair of eigenvectors of the local scattering matrix $S$ (7.7), under what conditions will $|v\rangle$ lie in that same subspace? WLOG consider the case where $(x,y)$ is even*

Let

$$\{v_1, v_2, v_3\} := \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}. \tag{7.11}$$

*Case 1:* Suppose the velocities at each neighboring site lives in the eigenspace of $\lambda = a - b$. Let $|v_i\rangle = a_i v_2 + b_i v_3$, where $i = 1$ denotes the left neighbor, $i = 2$ the right neighbor, and $i = 3$ the neighbor above. We advect locally and determine

$$|v\rangle = \begin{pmatrix} a_1 + b_1 \\ -a_2 \\ -b_3 \end{pmatrix}$$

The scattering will preserve the subspace if and only if $|v\rangle \in Span\{v_2, v_3\}$. Solving the following augmented matrix will give us the exact condition when this happens.

$$\left(\begin{array}{cc|c} 1 & 1 & a_1 + b_1 \\ -1 & 0 & -a_2 \\ 0 & -1 & -b_3 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & a_1 + b_1 \\ 0 & 1 & (a_1 - a_2) + b_1 \\ 0 & 0 & (a_1 - a_2) + (b_1 - b_3) \end{array}\right).$$

Thus the subspace is preserved if the following holds:

$$(a_1 - a_2) + (b_1 - b_3) = 0 \tag{7.12}$$

*Case 2:* Suppose $|v_i\rangle = a_i v_1 + b_i v_2$.

$$|v\rangle = \begin{pmatrix} a_1 + b_1 \\ a_2 - b_2 \\ a_3 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & a_1 + b_1 \\ 1 & -1 & a_2 - b_2 \\ 1 & 0 & a_3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & a_1 + b_1 \\ 0 & 2 & (a_1 - a_2) + (b_1 + b_2) \\ 0 & 1 & (a_1 - a_3) + b_1 \end{array}\right)$$

$$\longrightarrow \left( \begin{array}{ccc|c} 1 & 1 & | & a_1 + b_1 \\ 0 & 2 & | & (a_1 - a_2) + (b_1 + b_2) \\ 0 & 0 & | & (\frac{1}{2}(a_1 + a_2) - a_3) + \frac{1}{2}(b_1 - b_2) \end{array} \right).$$

Thus the subspace is preserved if the following holds:

$$(\frac{1}{2}(a_1 + a_2) - a_3) + \frac{1}{2}(b_1 - b_2) = 0 \tag{7.13}$$

*Case 3:* Suppose $|v_i\rangle = a_i v_1 + b_i v_3$.

$$|v\rangle = \left( \begin{array}{c} a_1 + b_1 \\ a_2 \\ a_3 - b_3 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 1 & 1 & | & a_1 + b_1 \\ 1 & 0 & | & a_2 \\ 1 & -1 & | & a_3 - b_3 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 1 & 1 & | & a_1 + b_1 \\ 0 & 1 & | & (a_1 - a_2) + b_1 \\ 0 & 0 & | & (2a_2 - a_3 - a_1) + (b_3 - b_1) \end{array} \right)$$

Thus the subspace is preserved if the following holds:

$$(2a_2 - a_3 - a_1) + (b_3 - b_1) = 0 \tag{7.14}$$

Case 1 seems to be the most interesting. It involves two difference terms which represent the same velocity subspace at next nearest neighbor vertices. If these velocity subspaces were to be candidates for our space of spinors they must be smooth and bounded in the continuum limit. This in fact is the key idea to the next theorem.

**Theorem 7.3.** *Consider a QRW on the honeycomb lattice as defined above and let $\Delta x$ be the lattice spacing. Then as $\Delta x \to 0$ there exit a $2mn$ dimensional subspace $\mathcal{H}' \subset \mathcal{H}$ that is invariant under the time evolution.*

*Proof.* Recall that a given $\psi(t, x, y) = (\alpha(t, x, y), \beta(t, x, y), \gamma(t, x, y))$ where $\alpha,\ \beta,\ \gamma : \mathbb{N} \times L \to \mathbb{C}$ then

$$\Psi(t) = \sum_{(x,y)} \psi(t, x, y)\ |(x, y)\rangle.$$

Consider the eigenvectors, $V := \{v_1, v_2, v_3\}$, equation (7.8), of the local scattering matrix $S$. Since $S$ is unitary $V$ forms a basis for $\mathcal{H}_s$, $\{|v_2\rangle, |v_3\rangle, |v_1\rangle\}$. Thus we may write

$$\psi(t, x, y) = \alpha(t, x, y)|v_1\rangle + \beta(t, x, y)|v_2\rangle + \gamma(t, x, y)|v_3\rangle. \tag{7.15}$$

Let $\Delta x$ be the lattice spacing length. As $\Delta x \to 0$ we have $\alpha, \beta, \gamma : \mathbb{N} \times \mathbb{R}^2 \to \mathbb{C}$. Since we interpret these functions as components of a probability amplitude we enforce that they be analytic, bounded and first derivatives be bounded.

