

Graphs, Zeta Functions, Diameters, and Cospectrality

Sarah Elghraoui

University of California, San Diego
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Faculty Mentor: Professor Audrey Terras

University of California, San Diego Department of Mathematics

Much research has been done on the matter of graph theory and specifically on the information on graphs available from the Ihara zeta function as well as from the properties of cospectral and isomorphic graphs (We will define these terms later on). This paper builds on the research previously done in the discussion of how they are related to each other as well as with regard to diameter, a graph invariant.

To proceed, we define the essential terminology to this paper. A *graph* G is an ordered pair $(V;E)$ where V is a set and E is a set of unordered pairs whose elements are taken from V . An element of the set V is called a *vertex* and an element of the set E is called an *edge*, connecting two vertices. A *loop* is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints. A *simple graph* is one having no loops or multiple edges. When two vertices u and v are endpoints of an edge, we say they are *adjacent*. The *adjacency matrix* of a simple graph is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (v_i, v_j) according to whether v_i and v_j are adjacent or not. For a simple graph with no self-loops, the adjacency matrix must have 0s on the diagonal. For an undirected graph, the adjacency matrix is symmetric. Consequently, all of its eigenvalues are real. For this paper, all graphs will be assumed to be connected simple undirected graphs.

A graph is said to be *k-regular* if all of its vertices are of degree k , where degree is determined by the number of edges coming out of the vertex. A 0-regular graph has no edges, a 1-regular graph consists of disconnected edges, and a 2-regular graph consists of disconnected cycles. The first interesting case is therefore 3-regular graphs.

A *path* is a sequence $C = a_1 \dots a_s$, where a_j is an oriented edge of X . A *closed path* is when the starting vertex is the same as the terminal vertex. The closed path $C = a_1 \dots a_s$ is called a *prime path* if you can only go around the path once and the path has no backtracking $a_{i+1} \neq a_i^{-1}$ and no tail $a_s \neq a_1^{-1}$.

For the closed path $C = a_1 \dots a_s$, the equivalence class $[C]$ means the following $[C] = \{a_1 \dots a_s, a_2 \dots a_s a_1, \dots, a_s a_1 \dots a_{s-1}\}$. Two closed paths are equivalent if we get one from the other by changing the starting vertex. A *prime* in the graph X is an equivalence class $[C]$ of prime paths. The length of the path C is $v(C) = s$, the number of edges in C .

The *Ihara zeta function* for a finite connected graph (without degree 1 vertices) is defined to be the following function of the complex number u , with $|u|$ sufficiently small:

$$\zeta_X(u) = \zeta(u, X) = \prod_{[P]} (1 - u^{v(P)})^{-1}$$

where the product is over all primes $[P]$ in X . Recall that $v(P)$ denotes the length of P .

Theorem (Ihara theorem generalized by Bass) Let A be the adjacency matrix of X and Q the diagonal matrix with the j th diagonal entry q_j such that q_j+1 is the degree of the j th vertex of X . Suppose that r is the rank of the fundamental group of X ; $r-1 = |E| - |V|$. Then we have the *Ihara determinant formula*

$$\zeta_x(u)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2).$$

The Ihara zeta function is interesting because of its similarity to the Riemann zeta function. With the zeros of the Riemann zeta function being of primary interest, its counterpart is the poles of the Ihara zeta function which are just the zeros of the reciprocal of the Ihara zeta function.

We say that two graphs G, H are *isomorphic* if there exists a one-to-one function f from $V(G)$ onto $V(H)$ such that g defined by $g((x, y)) = (f(x), f(y))$ for all (x, y) belongs to $E_d(G)$ is a one-to-one function from $E_d(G)$ onto $E_d(H)$. Such a function f is a *graph isomorphism*.

Proceeding with setting the terminology, the set of graph eigenvalues of the adjacency matrix is called the *spectrum* of the graph. *Cospectral graphs* are graphs whose adjacency matrices share the same graph spectrum.

A *Graph invariant* is a property of graphs that depends only on the abstract structure, not on graph representations such as particular labelings or drawings of the graph. A graph invariant $I(G)$ is called *complete* if the identity of the invariants $I(G)$ and $I(H)$ implies the isomorphism of the graphs G and H .

