

# Induced Forests in Planar Graphs

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May 28, 2010

# Contents

|          |   |           |
|----------|---|-----------|
| <b>1</b> | <b>Introduction</b>                     | <b>3</b>  |
| 1.1      | Graphs . . . . .                        | 3         |
| 1.2      | Planar Graphs . . . . .                 | 4         |
| 1.3      | Induced forests . . . . .               | 5         |
| 1.4      | Organization . . . . .                  | 8         |
| <b>2</b> | <b>Euler’s Formula</b>                  | <b>9</b>  |
| 2.1      | The girth of a graph . . . . .          | 9         |
| 2.2      | Kuratowski’s theorem . . . . .          | 10        |
| <b>3</b> | <b>Coloring Planar Graphs</b>           | <b>12</b> |
| 3.1      | The 4-color theorem . . . . .           | 13        |
| 3.2      | The 5-color theorem . . . . .           | 13        |
| <b>4</b> | <b>Combinatorial Duality</b>            | <b>15</b> |
| 4.1      | Whitney’s theorem . . . . .             | 16        |
| 4.2      | Duals of triangulations . . . . .       | 17        |
| 4.3      | Bonds and cycles . . . . .              | 18        |
| <b>5</b> | <b>Induced forests of planar graphs</b> | <b>19</b> |
| 5.1      | Lemmas . . . . .                        | 20        |
| 5.2      | Proof of Theorem 1.2 . . . . .          | 21        |
| 5.3      | Proof of Theorem 1.1 . . . . .          | 22        |

# 1 Introduction

One of the central research areas in the theory of graphs is planarity, dating back to early results of the ancient greeks on platonic solids. The first discernable theorem on planar graphs in modern mathematics is attributed to Euler, concerning traversing the bridges of Königsberg without walking the same street twice. A century later, it was conjectured by a student of de Morgan that every planar graph is colorable with four colors, and this question was considered by a number of distinguished mathematicians for over a century, until the conjecture was finally verified by Appel and Haken [6]. Planar graphs developed alongside the intrinsic interest in the four-colour problem as an area worthy of research in its own right, and has a number of surprising connections to other areas of mathematics.

This thesis studies the particular problem of induced forests in planar graphs, motivated by a well-known conjecture of Albertson [3] and Borodin [7]. To describe the conjecture and our results, we require the following graph theoretic terminology.

## 1.1 Graphs

A graph is a pair  $(V, E)$  where  $V$  is a set of *vertices* and  $E$  is a set of unordered pairs of elements of  $V$  called *edges*. If  $G$  is a graph we write  $V(G)$  for its set of vertices and  $E(G)$  for its set of edges. If  $\{u, v\} \in E$ , then we say that  $u$  and  $v$  are *adjacent* in the graph. The neighborhood of a vertex  $v \in V$  is the set of  $w$  such that  $\{v, w\} \in E$ , and we write  $\Gamma(v)$  for the neighborhood of  $v$ .

The degree of  $v$  is denoted by  $d(v)$  and is  $|\Gamma(v)|$ . One of the basic results in graph theory is the so-called handshaking lemma,

$$\sum_{v \in V} d(v) = 2|E| \tag{1}$$

A *walk* in a graph is an alternating sequence  $v_1e_1v_2e_2 \dots v_ke_kv_1$  such that  $v_i \in V$  and  $e_i \in E$  and  $e_i = \{v_i, v_{i+1}\}$  for  $i < k$  and  $e_k = \{v_k, v_1\}$ . Note that both vertices and edges can be repeated. A closed walk is a walk whose first and last vertices are identical. The length of a walk  $v_1e_1v_2e_2 \dots v_ke_kv_1$  is  $k$ . If all the vertices of a walk are distinct, then it is called a *path*, and if all the vertices of a closed walk are distinct except the first and last, then the walk is called a *cycle*. A graph is connected if any two vertices are the ends of some walk. The span of a set  $L$  of edges in a graph is the graph whose vertex set is the union of all vertices in edges of  $L$  and whose edge set is  $L$ . If  $U$  is a set of vertices in a graph, then the subgraph induced by  $U$  is the graph with vertex set  $U$  and consisting of all edges between vertices of  $U$ . A subgraph of a graph  $G$  is induced if it is induced by some set  $U$  of vertices of  $G$ . A component of a graph is a maximal (with respect to edges) connected subgraph.