Let $(a, b) \in L$. Expand $\alpha(x, y)$ about $(a, b)$

$$\begin{aligned} \alpha(t, x, y) &= \sum_{i,j=0}^{\infty} \frac{(x - a)(y - b)}{i!j!} \frac{\partial^{i+j}}{\partial x^i \partial y^j} \alpha|_{(a,b)} \\ &= \alpha(t, a, b) + (x - a)\alpha_x(t, a, b) + (y - b)\alpha_y(t, a, b) \\ &\quad + \frac{1}{2}((x - a)^2 \alpha_{xx}(t, a, b) + 2(x - a)(y - b)\alpha_{xy}(t, a, b) + (y - b)^2 \alpha_{yy}(t, a, b)) + \cdots . \end{aligned} \tag{7.16}$$

Let $(c,d) \in E((a,b))$ thus $|a-c| \leqslant \Delta x$ and $|b-d| \leqslant \Delta x$. Choose $M \in \mathbb{R}$ so that $\alpha_x \leq M$ and $\alpha_y \leq M$. Then

$$\begin{aligned}
\alpha(t,c,d) - \alpha(t,a,b) &= (c-a)\alpha_x(t,a,b) + (d-b)\alpha_y(t,a,b) \\
&\quad + \frac{1}{2}((c-a)^2\alpha_{xx}(t,a,b) + 2(c-a)(d-b)\alpha_{xy}(t,a,b) + (d-b)^2\alpha_{yy}(t,a,b)) + \cdots \\
&\leq \Delta x\alpha_x(t,a,b) + \Delta x\alpha_y(t,a,b) + O(\Delta x^2) \\
&\quad \text{for small } \Delta x, \text{ we may drop the } O(\Delta x^2) \text{ term} \\
&\leq \Delta x M
\end{aligned} \tag{7.17}$$

Thus $\Delta x \to 0 \implies \alpha(t,c,d) - \alpha(t,a,b) = 0$. A similar computation shows that $\Delta x \to 0 \implies \beta(t,c,d) - \beta(t,a,b) = 0$. From 7.12 this is true if and only if the subspace

$$\mathcal{H}' = \text{span}\{|(x,y),v_2\rangle,\ |(x,y),v_3\rangle \mid \text{for all } (x,y) \in L\}$$

is preserved by the time evolution of the QRW. By construction $\dim(\mathcal{H}') = 2mn$.

□

# 8 Plane waves and dispersion

We say a quantum random walk is homogeneous if it commutes with translations. In one dimension, defining translations is natural: $T|x,\alpha\rangle = |x+1,\alpha\rangle$. In two dimensions we expect at least two nontrivial translations. Further difficulties appear in two dimensions for example when one considers the hexagonal lattice. In this case, with one step translations we lose the notions of vertex adjacency and orientation in the even odd vertex spin spaces. For a proper definition we consider translations to next nearest neighbors, i.e., the vertex $(0,0)$ has as its next nearest neighbors the set$\{(2,0),(1,1),(-1,1),(-2,0),(-1,-1),(1,-1)\}$. The following definitions then become very natural:

**Definition 8.1.** Translations on a hexagonal lattice are generated by two operators $T_1, T_2 \colon \mathcal{H} \to \mathcal{H}$ where