The *distance* between two vertices x, y in X is the minimum number of edges in a path connecting x to y . If our graph is connected, such a path will exist. The maximum distance over all pairs x, y of vertices in X is the *diameter* of X , in other words it is the longest of the shortest path lengths between pairs of vertices.. The paths in the definition of the diameter, unlike primes, are non-closed paths. Diameter is a graph invariant.

For an undirected graph G on n vertices, we can find an upper bound on the diameter $D(G)$ by using eigenvalues of the Laplacian [1,2] as follows:

$$D(G) \leq \left\lceil \frac{\log(n-1)}{\log \frac{\lambda_{n-1} + \lambda_1}{\lambda_{n-1} - \lambda_1}} \right\rceil + 1$$

where $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n+1}$ denote the eigenvalues of the Laplacian of G .

Theorem: A connected graph G with diameter d has at least $d+1$ distinct eigenvalues. [1]

Proof: Let x and y be vertices of G such that the distance between them is d , and suppose that $x = w_0, w_1, \dots, w_d = y$ is a path of length d . Then, for each $i \in \{1, 2, \dots, d\}$, there is at least one path of length i , but no shorter path joining w_0 to w_i . Consequently A^i has a non-zero entry in a

position where the corresponding entries of $I, A, A^2, \dots, A^{i-1}$ are zero. It follows that A^i is not linearly dependent on $\{I, A, \dots, A^{i-1}\}$ and that $\{I, A, \dots, A^d\}$ is a linearly independent set in $A(G)$. Since this set has $d+1$ members, the dimension of $A(G)$ is $d+1$. Therefore, it must have at least $d+1$ distinct eigenvalues. □

Now that a few of the necessary definitions have been conveyed, we will focus on the information that can (or cannot) be extracted from the Ihara zeta function, and will also discuss the property of diameter and its relationship to the Ihara zeta function.

First, we will begin this discussion with the relationship of graph invariants on isomorphic graphs. Easily computable graph invariants are instrumental for fast recognition of graph isomorphism, or rather non-isomorphism, since for any invariant at all, two graphs with different values cannot (by definition) be isomorphic. Two graphs with the same invariants may or may not be isomorphic, however. Therefore, when we consider diameter, it a property preserved through isomorphism.

