Chemical Reaction Networks: Comparing Deficiency, Determinant Expansions, and Sign Patterns (ROUGH DRAFT)

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ABSTRACT

This paper will treat three topics motivated by chemical reaction networks with mass-action kinetics; they are commonly referred to as deficiency, determinant expansions, and sign patterns. The dynamics of a chemical reaction network are governed by a non-linear system of ordinary differential equations

$$\frac{dx}{dt} = f(x),$$

and what is observed in experiments is often equilibria states, $\{x : \ f(x) = 0\}$ of this differential equation.

A major issue is counting the number of equilibria, and in particular determining if there is a unique equilibrium. There are two main hypotheses which insure that a chemical reaction has a unique equilibrium. The most classical one is known as the deficiency 0 condition, while more recently a condition on the determinant of the Jacobian of f has been fruitful. Since either of these conditions on f separately imply that there is a unique equilibria, it is natural to ask: does one condition imply the other? In this thesis we demonstrate that the answer is no. Indeed we give concrete examples showing that the "deficiency" and key properties of the Jacobian's determinant expansion have no simple bearing on each other.

The thesis also contains a study of the two standard ways of representing f for a chemical reaction network; we refer to them as the Stoichiometric Representation and the Complexes Representation. We show that a system which has one of these representations also has the other and we layout precisely the correspondence between them. At some level this is in the chemistry literature, but we could not find a reference which did this thoroughly at a high level of generality.

The thesis concludes with another topic, that of "sign patterns" of the Jacobian of f. Through various examples, we show sign patterns also have no relation to the deficiency of a chemical network. Additionally, we extend some known theoretical results on sign patterns of Jacobians and give an approach and results showing how to proceed on networks which fail to have a sign pattern.

CONTENTS

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1. INTRODUCTION

This paper will treat three topics motivated by chemical reaction networks with mass-action kinetics; these are commonly known as deficiency, determinant expansions, and sign patterns. The dynamics of a chemical reaction network are governed by a non-linear system of ordinary differential equations

$$\frac{dx}{dt} = f(x).$$

First, we study chemical reaction networks and the two standard ways of representing f for a chemical reaction network, which we call the Stoichiometric Representation and the Complexes Representation, and layout precisely the correspondence between the two systems. A major issue is counting the number of equilibria of the differential equation, and in particular determining if there is a unique equilibrium. There are two main hypotheses which insure that a chemical reaction has a unique equilibrium. The most classical one is known as the deficiency 0 condition, while more recently a condition on the determinant of the Jacobian of f has been fruitful. Since either of these conditions on f separately imply that there is a unique equilibria, it is natural to ask: does one condition imply the other? We demonstrate that the answer is no. Lastly, we extend some theoretical results on "sign patterns" of the Jacobian of f, and then show that sign patterns have no relation to the deficiency of a chemical network.

Remark 1.1. *Throughout this work, whenever we refer to a chemical reaction network, we always assume it has **mass-action kinetics** (see [CHWprept]) without explicitly saying so. Mass-action kinetics is motivated by chemistry considerations, and a full discussion of it is beyond the scope of this work. We refer the reader to [F79] for a comprehensive account of mass-action kinetics.*

1.1. **Differential Equations of Chemical Reaction Networks.** Our main objective in each section is to analyze systems of differential equations associated to chemical reaction networks. To this end, we first consider differential equations that act on the non-negative orthant, $\mathbb{R}^d_{\geq 0}$ in $\mathbb{R}^d$:

$$\frac{dx}{dt} = f(x), \tag{1.1}$$

where $f : \mathbb{R}^d_{\geq 0} \to \mathbb{R}^d$. The differential equations we will study are when (1.1) has a special form, and differential equations of this form can be written two different ways which we now present.

1.1.1. *Stoichiometric Representation of a Chemical Reaction Network.* A special form of equation (1.1) above is

$$\frac{dx}{dt} = f(x) = Sv(x) \tag{1.2}$$

where S is an integer $d \times d'$ matrix (commonly referred to as the **Stoichiometric matrix**), $x \in \mathbb{R}^d_{\geq 0}$, and $v(x) = v : \mathbb{R}^d_{\geq 0} \to \mathbb{R}^{d'}_{\geq 0}$ is a vector consisting of d' strictly monotonically increasing, smooth functions. Now let $U = U(x) := Jacobian(v(x)) = v'(x)$, a

matrix-valued function of partial derivatives of the entries of $v(x)$. In other words, if we write $v(x) = (v_1(x), \ldots, v_{d'}(x))^T \in \mathbb{R}^{d'}$, then U is a matrix function such that,

$$(1.3) \qquad\qquad U_{ij} = U_{ij}(x) = \frac{\partial v_i}{\partial x_j}$$

where U_{ij} denotes the entry of U appearing in row i and column j.