$$T_1|(a,b),\alpha\rangle = |(a+1,b+1),\alpha\rangle \tag{8.1}$$

$$T_2|(a,b),\alpha\rangle = |(a+1,b-1),\alpha\rangle \tag{8.2}$$

Notice

1. $(T_1+T_2)|(a,b),\alpha\rangle = |(a+2,b),\alpha\rangle$ (this is the familiar 1-dimensional 2-step translation)
2. $T_1 \circ T_2 = T_2 \circ T_1$ (translations commute)
3. $T_1$ and $T_2$ are unitary: $\langle(x,y),\alpha|T_1|(a,b),\alpha\rangle = \langle(x,y),\alpha|(a+1,b+1),\alpha\rangle = \langle(x-1,y-1),\alpha|(a,b),\alpha\rangle$. $\Rightarrow T_1^\dagger|(x,y),\alpha\rangle = |(x-1,y-1),\alpha\rangle$. Thus $T_1^\dagger = T_1^{-1}$. A similar argument can be made to show $T_2^\dagger = T_2^{-1}$.
4. $T_1$ and $T_2$ share an orthonormal set of eigenvectors $\Psi^{(k_1,k_2)}$, where
   - $T_1\Psi^{(k_1,k_2)} = e^{ik_1}\Psi^{(k_1,k_2)}$
   - $T_2\Psi^{(k_1,k_2)} = e^{ik_2}\Psi^{(k_1,k_2)}$
   - for eigenvalues given by $\{k_1,k_2\} \in \{2\pi\frac{d}{D} \mid D = lcm(m,n) \text{ and } d \in \mathbb{Z}, 1 \leqslant d \leqslant D\}$. We call these the wave numbers

Since our quantum random walk is homogeneous, i.e., commutes with translations, it inherits the eigenvectors of the translation operators

$$U\Psi^{(k_1,k_2)}(t) = \Psi^{(k_1,k_2)}(t+1) = e^{-i\omega_{k_1,k_2}}\Psi^{(k_1,k_2)}(t), \tag{8.3}$$

where we refer to the $\omega$'s as the frequencies. Recall if $x+y=2n$ then

$$U\psi(t,x,y) = \psi(t+1,x,y) = w_1\psi(t,x-1,y) + w_2\psi(t,x+1,y) + w_3\psi(t,x,y+1)$$

and if $x+y=2n+1$ then

$$U\psi(t,x,y) = \psi(t+1,x,y) = w_1\psi(t,x+1,y) + w_2\psi(t,x-1,y) + w_3\psi(t,x,y-1)$$

for $w_1 = Sr_{11}$, $w_2 = Sr_{22}$ and $w_3 = Sr_{33}$ as in equation (7.10). Here $S$ is the $3\times 3$ scattering

matrix and the $(r_{kk})_{ij} = \delta_{ik}\delta_{kj}$ Thus we have if $x + y = 2n$ then

$$\begin{aligned}
U^2\psi^{(k_1,k_2)}(t,x,y) &= U\psi^{(k_1,k_2)}(t+1,x,y) \\
&= w_1\psi^{(k_1,k_2)}(t+1,x-1,y) + w_2\psi^{(k_1,k_2)}(t+1,x+1,y) + w_3\psi^{(k_1,k_2)}(t+1,x,y+1) \\
&= w_1U\psi^{(k_1,k_2)}(t,x-1,y) + w_2U\psi^{(k_1,k_2)}(t,x+1,y) + w_3U\psi^{(k_1,k_2)}(t,x,y+1) \\
&= w_1[w_1\psi^{(k_1,k_2)}(t,x,y) + w_2\psi^{(k_1,k_2)}(t,x-2,y) + w_3\psi^{(k_1,k_2)}(t,x-1,y-1)]+ \\
&\quad w_2[w_1\psi^{(k_1,k_2)}(t,x+2,y) + w_2\psi^{(k_1,k_2)}(t,x,y) + w_3\psi^{(k_1,k_2)}(t,x+1,y-1)]+ \\
&\quad w_3[w_1\psi^{(k_1,k_2)}(t,x+1,y+1) + w_2\psi^{(k_1,k_2)}(t,x-1,y+1) + w_3\psi^{(k_1,k_2)}(t,x,y)] \\
&= [w_1^2 + w_2^2 + w_3^2 + w_1w_2(e^{-ik_1} + e^{-ik_2}) + w_1w_3e^{-ik_1}+ \\
&\quad w_2w_1(e^{ik_1} + e^{ik_2}) + w_2w_3e^{ik_2} + w_3w_1e^{ik_1} + w_3w_2e^{-ik_2}]\psi^{(k_1,k_2)}(t,x,y) \\
&=: D(k_1,k_2)\psi^{(k_1,k_2)}(t,x,y) \\
&= e^{-i2\omega_{k_1,k_2}}\psi^{(k_1,k_2)}(t,x,y).
\end{aligned} \tag{8.4}$$

A similar calculation can be done for $x + y = 2n + 1$.