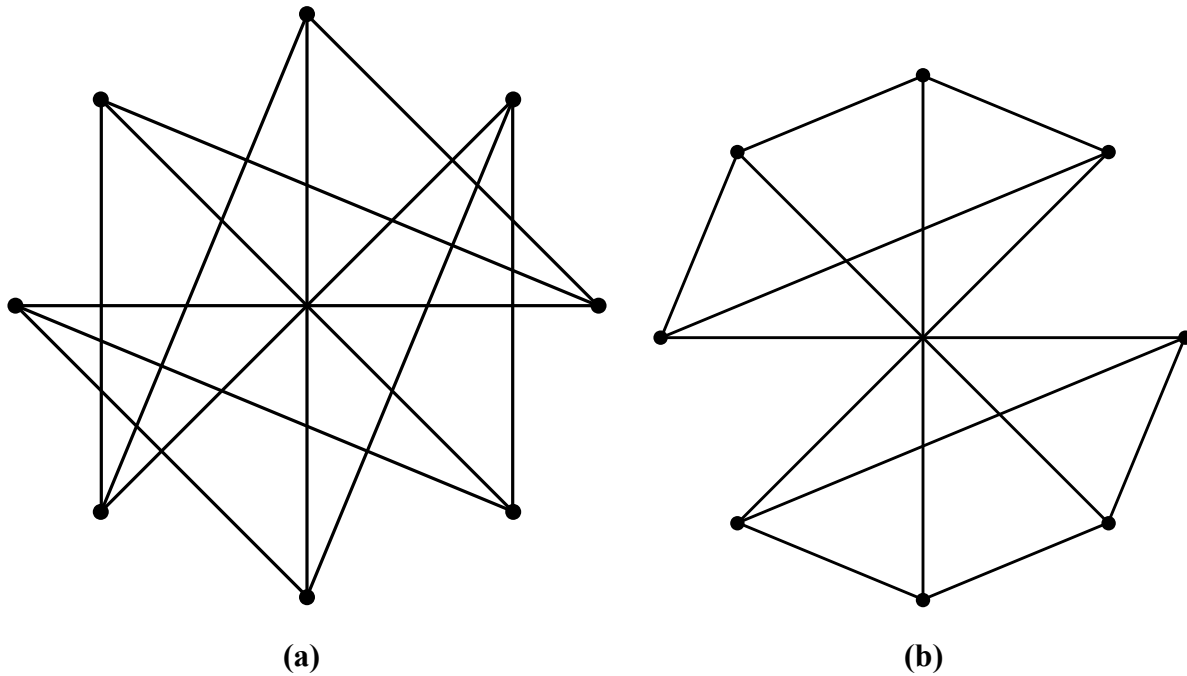
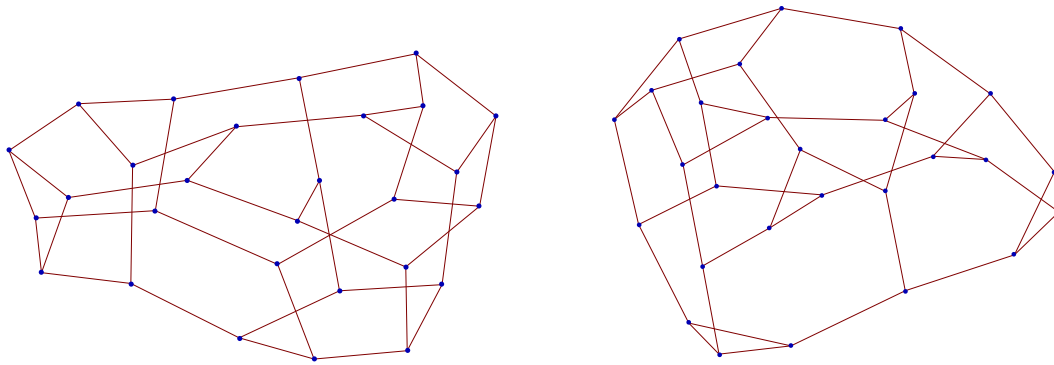


Figure 1. Two Isomorphic Graphs

Both of the above graphs in Figure 1 have a diameter of 3. However, graphs with the same diameter do not imply that it is isomorphic. Consider the graphs in Figure 2:



(a) (b)
Figure 2. Both are 3-regular graphs with Diameter 6

However when we enter the command to check for a graph isomorphism between the graphs in Figure 2 in Mathematica, we get FALSE. Therefore, diameter is not a complete graph invariant since it does not imply graph isomorphism.

Matthew Horton came to several conclusions regarding the Ihara zeta function and its relationship to a few of the various properties (see his paper for the justifications) [2]:

Information about G	How the information can be recovered from ζ_G
# of edges, $ E $	$= \frac{\deg(\zeta_G(u)^{-1})}{2}$
# of vertices, $ V $	$= \frac{\deg(\zeta_G(u)^{-1})}{2} - \text{order}_{u=1}(\zeta_G(u)) + 1$ unless $\zeta_G(u)^{-1} = (1-u^n)^2$ for some n in which case $ V =n$
G is isomorphic to H	cannot, in general, be determined from $\zeta_G(u)$, $\zeta_H(u)$ alone

Figure 3. Table of Information about G Recoverable from ζ_G

Two graphs with the same Ihara zeta function are not necessarily isomorphic, see for example the graphs constructed by Stark and Terras in Figure 4. [3]