For our purposes, we are only interested in the non-zero entries of matrix U. Thus, we will write the entry of U appearing in row i and column j as U_{ij} iff $\frac{\partial v_i}{\partial x_j} \neq 0$ and 0 otherwise.

By the linearity of the derivative, we see that in the equation (1.2),

$$(1.4) \qquad\qquad f'(x) = (Sv(x))' = Sv'(x) = SU$$

Of particular interest is when the equation in (1.2) has **Reaction Form (RF)**. This means:

$$S_{ij} \geq 0 \text{ iff } U_{ji} = 0 \text{ and } S_{ij} < 0 \text{ iff } U_{ji} > 0,$$

where $U_{ji} > 0$ means $\frac{\partial v_j}{\partial x_i}(x) > 0$ for each $x \in \mathbb{R}_{>0}^d$. Sometimes we will say that the Stoichiometric matrix S is RF because the entries of U depend entirely on the matrix S. Notice that the RF condition implies each entry of U is non-negative on the non-negative orthant $\mathbb{R}_{\geq 0}^d$ in $\mathbb{R}^d$. RF differential equations arise naturally in chemical reactions as we discuss in Section 2 and give examples. When RF differential equations do arise from a chemical reaction network, we call this system of differential equations the **Stoichiometric Representation** of a reaction network (see §2.2).

We now turn to a different representation of RF differential equations that is commonly studied in relation to chemical reaction networks.

1.1.2. *Complexes Representation of a Chemical Reaction Network.* In addition to the Stoichiometric Representation of a chemical network, another common way of writing differential equations associated to a chemical network is

$$(1.5) \qquad\qquad \frac{dx}{dt} = Y A_\kappa \Psi(x), \qquad x \in \mathbb{R}_{\geq 0}^d$$

where Y is a $d \times n$ matrix of integers, A_κ is a $n \times n$ constant matrix such that the entries in each of its columns sum to 0, and $\Psi(x)$ is a vector consisting of n monomials. We call this system of differential equation the **Complexes Representation** of a chemical network (see §2.3).

In Theorem 2.8 and Proposition 2.9 we show that the Stoichiometric Representation and the Complexes Representation represent the exact same system of differential equations, and we provide an algorithm for decomposing the Stoichiometric Representation $\frac{dx}{dt} = Sv(x)$ into the Complexes Representation $\frac{dx}{dt} = Y A_\kappa \Psi(x)$, and vice versa. Thus, we are justified in our earlier statement that the Complexes Representation is another way of writing RF differential equations when such equations arise from a chemical reaction network. Of particular interest is the number and type of equilibria for these differential equations. The **deficiency 0** method (see §3) is used to analyze equilibria for the Complex Representation, while the **core determinant** method (see §3) is used to analyze

equilibria of the Stoichiometric Representation. We now give a brief introduction to these two methods.

1.2. **Deficiency versus Determinant of the Jacobian f'(x)= SU.** Two powerful tools used to analyze equilibria for RF differential equations arising from a chemical reaction network are the **deficiency** and the **core determinant** associated to the network. We now provide some brief definitions and motivation for these two concepts.

Definition 1.2. For the Complex Representation in (1.5) corresponding to a chemical reaction network, the **deficiency** of the network is defined to be

$$\dim(\ker Y \cap \operatorname{Im} A_\kappa).$$

Definition 1.3. For the matrices $S \in \mathbb{R}^{d \times d'}$, $U \in \mathbb{R}^{d' \times d}$ described in §1.1.1, with $r := \operatorname{rank}(S)$, the core determinant is defined to be