## 1.2 Planar Graphs

A *planar graph* is a graph  $(V, E)$  whose vertex set  $V$  can be represented by a collection of distinct points in the plane such that two vertices  $u$  and  $v$  which adjacent in the graph are joined by a simple curve in the plane with endpoints  $u$  and  $v$ , and such that no two curves share any points except

possibly their endpoints. If a graph is drawn in the plane in this way, then the drawing is referred to as a *plane embedding* of the graph. If we remove the points in a plane embedding of a planar graph from  $\mathbb{R}^2$ , then we are left with a collection of disjoint topologically connected regions of  $\mathbb{R}^2$  which are called the *faces* of the plane embedding. The *infinite face* is the unique face with infinite area. If  $f$  is a face in a plane embedding of a planar graph, then  $\partial f$  is the topological boundary of  $f$ . The (combinatorial) *boundary* of a face  $f$  is the minimal closed walk in the graph containing all points and curves on the boundary of  $f$ . The *degree* of a face  $f$  is the length of the boundary of  $f$  and is denoted  $d(f)$ . Note that in general, the boundary walk of a face is not necessarily a cycle – for instance in the case of the plane embedding of a tree, the unique boundary walk of the infinite face has many repeated edges. An elementary combinatorial fact, following the proof of the handshaking lemma, is that if  $F$  is the set of faces in a plane embedding, then

$$\sum_{f \in F} d(f) = 2|E| \tag{2}$$

We shall see using the notion of combinatorial duality that this is precisely equivalent to the handshaking lemma.

### 1.3 Induced forests

A *forest* is a graph with no cycles, and a *tree* is a connected forest. An *induced forest* in a graph is an induced subgraph containing no cycles. Of substantial interest in graph theory is the question of removing a minimum number of vertices in a graph to destroy all cycles – this is occasionally referred to as

the *decycling number*. If a graph has  $n$  vertices, then the decycling number added to the maximum size of an induced forest equals  $n$ , since they are complementary notions. As an instance of this problem, the largest induced forest in the (planar) graph below is shown. The decycling number must therefore equal fifteen.

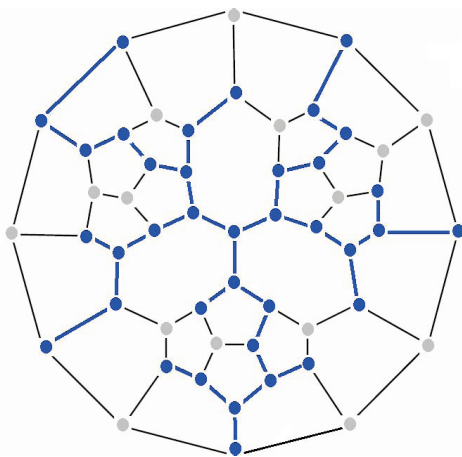


Figure : Induced forests

This problem is interesting from both a combinatorial and algorithmic perspective. We discuss a new approach to this problem in planar graphs, using combinatorial duality, which provides some partial new results on some well-known conjectures on induced forests in planar graphs. The motivation for this section is a conjecture of Albertson and Berman [4] and, independently Akiyama and Watanabe [2] which states that every planar graph contains an induced forest on at least half of its vertices:

**Conjecture.** *Let  $G$  be a planar graph on  $n$  vertices. Then  $G$  contains an induced forest on at least  $\frac{n}{2}$  vertices.*

This conjecture remains open and appears to be very difficult to prove. If the conjecture is true, an important consequence is that every planar graph has an independent set on at least one quarter of its vertices. The latter statement follows immediately from the 4-color theorem (there must be a set of at least  $\frac{n}{4}$  of some color in every 4-coloring, and those vertices have no edges between them). Finding a proof independent of the 4-color theorem that every planar graph has such a large independent set has attracted the attention of many researchers. The induced forest conjecture has been studied by numerous researchers [4, 2, 8, 11, 15, 16] and has been proved for planar graphs of girth at least five in [8] and later for triangle-free planar graphs in [19] via discharging methods. In this paper, we give a short proof of a substantially stronger result. Here and in what follows, a partition of a planar graph into two induced forests refers to a partitioning of the vertex set into two sets, each of which induces a forest.

**Theorem 1.1** *Every triangle-free planar graph has a vertex partition into two induced forests.*

This result was proved by lengthy discharging methods by Raspaud and Wang [17]. It is also a strengthening in triangle-free graphs of the classical result that every planar graph can be partitioned into at most three induced linear forests [16] – these are forests of which every component is a path. In

addition, we obtain the following result for all planar graphs. A *pseudoforest* is a graph such that every block – every maximal 2-connected subgraph – has two or three vertices. A *pseudotree* is a connected pseudoforest. The block decomposition of a graph ensures that the graph has a tree-like structure: it consists of triangles and edges joined in a tree-like fashion at cutvertices. We prove the following result, which will be shown to imply Theorem 1.1.

**Theorem 1.2** *Let  $G$  be a 4-connected planar graph and let  $\partial$  be any set of three vertices on the boundary of some face of  $G$  and  $x \in \partial$ . Then there is a partition  $V(G) = A \cup B$  such that both  $A$  and  $B$  induce pseudoforests and  $x$  is in one of the pseudoforests while the vertices in  $\partial \setminus \{x\}$  are in the other pseudoforest*

In the case that  $G$  is maximal planar and 4-connected, we actually get pseudotrees in Theorem 1.2. The methods we use to prove these two new theorems actually generalize to giving the best known bounds on the size of induced forests in planar graphs with  $m$  edges,  $n$  vertices, and girth  $g$ . These results are beyond the scope of the current discussion.