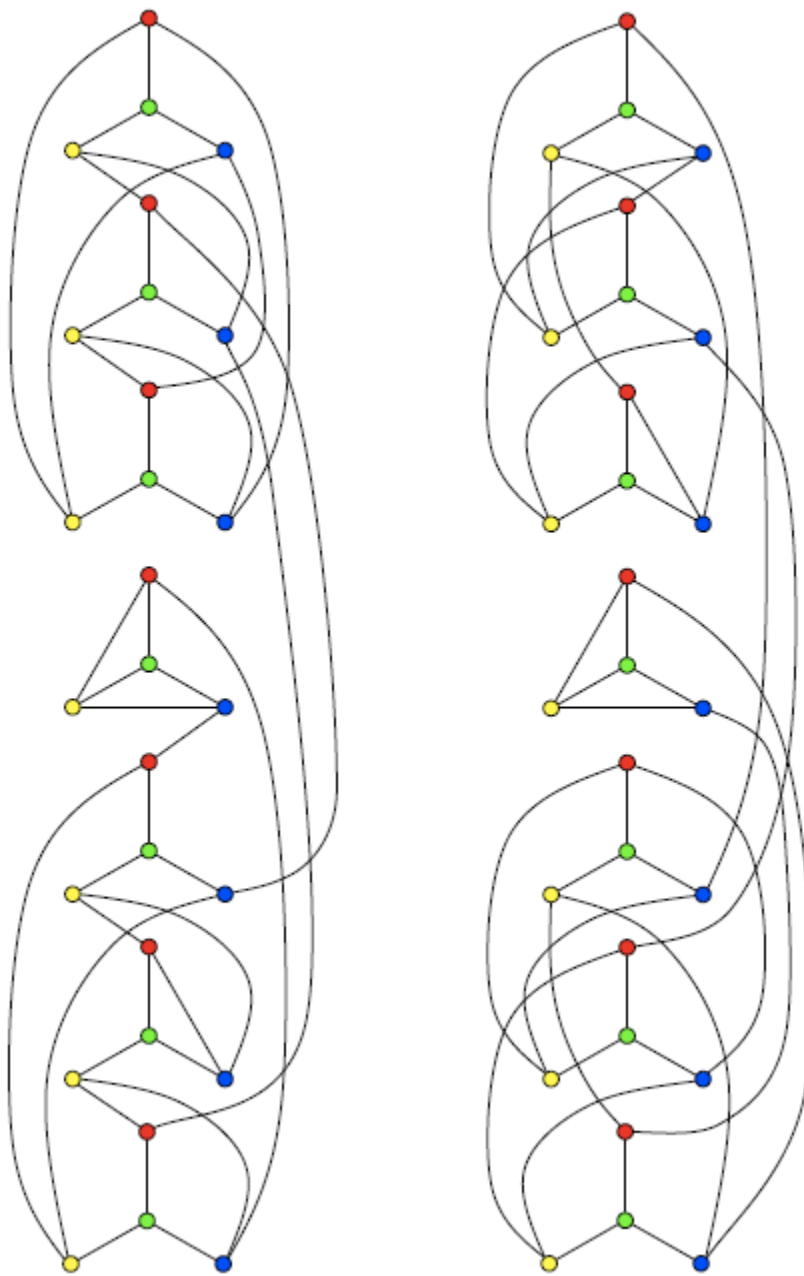


Figure 4.

The see more clearly why the above graphs are not isomorphism, see another drawing of the same two graphs in Figure 5 [4]:

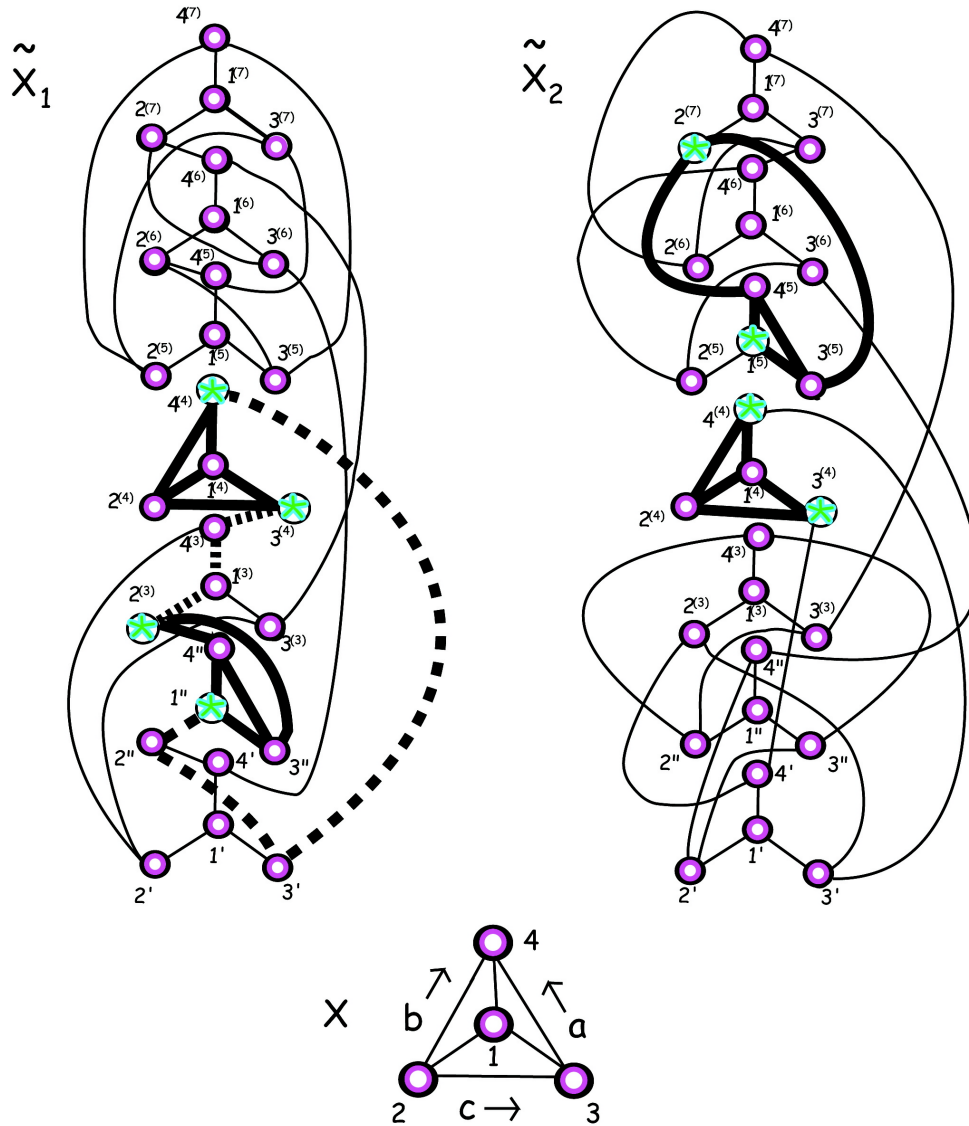
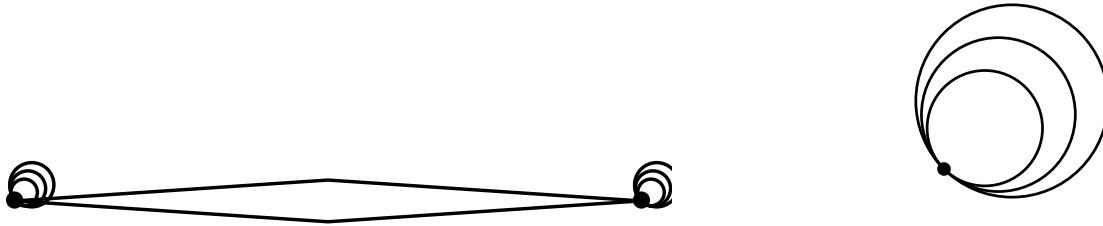


Figure 5.

There are exactly four triangles in each graph (shown by thick solid lines) and they are connected in pairs in both graphs. The starred vertices are the two vertices not on common edges. In \tilde{X}_1 we can go in 3 steps (as illustrated in the dotted lines) from a starred vertex in one pair to a starred vertex in the other pair, and in fact in 2 different ways. This cannot be done at all in \tilde{X}_2 . [4]

Since the information about the graph used to create the Ihara zeta function concerns the lengths of closed paths in the graph, the greatest successes in extracting information from the Ihara zeta function concern information about the lengths of closed paths in the graph. [2]

With diameter being a non-closed path, we ask, do graphs with the same Ihara zeta function have the same diameter? The answer is, no. See Figure 6.



(a)

(b)

Figure 6. Czarneski Graphs with the same Ihara zeta function [5]

Figure 6(a) has a diameter of 1 and Figure 6(b) has a diameter of 0, but both graphs have the same Ihara zeta function.

Cospectral graphs need not be isomorphic, but isomorphic graphs are always cospectral. For general graphs, the Ihara-Salberg zeta function can be useful as a tool for distinguishing graphs since its possible to have cospectral graphs with different zeta functions. For k-regular graphs, however, being cospectral is equivalent to having the same zeta function. [6]

Observation: If two graphs are regular and cospectral, then they have the same Ihara zeta function.