$$D(k_1,k_2) = \begin{pmatrix} a^2 + b^2\left(2e^{ik_1} + e^{ik_2}\right) & b\left(be^{-ik_2} + a\left(1 + e^{-ik_1} + e^{-ik_2}\right)\right) & b\left(a + ae^{-ik_1} + be^{ik_2}\right) \\ b\left(be^{ik_1} + a\left(1 + e^{ik_1} + e^{ik_2}\right)\right) & a^2 + b^2\left(e^{-ik_1} + 2e^{-ik_2}\right) & b\left(a + be^{-ik_1} + ae^{ik_2}\right) \\ b\left(a\left(1 + e^{ik_1}\right) + b\left(e^{ik_1} + e^{ik_2}\right)\right) & b\left(a + ae^{-ik_2} + b\left(e^{-ik_1} + e^{-ik_2}\right)\right) & a^2 + b^2\left(e^{-ik_1} + e^{ik_2}\right) \end{pmatrix} \tag{8.5}$$

where $a = \cos\theta$ and $b = e^{i\cos^{-1}(-\frac{\sqrt{2}}{4}\tan\theta)}\sin\theta$. To find the dispersion relation, we solve the characteristic equation

$$\det(D(k_1,k_2) - e^{-2i\omega}I) = 0. \tag{8.6}$$

Although we have not succeeded in finding a closed form dispersion relation, we plot some numerical data in figure 8.2. We note that the graph seems to show our model produces a discrete model for the small mass Dirac equation.

**Figure 8.2.** (Mathematica code for dispersion relation)

```
(* S is the scaterring matrix, a = Cos[θ],
b = e^(iβ(θ)) Sin[θ] and β(θ)= ArcCos[-Sqrt[2]/4 Tan[θ]]*)
S = {{a, b, b}, {b, a, b}, {b, b, a}}; r11 = {{1, 0, 0}, {0, 0, 0}, {0, 0, 0}}; r22 = {{0, 0, 0}, {0, 1, 0}, {0, 0, 0}}; r33 = {{0, 0, 0}, {0, 0, 0}, {0, 0, 1}};
w1 = S.r11; w2 = S.r22 ; w3 = S.r33;

(*d[m,n,a,b] = D(k1,k2), with m=k1 and n=k2*)
d[m_, n_, a_, b_] = w1.w1 + w2.w2 + w3.w3 + (Exp[-I m] + Exp[-I n]) w1.w2 +
   w1.w3 (Exp[-I m]) + w2.w1 (Exp[I m] + Exp[I n]) + w2.w3 (Exp[I n]) +
   w3.w1 (Exp[I m]) + w3.w2 (Exp[-I n]);
disp[m_, n_, θ_] = d[m, n, Cos[θ], Exp[I ArcCos[-Sqrt[2] / 4 Tan[θ]]] Sin[θ]];
```

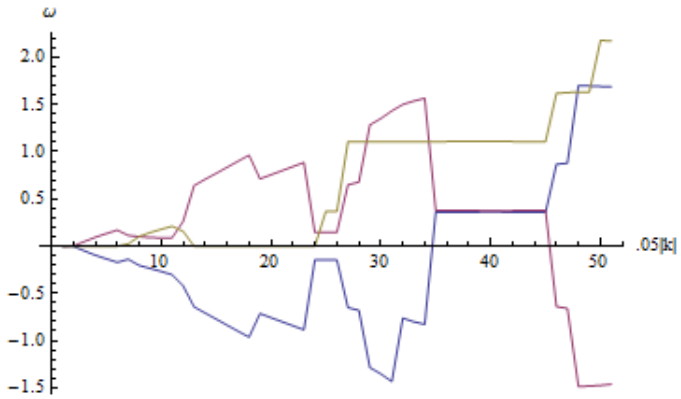
```
(* k1 = √3 / 2 x + 3/2 y = r((Sqrt[3]/2)Cos[t]+(3/2)Sin[t]) and
     k2 = √3 / 2 x - 3/2 y = r((Sqrt[3]/2)Cos[t]-(3/2) Sin[t])
      Notice the domain of β(θ) is ArcTan[-2 √2] ≤ θ ≤ ArcTan[2√2].
       The  θ → ArcTan[2√2] ≈ 1.2309594,
corresponds to the massless particle case*)
eigd[r_, t_] =
  N[disp[r ((Sqrt[3] / 2) Cos[t] + (3 / 2) Sin[t]), r ((Sqrt[3] / 2) Cos[t] - (3 / 2) Sin[t]),
    1.2309594]];

(*Solve for the eignevalue arguments and average over 0<θ<2Pi  *)
Table[
 Table[Mean[Transpose[Table[Arg[Eigenvalues[eigd[r, t]]], {t, 0, 2 Pi, Pi / 8}]][[i]]],
  {r, .01, 2.51, .05}], {i, 1, 3}]
```