## 1.4 Organization

In the next section, we discuss the celebrated Euler’s Formula for planar graphs, which will be used in subsequent material. One of the key tools in this thesis is the notion of combinatorial duality, which is discussed in Section 5. Theorems 1.1 and 1.2 will be proved in Section 6.



## 2 Euler's Formula

A *triangulation* is defined as a planar graph such that the addition of an edge between two non-adjacent vertices results in a non-planar graph. In a triangulation, it is simple to check that every face has a boundary walk that is a cycle of length three – a triangle. The number of edges and faces in a triangulation with  $n$  vertices is invariant i.e. it is independent of the plane embedding. This is a basic consequence of the famous *Euler's Formula*:

**Proposition 2.1** *Let  $G$  be a connected plane embedding with  $n$  vertices,  $\phi$  faces and  $e$  edges. Then*

$$n - e + \phi = 2.$$

This formula is easy to prove by induction on  $e$ : if we remove an edge of the graph in a cycle, then the faces on either side of that edge merge into one face and the formula stands. If the graph has no cycles, it must be a tree and then  $e = n - 1$  and  $\phi = 1$  so the formula holds once more.

### 2.1 The girth of a graph

To check the number of edges and faces in a triangulation on  $n \geq 3$  vertices, we observe from (2) that since  $d(f) = 3$  for every face in a plane embedding of a triangulation, we have  $3\phi = 2e$ . Therefore by Euler's Formula,

$$e = \phi + n - 2 = \frac{2e}{3} + n - 2$$

and therefore  $e = 3n - 6$ . Similarly, one obtains  $\phi = 2n - 4$ . More generally, the following result holds: here the girth of a graph containing at least one cycle is the length of a shortest cycle in the graph.

**Proposition 2.2** *Let  $G$  be a plane embedding with  $n$  vertices and  $e$  edges and girth  $g$ . Then*

$$e \leq \frac{g}{g-2}(n-2). \quad (3)$$

**Proof.** From (2), since  $d(f) \geq g$  for every face of  $G$  we have  $g\phi \leq 2e$ . Combining this with Euler's Formula, we obtain

$$e = \phi + n - 2 \leq \frac{2e}{g} + n - 2.$$

Solving for  $e$  gives the result. ■

## 2.2 Kuratowski's theorem

Euler's Formula in general can be used as a simple certificate that a graph is not planar. For instance we see from the above that a triangulation on  $n \geq 3$  vertices has at most  $3n - 6$  edges, so any graph on  $n$  vertices with more than  $3n - 6$  edges cannot be planar. As another example, using (2.2), a planar graph with girth  $g = 7$  and  $n$  vertices has at most  $\frac{7}{5}(n - 2)$  edges. The graph below has  $n = 14$  and girth  $g = 7$  so if it were planar would have less than  $\frac{7}{5} \cdot 12 < 17$  edges. However the graph clearly has  $\frac{1}{2} \cdot 14 \cdot 3 = 21$  edges by (1), so it cannot be planar.

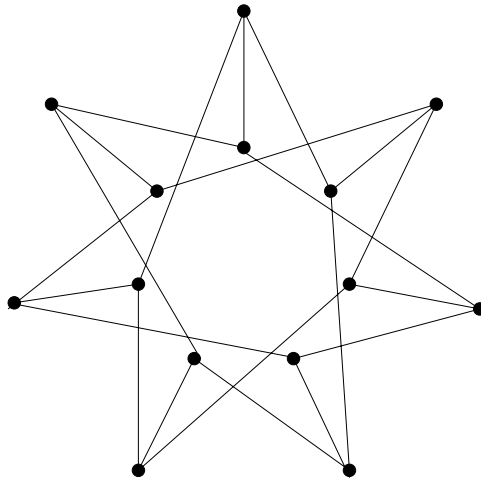


Figure : A non-planar graph

Of course, there are  $n$ -vertex graphs with fewer than  $3n - 6$  edges which are not planar, so we do not have a necessary and sufficient certificate for planarity. Such a certificate is provided by Kuratowski's Theorem [12], which we do not state here. That theorem is based on the fact that the complete graph on five vertices (five vertices all joined by edges) and the complete bipartite graph with three vertices in each part (three vertices completely joined to three other vertices) are not planar, and that all non-planar graphs can be reduced in a simple way to one of those two graphs. These graphs are shown below.

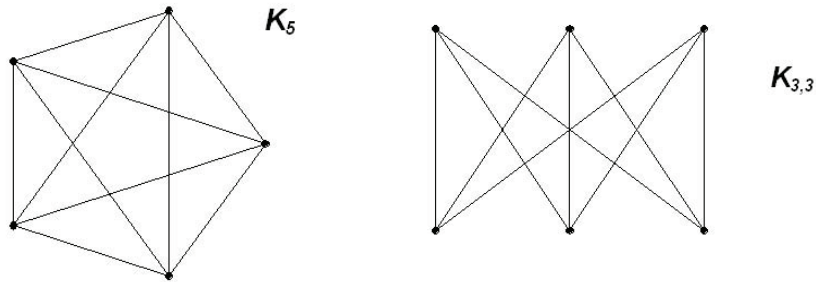


Figure : Kuratowski's Theorem

There are now many equivalent formulations of Kuratowski's Theorem.