$$(1.6) \qquad cd(S) := \lim_{t \to 0} \frac{1}{t^{d-r}} \det(SU - tI).$$

Since S is a constant matrix, the expression $cd(S)$ is a polynomial in the functions $U_{ij}(x)$, which are the entries of the matrix function $U(x) = v'(x)$ as described in §1.1.1. Hence, it makes sense to write $cd(S) = cd(S)(x)$ for $x \in \mathbb{R}_{\geq 0}^d$. Also, we will say that $cd(S)$ has a **Determinant Sign (DetSign)** iff all the terms in the determinant expansion have the same sign. Notice that since each entry $U_{ij}(x)$ of matrix U is a non-negative function on the positive orthant $\mathbb{R}_{\geq 0}^d$, the core determinant having a DetSign implies $cd(S)(x)$ is sign invariant for each $x \in \mathbb{R}_{\geq 0}^d$.

Definition 1.4. Let S be the $d \times d'$ Stoichiometric matrix defined in §1.1.1. A **(positive) stoichiometric compatibility class** is a nonempty set of the form $\mathbb{R}_{\geq 0}^d \cap (\operatorname{Im} S + c)$ for some $c \in \mathbb{R}_{\geq 0}^d$.

There are two important results regarding the equilibria of a chemical network when the deficiency of the network is 0 or $cd(S)$ has a DetSign. One of the results introduces a concept called **weak reversibility** of a reaction network, which we refer the reader to [G03] for a full discussion. We state both of these results in the following two theorems.

Theorem 1.5. *A weakly reversible mass-action chemical reaction network with deficiency 0 contains one unique equilibrium point in each positive stoichiometric compatibility class.*

Theorem 1.6. *Suppose $\frac{dx}{dt} = f_b(x) := S v^b(x)$ has reaction form with $v^b(x)$ once continuously differentiable in x and depending continuously on a parameter $0 \leq b \leq 1$. Suppose each component $v_j^b(x)$ is monotone nondecreasing. Suppose M is a compact, positive stoichiometric compatibility class. Suppose cd(S) has a Determinant Sign.*

If there are no zeros $f_b(x) = 0$ for any b and any x on the boundary of M, then the number of zeros for f_b in the interior of M is independent of b.

Theorem 1.5 (see [G03]) and Theorem 1.6 (see [HKG08]) reveal that a network having 0 deficiency or the core determinant having a DetSign imply similar strong results regarding the uniqueness of equilibria for the chemical network.

A natural question to ask is whether the deficiency of a chemical network correlates to the core determinant having a DetSign; in other words, is there any overlap between the deficiency 0 condition and the DetSign condition? Surprisingly, the answer is no. *We show that a network having 0 deficiency has no correlation to whether the core determinant has a Determinant Sign (§3 Example 3.8 and 3.9).* In the same section, we also compare deficiency to other determinant variations.

1.3. **Sign Pattern of** $SU = f'(x)$**.** The final topic presented in this thesis, see §4, concerns when the matrix SU described in §1.1.1 is a sign definite matrix [BS95]. The subject of sign definite matrices considers classes of matrices having a fixed sign pattern (two matrices are in a given class iff each of their entries has the same sign (or is 0)), then one studies determinants. In the case of RF differential equations, one approach to analyzing equilibria for the system is to study the core determinant expansion of SU described earlier (see [HKG08]). In general, one could use sign definite matrix theory to analyze the determinant of a matrix, but in the case of RF differential equations, the matrix SU is defined to be a matrix-valued function, and hence, will often not be sign definite. Nevertheless, we may extend the definition of a fixed sign pattern to the matrix SU so that sign definite matrix theory would apply. This will be the motivation behind the next set of definitions.

Definition 1.7. With the notation in (1.3) and (1.4) in §1.1.1, we will say that an entry of the matrix SU in row i and column j,

$$(SU)_{ij} = \sum_{k=1}^{d'} S_{ik} U_{kj}$$

has a **sign** if $S_{ik} \leq 0$, $\forall k \in \{1, \ldots, d'\}$ or $S_{ik} \geq 0$, $\forall k \in \{1, \ldots, d'\}$; i.e., $(SU)_{ij}$ has a sign means that it is a non-negative or non-positive linear combination of the entries of matrix U. The matrix SU is said to have a **sign pattern (SP)** if each entry of SU has a sign. Thus, saying that SU has a sign pattern means that every entry of the matrix SU is a positive or negative linear combination of the entries of matrix U. If an entry does have a sign, we will say that the sign of that entry is minus (-) if it is a negative linear combination, plus (+) if it is a positive linear combination, and 0 otherwise. If an entry happens to be constant, we will still say that the entry has a sign, and it will be plus, minus, or 0 depending on whether the entry is positive, negative or 0. If two matrices have a sign pattern, they are said to have the same sign pattern iff every entry of one matrix has the same sign as the corresponding entry of the other matrix. Here are a few examples.