Out[21]=
```
{{-0.000820053, -0.00289666, -0.0503278, -0.095795, -0.134452, -0.171622, -0.138948,
  -0.204697, -0.23598, -0.268459, -0.302178, -0.421803, -0.644314, -0.708873,
  -0.773065, -0.837048, -0.900987, -0.965039, -0.714487, -0.757564, -0.80069,
  -0.843908, -0.887248, -0.1453, -0.14728, -0.148529, -0.651896, -0.680252,
  -1.2859, -1.35386, -1.43501, -0.763393, -0.80109, -0.829126, 0.36056,
  0.357467, 0.357919, 0.360029, 0.361868, 0.362617, 0.362251, 0.361054, 0.359423,
  0.35785, 0.356963, 0.867985, 0.879147, 1.70188, 1.6975, 1.69319, 1.68897},
 {-0.000526547, 0.00154887, 0.0489767, 0.0944373, 0.133081, 0.170205, 0.113278,
  0.100124, 0.0906456, 0.088103, 0.0894017, 0.263072, 0.642992, 0.707554,
  0.771748, 0.835732, 0.899669, 0.96372, 0.713165, 0.756239, 0.799362,
  0.842576, 0.885911, 0.143958, 0.145933, 0.147177, 0.650539, 0.67889, 1.28454,
  1.35249, 1.43365, 1.50125, 1.53896, 1.567, 0.377327, 0.38042, 0.379961,
  0.377842, 0.375999, 0.375248, 0.375616, 0.376817, 0.378454, 0.380032,
  0.380925, -0.640378, -0.659708, -1.48718, -1.48343, -1.47547, -1.46364},
 {-0.000435673, -0.000434489, -0.000431224, -0.000424587, -0.00041068,
  -0.000364803, 0.0238871, 0.102791, 0.143552, 0.178574, 0.210994, 0.156949,
  -0.000458944, -0.000461674, -0.000462298, -0.000460992, -0.000457829, -0.000452685,
  -0.000445142, -0.000434268, -0.000418004, -0.000390957, -0.000331509, 0.00044419,
  0.369195, 0.369808, 1.10794, 1.10817, 1.10823, 1.10827, 1.10829, 1.1083, 1.10832,
  1.10834, 1.10838, 1.10842, 1.10846, 1.10848, 1.10848, 1.10846, 1.10844, 1.10841,
  1.1084, 1.10839, 1.10839, 1.6187, 1.62693, 1.63183, 1.63313, 2.1802, 2.17448}}
```

In[22]:=
```
ListLinePlot[%, AxesLabel → {".05|k|", "ω"}]
```

Out[22]=



# 9 Discussion

The power of quantum random walks has been implemented in the setting of quantum computing theory. Childs and Goldstone [3] noticed that for the spatial search problem, a quantum random walk algorithm, on a square lattice modeling a discrete Dirac equation, matched or showed improvement as compared to other quantum algorithms. Our result is potentially useful for implementing a quantum algorithm on a hexagonal lattice.

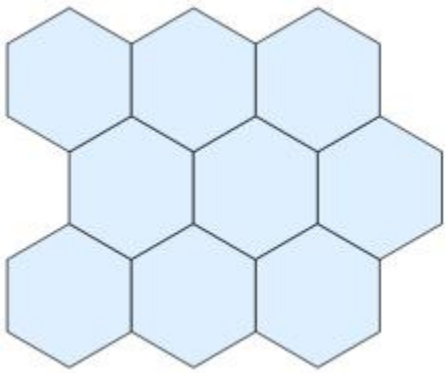
Graphene is a two dimensional carbon material whose structure is a hexagonal lattice. Recently graphene has received a fair amount of attention. The Nobel Prize in Physics was awarded to Andre Geim and Konstantin Novoselov "for groundbreaking experiments regarding the two-dimensional material graphene" [19]. From the tight-binding electron model one can realize electron transfer in graphene as a discrete Dirac equation [13]. A future direction of research will be to reconcile this tight-binding electron model with our QRW model.

# 10 Appendix

**Example 10.1.** (Generating the hexagonal lattice in Mathematica)

```
In[26]:= h[x_, y_] :=
          Polygon[Table[{2 (Cos[2 Pi k / 6 - Pi / 6] + x), 2 (Sin[2 Pi k / 6 - Pi / 6] + y)}, {k, 6}]]

In[37]:= Graphics[{EdgeForm[Opacity[.7]], LightBlue,
           Table[h[i + (1 + (Sqrt[3] / 2)) - (1 + (Sqrt[3] / 2))^(Mod[(2 / 3) j + 1, 2]), j],
            {i, 0, 5, Sqrt[3]}, {j, 0, 4, 3 / 2}]}]
```



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