### 3 Coloring Planar Graphs

Let  $[k] := \{1, 2, \dots, k\}$ . A graph  $(V, E)$  is *k-colorable* if there is a map  $\chi : V \rightarrow [k]$  such that whenever  $\{u, v\} \in E$ ,  $\chi(u) \neq \chi(v)$ . In other words, there is an assignment of *colors* to the vertices of the graph such that no two adjacent vertices receive the same color. The smallest  $k$  such that a graph is  $k$ -colorable is called the *chromatic number* of the graph. This invariant is the subject of a remarkable amount of research outside of planar graph theory, and is known to be extremely difficult to determine for an arbitrary graph. It has sparked a number of new methods in graph theory, perhaps the most famous being the topological method of Lovász [13], used to give a beautiful proof of Kneser's conjecture.

### 3.1 The 4-color theorem

Perhaps the most famous problem in all of graph theory is the four color problem. In the mid nineteenth century, a student of de Morgan conjectured that every planar graph is 4-colorable. It is not hard to produce many planar graphs which are not 3-colorable, and so this potential result would be tight. The problem attracted much attention for over a century, until it was finally solved by Appel and Haken [6]:

**Theorem 3.1** *Every planar graph is 4-colorable.*

It is considered by some mathematicians a disadvantage that the proof by Appel and Haken is computer aided, however recent simplifications of their proof are sufficient to convince most mathematicians that the proof is correct. A short proof of the four color theorem, without the aid of a computer, remains elusive. We remark that the 4-color theorem is surprisingly related to areas of mathematics outside combinatorics.

### 3.2 The 5-color theorem

An effective approach to coloring planar graphs is provided by Euler's Formula. We sketch the proof that every planar graph is 5-colorable.

**Theorem 3.2** *Every planar graph is 5-colorable.*

**Proof.** By (2.1), every planar graph contains a vertex of degree at most five, otherwise (2.1) would give  $e \leq 3n - 6$  with  $g = 3$ , whereas if all vertices had

degree at least six, then (1) gives the contradiction

$$e \geq \frac{1}{2} \sum_{v \in V} d(v) \geq \frac{1}{2} \cdot 6n = 3n.$$

We use the existence of a vertex of degree at most five to come up with a five-coloring of a planar graph by induction. Note that every planar graph on at most five vertices is 5-colorable, since assigning a different color to every vertex provides a coloring. So we assume  $n$ , the number of vertices, is at least six. If  $(V, E)$  is a planar graph and  $v$  is the vertex of smallest degree in  $V$ , we consider the case  $d(v) \leq 4$  and  $d(v) = 5$  separately. If  $d(v) \leq 4$ , we remove  $v$  together with all edges on  $v$ . Inductively, there is a 5-coloring of the remaining graph. In particular, the neighbors of  $v$  use in total at most four colors, since  $d(v) \leq 4$ . This implies the existence of a remaining color which may be assigned to  $v$  to obtain a 5-coloring of the whole graph.

The second case we consider is  $d(v) = 5$ . Let  $\Gamma(v) = \{v_1, v_2, v_3, v_4, v_5\}$  be the neighbors of  $v$ . If all pairs of neighbors of  $v$  form edges, then the graph could not be planar, since it contains  $K_5$  with vertex set  $\Gamma(v)$ , and by Kuratowski's Theorem [12],  $K_5$  is not planar. So we may assume that  $v_1$  and  $v_2$  are not adjacent. Now we perform the following deformation of the plane embedding. We move  $v_1$  and  $v_2$  continuously onto  $v$  to obtain a new planar graph  $\hat{G}$ , as shown in the figure below. The vertices  $v_1$  and  $v_2$  are red,  $v$  is blue, and  $v_1$  and  $v_2$  are “dragged” along with all their edges onto  $v$ .