Example 1.8. Consider:

$$(1.7) \qquad SU = \begin{bmatrix} -U_{11} - 2U_{21} & U_{12} + 4U_{22} \\ U_{11} + U_{21} & 0 \end{bmatrix}$$

This matrix has as a sign pattern since the sign of $(SU)_{11}$ is minus, the sign of $(SU)_{21}$ is plus, the sign of $(SU)_{12}$ is plus, and the sign of $(SU)_{22}$ is 0. Now consider:

$$(1.8) \qquad \tilde{S}\tilde{U} = \begin{bmatrix} -\tilde{U}_{11} + 2\tilde{U}_{21} & \tilde{U}_{12} + 4\tilde{U}_{22} \\ \tilde{U}_{11} + \tilde{U}_{21} & 0 \end{bmatrix}$$

Since entry $(\tilde{S}\tilde{U})_{11}$ is neither a positive nor negative linear combination of the entries of U, it does not have a sign, and hence, $\tilde{S}\tilde{U}$ does not have a sign pattern. With all these definitions, the matrix in (1.7) has the same sign pattern as:

$$\begin{bmatrix} -1 & +1 \\ +1 & 0 \end{bmatrix}$$

∎

In §4 Corollary 4.2, we extend results in [HKG08] to determine necessary and sufficient conditions for an entry of SU to have a sign. We also show how under certain conditions, that sign pattern analysis may still apply to SU even if it does not have a sign pattern. Finally, we show that deficiency 0 and having a sign pattern are not correlated (see §4.2 Example 4.6 and 4.7).

2. Chemical Reaction Networks

The goal of this section is to provide the necessary background and theoretical results for comparing deficiency and determinant expansions, which we do in §3.

This section first introduces the basic setup and definitions, accompanied by examples, of chemical reaction networks. We then study the two standard ways of representing differential equations arising from a chemical reaction network; we refer to them as the Stoichiometric Representation and the Complexes Representation. We show that a system which has one of these representations also has the other, and we layout precisely the correspondence between them. Also, we provide an algorithm in Proposition 2.4 of how to construct chemical reaction networks when starting with a certain type of matrix; we use this result repeatedly in §3 when we analyze deficiency versus determinant expansions.

2.1. **Basic Setup of Chemical Reaction Networks.** As previously mentioned, certain classes of chemical reaction networks are modeled naturally by RF systems of differential equations. A comprehensive and intuitive account of how differential equations are built from chemical reaction networks can be found in [F79]. In this paper, we will be content with only a formal mathematical treatment of chemical reaction networks and the differential equations associated to them.

Suppose we are given a certain set of chemical **reactions** $\mathcal{R}$ with $|\mathcal{R}| < \infty$. Set

$$\mathcal{S} = \{S_1, S_2, ..., S_n\},$$

the ordered set of all **species** which occur in all elements (i.e. chemical reactions) in $\mathcal{R}$. By definition, $r \in \mathcal{R}$ means

$$(2.1) \qquad r = \sum_{i=1}^{n} a_i S_i \rightarrow \sum_{i=1}^{n} b_i S_i,$$

with $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}_{\geq 0}$ and satisfy $a_i \neq 0 \Rightarrow b_i = 0$ and $b_j \neq 0 \Rightarrow a_j = 0$ for any $i, j \in \{1, \ldots, n\}$. Each reaction has exactly one input and exactly one output, which we call the **complexes**. Thus, a complex is just a non-negative, integer linear

combination of species, and we may think of a reaction as a relation on 2 complexes that is given by an arrow ($\to$). Indeed, the input of a reaction is the complex on the left side of the arrow, and the output to a reaction is the complex on the right side of the arrow. In reaction r of (2.1), we have complexes, $\sum_{i=1}^{n} a_i S_i$, and $\sum_{i=1}^{n} b_i S_i$. We will denote $\mathcal{C}$ as the set of all such complexes. Indeed, since each reaction has exactly one input and one output, and each complex must contain at least one species, we have the relation, $|S| \leq |C| \leq 2|R|$. There is a natural injective, map, $\varphi : \mathcal{C} \to \mathbb{Z}_{\geq 0}^n$ defined by