Proof: Using the spectral theorem on A , we have
$$\zeta(X,u)^{-1} = (1-u^2)^{r-1} \prod_{\lambda \in \text{Spec} A} (1-u\lambda + qu^2)$$

Clearly 2 graphs with the same spectrum have the same Ihara zeta function.



What happens when we take out the condition of regular in the above observation? Let us consider the below cospectral, but irregular figures [7].

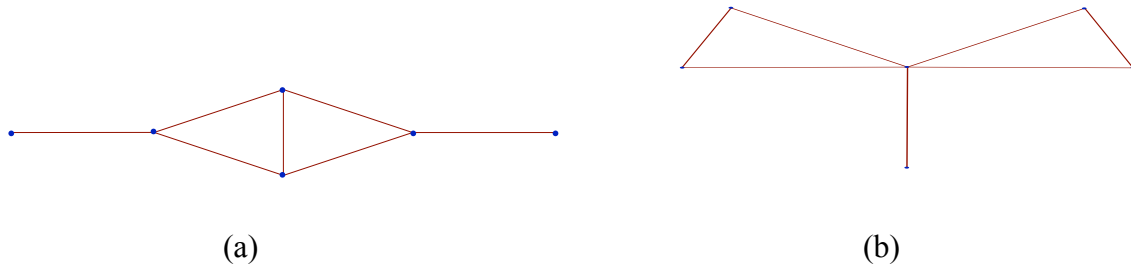


Figure 7. Two cospectral (and non-isomorphic) connected graphs on 6 nodes, both having the characteristic polynomial $\lambda^6 - 7\lambda^4 - 4\lambda^3 + 7\lambda^2 + 4\lambda - 1$

We can use a method from [4] which is too complicated to explain here to compute the Ihara zeta functions of the respective graphs and see if they are indeed the same. We will again be using the spectral formula for the Ihara zeta.

$$\text{Det} \begin{bmatrix} 1 - u^3 & -u & 0 & -u^3 \\ -u^3 & 1 - u^3 & -u^3 & 0 \\ 0 & -u^3 & 1 - u^3 & -u^3 \\ -u^3 & 0 & -u & 1 - u^3 \end{bmatrix}$$

For the graph 7(a)

yielding $1 - 4 u^3 - 2 u^4 + 4 u^6 + 4 u^7 + u^8 - 4 u^{10}$

$$\text{Det} \begin{bmatrix} 1 - u^3 & -u & 0 & -u^3 \\ -u^5 & 1 - u^3 & -u^3 & 0 \\ 0 & -u^3 & 1 - u^3 & -u^5 \\ -u^3 & 0 & -u & 1 - u^3 \end{bmatrix}$$

whereas for graph 7(b)

yielding $1 - 4 u^3 + 2 u^6 + 4 u^9 - 3 u^{12}$

The question of when two graphs have the same zeta function is equivalent to the question of when two graphs are isospectral with respect to the T matrix (edge adjacency matrix). [5]

The question we pose is, do co-spectral graphs have the same diameter? No. The graphs in Figure 7 are a counter example. Figure 7(a) has diameter 4 while Figure 7(b) has diameter 2.

REFERENCES

- [1] Norman Biggs. Algebraic Graph Theory, page 10. Cambridge University Press, 1974.
- [2] Matthew Horton. Ihara zeta functions of irregular graphs, page 24. 2006.
- [3] Harold M. Stark and Audrey A. Terras. Zeta functions of finite graphs and coverings, II. *Advances in Mathematics*, (154):32–195, 2000.
- [4] Audrey Terras. A Stroll Through the Garden of Graph Zeta Functions, pages 111-112. 2009.
- [5] Yaim Cooper. Properties determined by the Ihara zeta function of a Graph, page 2. *European Journal of Combinatorics*, 2009.
- [6] C. K. Storm, The zeta function of a hypergraph, page 19. *Electronic Journal of Combinatorics* 13, 2006.
- [7] Frank Harary. Cospectral Graphs and Digraphs, page 323. *Bulletin London Mathematical Society*, 3, 1971.