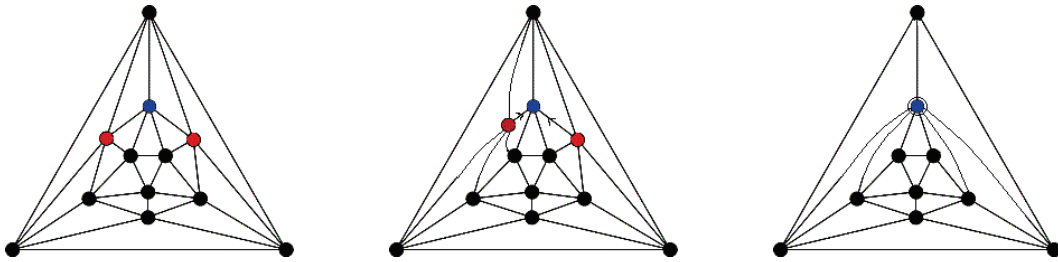


Figure : 5-Color Theorem

By induction,  $\hat{G}$  admits a 5-coloring  $\hat{\chi}$ , and we may assume  $\hat{\chi}(v) = 1$  and  $\hat{\chi}(v_3), \hat{\chi}(v_4), \hat{\chi}(v_5) \in \{2, 3, 4\}$ . Now define a coloring  $\chi^*$  of the original planar graph  $G$  as follows. If  $x \notin \{v, v_1, v_2\}$ , define  $\chi(x) = \hat{\chi}(x)$  – note that  $x \in V(\hat{G})$ . If  $x = v_1$  or  $x = v_2$ , define  $\chi(x) = \hat{\chi}(v) = 1$  – the coloring is fine since  $v_1$  and  $v_2$  are not adjacent in  $G$ . Finally, we see that the colors used in  $\Gamma(v)$  are contained in  $\{1, 2, 3, 4\}$ , so we can define  $\chi(v) = 5$  to obtain a 5-coloring of  $G$ . ■

The simple approach given here does not work for the 4-color theorem, and indeed hundreds of different configurations have to be eliminated when attempting the 4-color theorem in this way. One of the early attempts at the 4-color theorem, due to Tait [20] at the end of the nineteenth century, was to recast the 4-color problem in terms of combinatorial duality.

## 4 Combinatorial Duality

If  $G$  is a plane embedding, the a *combinatorial dual* of  $G$ , denoted  $G^*$ , is a new plane embedding defined as follows. Into the interior of each face  $f$  of  $G$ ,

insert a new point  $v_f$ . If  $f$  and  $g$  are distinct faces sharing a boundary curve, join  $v_f$  to  $v_g$  with a simple curve through some interior point of the common boundary curve of  $f$  and  $g$ , and for each curve which is on the boundary of only one face  $f$ , join  $v_f$  to  $v_f$  with a simple closed curve through an interior point of that curve. We ensure that no two of the added curves share any interior points. Then we remove all points and curves in  $G$  to obtain  $G^*$ . An example is shown below:

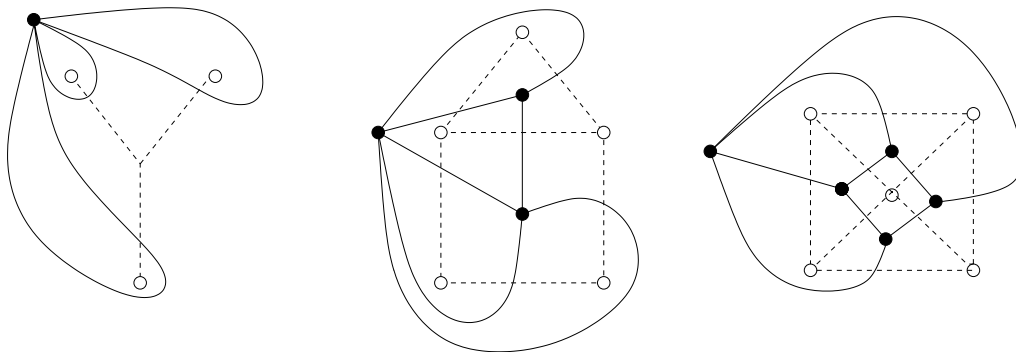


Figure : Duality

The combinatorial dual of a plane embedding is not necessarily unique.

#### 4.1 Whitney's theorem

However, a theorem of Whitney [23] gives the following results on drawing planar graphs. A graph is said to be  $k$ -connected if the removal of any set of at most  $k - 1$  vertices and all edges containing those vertices always gives a connected graph. For instance, the graph in the above picture is 2-connected, but certainly not 3-connected since there are two vertices whose removal



disconnects the graph. The key to drawing planar graphs and uniqueness of combinatorial duals is 3-connectivity:

**Theorem 4.1** *Let  $G$  be 3-connected planar graph. Then  $G$  has a plane embedding such that the topological boundary of every face is a convex polygon. Furthermore,  $G$  has a unique plane embedding up to deformation of the edges, and that plane embedding has a unique combinatorial dual.*

For instance, it is a simple task to check that a triangulation on at least four vertices is 3-connected, and therefore maximal plane embeddings have unique combinatorial duals.

## 4.2 Duals of triangulations

In a triangulation, we observed that every face boundary is a triangle, and that every triangulation on at least four vertices is 3-connected. By Theorem 4.1, the dual  $G^*$  of a triangulation  $G$  is unique. Moreover, the dual is cubic – every vertex of the dual has degree three – and also 3-connected. An approach to the 4-color theorem was suggested by Tait [20]. A graph is *hamiltonian* if it contains a cycle through all its vertices – a *hamiltonian cycle*. Tait [20] showed that if every cubic 3-connected planar graph is hamiltonian, then every planar graph is 4-colorable. Unfortunately, as discovered roughly fifty years later by Tutte [21], there are cubic three-connected planar graphs which are not hamiltonian. An example with forty-four vertices is shown below, and the smallest examples have thirty-eight vertices [10]. In fact, Robinson and

Wormald [18] later showed that the proportion of cubic 3-connected planar graphs on  $n$  vertices which are not hamiltonian tends to one as  $n \rightarrow \infty$ .