$$\varphi(\sum_{i=1}^{n} a_i S_i) = (a_1, \ldots, a_n)^T$$

From now on we will identify a complex with its image under φ; i.e., a complex will be a vector of non-negative integers determined by its representation as a linear combination of species. To proceed, let

$$\mathcal{C} = \{y_1, y_2, \ldots, y_k\}$$

be some ordering of the complexes. Also, for each reaction $y_i \to y_j \in \mathcal{R}$, $i, j \in \{1, \ldots, k\}$, we associate a positive rate constant $\kappa_{i \to j} \in \mathbb{R}_{>0}$. Sometimes, we shall write $\kappa_{y_i \to y_j}$ instead of $\kappa_{i \to j}$ whenever it is more convenient. Let

$$\mathcal{K} = \{\kappa_{i \to j} \in \mathbb{R}_{>0} : y_i \to y_j \in \mathcal{R}\}$$

be the set of all reaction rate constants. Thus, $|\mathcal{K}| = |\mathcal{R}|$ by construction. We will also insist on two more special properties of $\mathcal{R}$:

$$(1) \ y \to y \notin \mathcal{R} \text{ for any } y \in \mathcal{C}$$

$$(2) \text{ For every } y_i \in \mathcal{C}, \text{ there exists } y_j \in \mathcal{C} \text{ such that } y_i \to y_j \in \mathcal{R} \text{ or } y_i \to y_j \in \mathcal{R}$$

To formalize this setup, we define a chemical reaction network where d species and n complexes participate in d' chemical reactions as a quadruple $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathcal{K})$ of ordered species ($\mathcal{S}$), ordered complexes $\mathcal{C}$, reactions ($\mathcal{R}$), and reaction rate constants ($\mathcal{K}$). As just described, $\mathcal{S}, \mathcal{C}, \mathcal{R}$, and $\mathcal{K}$ are given by

$$\mathcal{S} = \{S_1, S_2, ..., S_d\}$$
$$\mathcal{C} = \{y_1, y_2, \ldots, y_k\}, y_i \in \mathbb{Z}_{\geq 0}^d, i = 1, \ldots, k$$
$$\mathcal{R} = \{y_{i_1} \to y_{j_1}, \ldots, y_{i_{d'}} \to y_{j_d}\}, \text{ where } y_{i_s}, y_{j_t} \in \mathcal{C} \text{ for } s, t = 1, \ldots, d$$
$$\mathcal{K} = \{\kappa_{i \to j} \in \mathbb{R}_{>0} : y_i \to y_j \in \mathcal{R}\}.$$

$\blacksquare$

We now present an example of a chemical reaction network.

Example 2.1. Suppose we had the set of reactions,
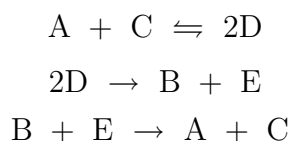
$$A + C \rightleftharpoons 2D$$
$$2D \to B + E$$
$$B + E \to A + C$$

Then we let our set of species be,

$$\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$$

with $S_1 = A$, $S_2 = B$, $S_3 = C$, $S_4 = D$, and $S_5 = E$. Notice that we have 3 complexes, $y_1 = S_1 + S_3 \sim (1, 0, 1, 0, 0)^T$, $y_2 = 2S_4 \sim (0, 0, 0, 2, 0)^T$, $y_3 = S_2 + S_5 \sim (0, 1, 0, 0, 1)^T$, so that our set of complexes is

$$\mathcal{C} = \{y_1, y_2, y_3\}.$$

We can write our set of reactions as,

$$\mathcal{R} = \{y_1 \to y_2, y_2 \to y_1, y_2 \to y_3, y_3 \to y_1\}.$$

This means we will only have rate constants $\kappa_{1\to2}$, $\kappa_{2\to1}$, $\kappa_{2\to3}$, and $\kappa_{3\to1}$. Thus,

$$\mathcal{K} = \{\kappa_{1\to2}, \kappa_{2\to1}, \kappa_{2\to3}, \kappa_{3\to1}\}.$$

∎

Now that we have the basic setup, we want to analyze systems of differential equations associated to a given chemical reaction network. There are two common ways to describe such differential equations, and we shall present each of these ways in the following two subsections.