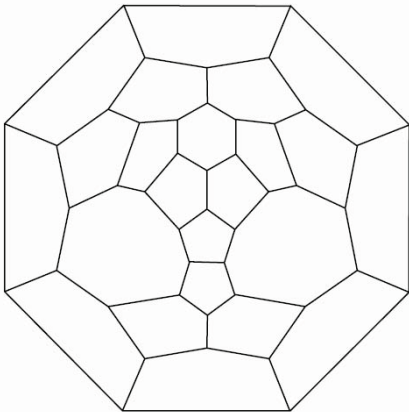


Figure : Nonhamiltonian planar graph

### 4.3 Bonds and cycles

The following observations on combinatorial duals will be used in this thesis. A *bond* in a graph is a minimum set of edges of the graph whose removal disconnects the graph. If  $G = (V, E)$  is a plane embedding and  $G^*$  a combinatorial dual to  $G$ , then there is natural a one-to-one correspondence between edges of  $G$  and edges of  $G^*$ , by construction. For convenience, if  $e \in E$  then we denote by  $e^*$  the unique edge of  $G^*$  corresponding to  $e$ , and if  $L$  is a set of edges let  $L^* = \{e^* : e \in L\}$ . Similarly if  $f$  is a face of  $G$ , let  $f^*$  denote the vertex of  $G^*$  corresponding to  $f$ , and if  $U$  is a set of faces of  $G$ , let  $U^* = \{f^* : f \in U\}$ . If  $C$  is a cycle in a plane embedding  $G$ , then we let

$\text{int}(C)$  denote the set of faces in the interior of  $C$  and let  $\text{ext}(C)$  denote the set of faces in the exterior of  $C$ . The following result is straightforward.

**Proposition 4.2** *Let  $G$  be a plane embedding and  $G^*$  a combinatorial dual of  $G$ . Then*

1. *A set  $L$  of edges of  $G$  is a bond if and only if  $L^*$  spans a cycle in  $G^*$*
2. *If  $C$  is a cycle in  $G$ , then  $\text{int}(C)^*$  and  $\text{ext}(C)^*$  induce connected subgraphs of  $G$*
3. *If no vertex of  $G$  is in the interior of  $C$ , then  $\text{int}(C^*)$  induces a tree in  $G^*$ .*

**Proof.** First we prove the second statement. Let  $I = \text{int}(C)$  and let  $F$  be the subgraph of  $G^*$  induced by  $I$ . Suppose  $F$  has distinct components  $F_1$  and  $F_2$  and let  $f$  be the common infinite face of  $F_1$  and  $F_2$ . Then  $f^*$  is a vertex of  $G$  in the interior of  $C$ . If  $f_1, f_2, \dots, f_k$  are the faces with  $f^*$  on their boundary in  $G$ , then  $f_1^*, f_2^*, \dots, f_k^*$  are vertices of a closed walk in  $F$ . However, that closed walk then contains vertices of  $F_1$  and vertices of  $F_2$ , contradicting that  $F_1$  and  $F_2$  are different components. Therefore  $F$  is connected. Similarly  $\text{ext}(C)^*$  induces a connected subgraph of  $G^*$ , and the proof of the second statement is finished. The proof of the remaining two statements are similar. ■

## 5 Induced forests of planar graphs

We prove Theorem 1.1 in this section.

## 5.1 Lemmas

We rely on the following result of Thomassen [20], building on an earlier theorem of Tutte [22], on paths in planar graphs. More recently, the result has been slightly strengthened by Sanders [19].

**Lemma 5.1** *Let  $G$  be a 2-connected planar graph and let  $x, y \in V(G)$  and  $e \in E(G)$ . Then there exists a path  $P \subseteq G$  with endpoints  $x$  and  $y$  such that  $e \in E(P)$  and for each component  $K$  of  $G - V(P)$ ,  $|\Gamma(K) \cap V(P)| \leq 3$ .*

This result clearly implies that any two vertices in a 4-connected planar graph are connected by a hamiltonian path. For our particular application, we need Lemma 5.1 in the case of planar graphs which have no non-trivial three-edge cuts: such graphs are called *cyclically 4-edge connected*. Specifically, a graph  $G$  is cyclically 4-edge connected if for any set  $E$  of at most three edges of  $G$ , some component of  $G - E$  is acyclic. Notice that if a cubic graph is cyclically 4-edge connected, then every 3-edge cut consists of three edges containing some vertex of  $G$ . A dominating cycle in a graph  $G$  is a cycle  $C$  such that every component of  $G - V(C)$  is an isolated vertex. An easy consequence of Lemma 5.1 is the following:

**Lemma 5.2** *Let  $G$  be a cyclically 4-edge connected cubic planar graph and let  $e, f$  be distinct edges of  $G$ . Then  $G$  contains a dominating cycle  $C$  such that  $e, f \in E(C)$ .*

**Proof.** Suppose  $f = \{x, y\}$  and let  $P$  be a path as in Lemma 5.1 from  $x$  to  $y$  and containing  $e$ . Then  $P$  together with  $f$  forms a cycle. Since  $G$  is cubic,

every component of  $G - V(C)$  is an isolated vertex, as required. ■

A corollary of Lemma 5.2 is a proof of a stronger statement than Theorem 1.1 in the case of 4-connected planar graphs. First we need a structural lemma:

**Lemma 5.3** *Let  $G$  be a connected graph containing no cycles of length at least four. Then  $G$  is a pseudotree.*

**Proof.** It is well-known that if a graph has no even cycles, then every vertex of degree at least three is a cutvertex. This means that every block of  $G$  is either a cycle or a complete graph of order two, and since  $G$  has no cycles of length more than three, every block has at most three vertices. It follows that  $G$  is a pseudotree. ■

## 5.2 Proof of Theorem 1.2

We add edges to  $G$  until  $G$  is a triangulation and in such a way that  $\partial$  is the boundary of a triangle. Let  $G^*$  be the combinatorial dual of  $G$ . Then  $G^*$  is a cubic planar graph. If  $G$  has a non-trivial bond  $E$  (non-trivial meaning not just consisting of the edges around one vertex), then the edges of  $G$  dual to the edges of  $E$  form a cycle by Proposition 4.2. However this cycle is then a separating cycle in  $G$ , and since  $G$  is 4-connected,  $|E| \geq 4$ . It follows that  $G^*$  is cyclically 4-edge connected. By Lemma 5.2,  $G^*$  contains a dominating cycle  $C^*$ . Furthermore, if  $e$  and  $f$  are the edges dual to the edges of  $\partial$  containing  $x$ , then by Lemma 5.2 we can ensure that  $C^*$  contains both  $e$  and

*f.* Suppose  $G^*$  is embedded in the plane without crossings. Let  $A$  denote the set of vertices of  $G$  dual to the faces of  $G^*$  which are in the interior of  $C^*$ , and let  $B$  be the set of vertices of  $G$  dual to the faces of  $G^*$  which are in the exterior of  $C^*$ . Clearly  $V(G) = A \cup B$  and furthermore we either have  $x \in A$  and  $\partial \setminus \{x\} \subset B$  or the reverse. We claim that  $A$  and  $B$  contain no cycles of length at least four, which will imply that  $A$  and  $B$  induced pseudoforests in  $G$ , by Lemma 5.3 and Proposition 4.2. In fact if  $G$  is a triangulation then these pseudoforests are actually pseudotrees by Proposition 4.2. Suppose  $C$  is a cycle of length at least four in  $G$  consisting of vertices in  $A$ . If  $E^*$  is the edge set dual edges to the edges of  $C$ , then  $E^*$  forms an edge-cut in  $G^*$ . Let  $F$  be the set of faces of  $G$  in the interior of  $C$ . Since  $C$  has length at least four,  $|F| \geq 2$  and the vertices of  $G^*$  dual to  $F$  induce a connected subgraph of  $G^*$  in the interior of  $C^*$ . However this contradicts the fact that every connected subgraph of  $G^*$  in the interior of  $C^*$  is an isolated vertex. Therefore every cycle in  $G$  through vertices of  $A$  or through vertices of  $B$  is a triangle. This completes the proof. ■

### 5.3 Proof of Theorem 1.1

Theorem 1.1 will be derived from Theorem 1.2. It is enough to prove the theorem for 2-connected graphs, since we may then apply the 2-connected case to every block of  $G$ . Since  $G$  is 2-connected, we may draw  $G$  so that each face of  $G$  is bounded by a cycle. Into each face  $f$  of  $G$  with boundary walk  $v_1v_2 \dots, v_rv_1$ , add vertices  $u_1, u_2, \dots, u_r, u$  and edges  $\{u_i, v_i\}, \{u_i, v_{i+1}\}$

for  $1 \leq i \leq r-1$  as well as the edge  $\{u_r, v_r\}, \{u_r, v_1\}$  and all the edges  $\{u, u_i\}$  for  $1 \leq i \leq r$ . Note that the resulting graph  $G^+$  is a triangulation. This is shown in the figure below, where the new vertices are in blue.

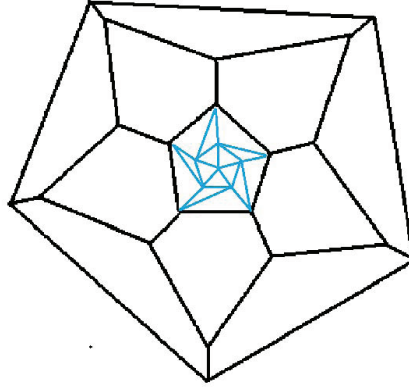


Figure : Proof of Theorem 1.1

Let  $T$  be a triangle in  $G^+$ . If  $V(T) \subset V(G^+) \setminus V(G)$ , then it is straightforward to check that  $G^+ - V(T)$  is 2-connected, so  $T$  cannot be a separating triangle. Similarly, if  $|V(T) \cap V(G)| = 1$ , then since  $G$  is 2-connected  $G^+ - V(T)$  is connected, so again  $T$  cannot be a separating triangle. Since  $G$  is triangle-free,  $T \not\subset G$  and therefore we consider the case  $|V(T) \cap V(G)| = 2$ . Suppose  $V(T) \cap V(G) = \{u, v\}$  and  $V(T) \cap V(G^+) = \{w\}$ . Then  $w$  is adjacent to  $u$  and  $v$  in  $G^+$ , which implies  $u$  and  $v$  are consecutive on the boundary walk of the face of  $G$  containing  $w$ . It follows that  $T$  has empty interior, and therefore  $T$  cannot be a separating triangle. We have shown that no triangle  $T$  in  $G^+$  is separating, and therefore  $G^+$  may be partitioned into two