2.2. The Stoichiometric Representation $\frac{dx}{dt} = Sv(x)$. Let $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathcal{K})$ be an arbitrary chemical reaction network, as introduced in §2.1, where we have

$$|\mathcal{S}| = d, \ |\mathcal{C}| = k, \ |\mathcal{R}| = |\mathcal{K}| = d'$$

Let us write our set of reactions as

$$\mathcal{R} = \{y_{i_1} \to y_{j_1}, \dots, y_{i_{d'}} \to y_{j_{d'}}\}, \ \text{where } y_{i_s}, y_{j_t} \in \mathcal{C} \text{ for } s, t = 1, \dots, d'.$$

We set S to be the $d \times d'$ stoichiometric matrix:

$$S = \left[\begin{array}{cccc} | & | & & | \\ y_{j_1} - y_{i_1} & y_{j_2} - y_{i_2} & \cdots & y_{j_{d'}} - y_{j_{d'}} \\ | & | & & | \end{array} \right]$$

where the t'th column of S is $y_{j_t} - y_{i_t}$, a vector with d components obtained by subtracting the vectors y_{j_t} and y_{i_t}.

Before proceeding, let us introduce some convenient notation. For $x = (x_1, \dots, x_d)^T \in \mathbb{R}^d$ and $w = (w_1, \dots, w_d)^T \in \mathbb{R}^d$, we denote,

$$x^w := \prod_{i=1}^{d} x_i^{w_i}$$

With this new notation, let $v : \mathbb{R}^d \to \mathbb{R}^{d'}$ be a vector of monomials with rate constants given by:

$$v(x) = \left[\begin{array}{c} \kappa_{i_1 \to j_1} x^{y_{i_1}} \\ \kappa_{i_2 \to j_2} x^{y_{i_2}} \\ \vdots \\ \kappa_{i_{d'} \to j_{d'}} x^{y_{i_{d'}}} \end{array} \right].$$

By writing $v(x) = (v_1(x), \dots, v_{d'}(x))^T$, we see that each monomial $v_t(x)$ is corresponding to the reaction $y_{i_t} \to y_{j_t}$ and depends only on the input y_{i_t} of the reaction. Also, if we

write $S = (S_{ij})$, then the entries of $v(x)$ are completely determined by the entries of S since a quick observation reveals that we have,

$$(2.2) \qquad v_t(x) = \kappa_{i_t \to j_t} \prod_{n=1}^{d} x_n^{|min\{0, S_{nt}\}|} = \kappa_{i_t \to j_t} x^{y_{i_t}} \text{ for } t = 1, \ldots, d'.$$

The differential equations that we associate to this network of chemical reactions are precisely,

$$(2.3) \qquad \frac{dx}{dt} = Sv(x), \quad x \in \mathbb{R}_{\geq 0}^d$$

Differential equations that arise from a chemical reaction network, and are written in the form of (2.3) are said to be in **Stoichiometric Form**. The differential equations $\frac{dx}{dt} = Sv(x)$ are said to be the **Stoichiometric Representation** of the chemical reaction network. An immediate consequence is that this construction actually gives us an RF system of differential equations.

Proposition 2.2. *The Stoichiometric Representation $\frac{dx}{dt} = Sv(x)$ in (2.3) is RF.*

Proof. Formal differentiation of formula (2.2) using the power rule (although with a slight abuse of notation) gives,

$$U_{tj} = \frac{\partial v_t}{\partial x_j} = \kappa_{i_t \to j_t} |min\{0, S_{jt}\}| \frac{\prod_{n=1}^{d} x_n^{|min\{0, S_{nt}\}|}}{x_j}$$

Hence, $U_{tj} = 0$ iff $S_{jt} \geq 0$ and $U_{tj} \neq 0$ iff $S_{jt} < 0$. This is precisely the definition of an RF system.

■

Example 2.3. We now present a concrete example taken from [CF06] of writing the Stoichiometric Representation of a chemical reaction network.

Consider the chemical network of reactions among species $A, B, M, N, R, X, Y,$ and Z :

$$(2.4) \qquad A + M \leftrightharpoons X, B + N \leftrightharpoons Y \to 2A + N, B + X \leftrightharpoons Z \to R + M$$

Our ordered set of species ($\mathcal{S}$), complexes ($\mathcal{C}$), and reactions ($\mathcal{R}$) are precisely

$$\mathcal{S} = \{A, B, M, N, R, X, Y, Z\}$$

$$\mathcal{C} = \{y_1, y_2, \ldots, y_8\} \text{ with}$$

$$y_1 = A + M \sim (1, 0, 1, 0, 0, 0, 0, 0)^T, y_2 = X \sim (0, 0, 0, 0, 0, 1, 0, 0)^T$$

$$y_3 = B + N \sim (0, 1, 0, 1, 0, 0, 0, 0)^T, y_4 = Y \sim (0, 0, 0, 0, 0, 0, 1, 0)^T$$

$$y_5 = 2A + N \sim (2, 0, 0, 1, 0, 0, 0, 0)^T, y_6 = B + X \sim (0, 1, 0, 0, 0, 1, 0, 0)^T$$

$$y_7 = Z \sim (0, 0, 0, 0, 0, 0, 0, 1)^T, y_8 = R + M \sim (0, 0, 0, 0, 1, 0, 1, 0)^T$$

$$\mathcal{R} = \{y_1 \rightleftarrows y_2, y_3 \rightleftarrows y_4, y_4 \to y_5, y_6 \rightleftarrows y_7, y_7 \to y_8\}$$

where $\sim$ denotes the image of a complex under the map ϕ in §2.1. Direct computation shows that our Stoichiometric matrix S is

$$S = \begin{bmatrix} | & | & | & | & | & | & | & | \\ y_2 - y_1 & y_1 - y_2 & y_4 - y_3 & y_3 - y_4 & y_5 - y_4 & y_7 - y_6 & y_6 - y_7 & y_8 - y_7 \\ | & | & | & | & | & | & | & | \end{bmatrix}$$

$$
= \begin{bmatrix}
-1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1
\end{bmatrix},
$$

and our vector $v(x)$ with $x = (x_1, \ldots, x_8)^T$ is given by

$$
v(x) = \begin{bmatrix}
\kappa_{A+M \to X} x^{y_1} \\
\kappa_{X \to A+M} x^{y_2} \\
\kappa_{B+N \to Y} x^{y_3} \\
\kappa_{Y \to B+N} x^{y_4} \\
\kappa_{Y \to 2A+N} x^{y_4} \\
\kappa_{B+X \to Z} x^{y_6} \\
\kappa_{Z \to B+X} x^{y_7} \\
\kappa_{Z \to R+M} x^{y_7}
\end{bmatrix}
= \begin{bmatrix}
\kappa_{A+M \to X} x_1 x_3 \\
\kappa_{X \to A+M} x_6 \\
\kappa_{B+N \to Y} x_2 x_4 \\
\kappa_{Y \to B+N} x_7 \\
\kappa_{Y \to 2A+N} x_7 \\
\kappa_{B+X \to Z} x_2 x_6 \\
\kappa_{Z \to B+X} x_8 \\
\kappa_{Z \to R+M} x_8
\end{bmatrix}.
$$

The differential equations in Stoichiometric Form associated to this chemical network are precisely $\frac{dx}{dt} = Sv(x)$.

■

In fact, we actually have a converse to Proposition 2.2 that will allow us to construct numerous RF systems of differential equations that correspond to chemical reaction networks when we start with a certain type of matrix.

Proposition 2.4. *Let $S \in \mathbb{Z}^{d \times d'}$, a $d \times d'$ matrix over the integers such that each column of S contains at least one positive and one negative entry. Then there exists a vector valued function $v : \mathbb{R}_{\geq 0}^d \to \mathbb{R}^{d'}$ such that the system of differential equations,*

$$
\frac{dx}{dt} = Sv(x)
$$

is RF and is the Stoichiometric Representation of some chemical reaction network.