induced pseudoforests  $P_1$  and  $P_2$  by Theorem 1.2. Removing all the vertices of  $V(G^+) \setminus V(G)$ , we obtain a partition of  $G$  into induced graphs  $P'_1 \subset P_1$  and  $P'_2 \subset P_2$ . Since  $G$  has no triangles,  $P'_1$  and  $P'_2$  are both induced forests, as required. ■

## References

- [1] Akiyama, J.; Era, H.; Gervacio, S. V.; Watanabe, M. Path chromatic numbers of graphs. *J. Graph Theory* 13 (1989), no. 5, 569–575.
- [2] Akiyama, J.; Watanabe, M. Maximum induced forests of planar graphs. *Graphs and Combinatorics* 3 (1987), 201–202.
- [3] Albertson, M. O. A lower bound for the independence number of a planar graph. *J. Combinatorial Theory Ser. B* 20 (1976), no. 1, 84–93.
- [4] Albertson, M. O.; Berman, D. M. A conjecture on planar graphs. *Graph Theory and Related Topics* (J. A. Bondy and U. S. R. Murty, eds.), (Academic Press, 1979), 357.
- [5] Alon, N.; Mubayi, D.; Thomas, R. *J. Graph Theory* 38 (2001), no. 3, 113–123.
- [6] Appel, K., and Haken, W., Every Planar Map is Four Colorable, *A.M.S. Contemporary Mathematics* 98 (1989), 236–240.



- [7] Borodin, O. V. A proof of B. Grünbaum's conjecture on the acyclic 5-colorability of planar graphs. (Russian) Dokl. Akad. Nauk SSSR 231 (1976), no. 1, 18–20.
- [8] Borodin, O. V.; Glebov, A. N. On the partition of a planar graph of girth 5 into an empty and an acyclic subgraph. (Russian) Diskretn. Anal. Issled. Oper. Ser. 1 8 (2001), no. 4, 34–53.
- [9] Broere, I.; Mynhardt, C. M. Generalized colorings of outerplanar and planar graphs. Graph theory with applications to algorithms and computer science (Kalamazoo, Mich., 1984), 151–161, Wiley-Intersci. Publ., Wiley, New York, 1985.
- [10] Holton, D., McKay, B. The smallest non-hamiltonian 3-connected cubic planar graphs have 38 vertices, J. Combinatorial Theory (B), 45 (1988) 315–319.
- [11] Hosono, K. Induced forests in trees and outerplanar graphs. Proc. Fac. Sci. Tokai Univ. 25 (1990), 27–29.
- [12] Kuratowski, K., Sur le problème des courbes gauches en topologie, Fund. Math. 15 (1930) 271–283.
- [13] László Lovász. Kneser's conjecture, chromatic number and homotopy. J. Combinatorial Theory, Ser. A, 25 (1978) 319–324.

- [14] Mihók, P. On vertex partition numbers of graphs. *Graphs and other combinatorial topics* (Prague, 1982), 183–188, Teubner-Texte Math., 59, Teubner, Leipzig, 1983.
- [15] Pelsmajer, M.J. Maximum induced linear forests in outerplanar graphs. *Graphs and Combinatorics*, 2003.
- [16] Poh, K. S. On the linear vertex-arboricity of a planar graph. *J. Graph Theory* 14 (1990), no. 1, 73–75.
- [17] Raspaud, A.; Wang, W., On the vertex-arboricity of planar graphs. *European Journal of Combinatorics* (2008) 1064-1075.
- [18] Richmond, L. Bruce; Robinson, R. W.; Wormald, N. C. On Hamilton cycles in 3-connected cubic maps. *Cycles in graphs* (Burnaby, B.C., 1982), 141–149, North-Holland Math. Stud., 115, North-Holland, Amsterdam, 1985.
- [19] Salavatipour, M. Large induced forests in triangle-free planar graphs. *Graphs Combin.* 22 (2006), no. 1, 113–126.
- [20] Thomassen, C. A theorem on paths in planar graphs. *J. Graph Theory* 7 (1983), no. 2, 169–176.
- [21] Tutte, W. On Hamiltonian circuits, *Journal of the London Mathematical Society* (2nd ser.) 21 (2) (1946) 98-101.
- [22] Tutte, W. A theorem on planar graphs. *Trans. Amer. Math. Soc.* 82 (1956), 99–116.

- [23] Whitney, H. A set of topological invariants for graphs. *American Journal of Mathematics*, 55 (1933) 235–321.