Proof. Let S be as in the statement of the Proposition. Write $S = S_+ - S_-$ where S_+, S_- have non-negative entries given by

$$
(S_+)_{ij} = \begin{cases} S_{ij} & \text{if } S_{ij} > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (S_-)_{ij} = \begin{cases} -S_{ij} & \text{if } S_{ij} < 0 \\ 0 & \text{otherwise} \end{cases}
$$

Our assumptions imply that each column of S_+, S_- is nonzero. Let $\mathcal{S} = \{S_1, \ldots, S_d\}$ be an arbitrary collection of species. Define the map $\phi : \mathbb{Z}^d \to \mathrm{span}_{\mathbb{Z}} \mathcal{S}$ by

$$
\phi(a) = \sum_{i=1}^d a_i S_i \text{ for } a = (a_1, \ldots, a_d)^T \in \mathbb{Z}^d
$$

Let $\{y_{i_1}, \ldots, y_{i'_d}\}$ be the ordered columns of S_- and $\{y_{j_1}, \ldots, y_{j'_d}\}$ be the ordered columns of S_+. By construction, the t'th column of S is precisely $y_{j_t} - y_{i_t}$. Now, identify the

columns of S_+ and S_- with their image under ϕ. Let $\mathcal{R}$ be the set of reactions,

$$\mathcal{R} = \{y_{i_1} \to y_{j_1}, \ldots, y_{i_{d'}} \to y_{j_{d'}}\}$$

If we let $\mathcal{K}$ be the set of rate constants that correspond to these reactions and $\mathcal{C} := \{y_{i_1}, \ldots, y_{i'_d}, y_{j_1}, \ldots y_{j'_d}\}$ be the set of complexes, we have a chemical reaction network given by $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathcal{C})$ whose Stoichiometric matrix is precisely S. Proposition 2.3 shows that the system of differential equations $\frac{dx}{dt} = Sv(x)$ for $x \in \mathbb{R}^d_{\geq 0}$ as constructed in (2.3) is RF.

∎

Differential Equations in Stoichiometric Form share many nice properties, and are studied extensively in the chemical literature. One may refer to [Pa06] and [F79] for some examples of these.

There is another way of representing this system of differential equations associated to a chemical reaction network, which has other significant properties. This will be the topic of the next section.

2.3. **The Complexes Representation** $\frac{dx}{dt} = YA_\kappa\Psi(x)$. Let $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathcal{K})$ be an arbitrary chemical reaction network as presented in §2.1, where we set

$$d = |\mathcal{S}|, \;\; k = |\mathcal{C}|, \;\; d' = |\mathcal{R}| = |\mathcal{K}|$$

Now let $Y : \mathbb{R}^{|\mathcal{C}|} \to \mathbb{R}^{|\mathcal{S}|}$ be the matrix defined by

$$Y := \begin{bmatrix} | & & | \\ y_1 & \cdots & y_k \\ | & & | \end{bmatrix}$$

As described before, each reaction, $y_i \to y_j$, $i, j \in \{1, \ldots, k\}$ has a corresponding rate constant $\kappa_{i \to j}$. Now let $A : \mathbb{R}^{|\mathcal{C}|} \to \mathbb{R}^{|\mathcal{C}|}$ be a matrix defined by:

$$A_{ij} = \begin{cases} \kappa_{i \to j} & \text{if } y_i \to y_j \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases},$$

where A_{ij} is the entry in row i and column j of matrix A. Notice that our restrictions on what types of reactions are permissable imply that A will have 0's along its diagonal. Now define the matrix $A_\kappa : \mathbb{R}^{|\mathcal{C}|} \to \mathbb{R}^{|\mathcal{C}|}$, such that,

$$A_\kappa = A^T - diag(A \cdot \mathbf{1}),$$

where $\mathbf{1}$ is a vector with $|\mathcal{C}|$ components consisting of 1's, and for any vector $v = \{v_1, \ldots, v_n\}^T \in \mathbb{R}^n$, $diag(v)$ is an $n \times n$ matrix defined by,

$$(diag(v))_{ij} = \begin{cases} v_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $diag(A \cdot \mathbf{1})$ is a matrix with row sums of matrix A along its diagonal and 0's everywhere else. Indeed the entry in row i and column j of matrix A_κ is:

$$(A_\kappa)_{ij} = \begin{cases} \kappa_{j \to i} & \text{if } i \neq j \text{ and } y_i \to y_j \in R \\ 0 & \text{if } i \neq j \text{ and } y_i \to y_j \in R \\ -\sum_{m=1}^{|\mathcal{C}|} A_{im} & \text{if } i = j \end{cases}$$

Finally we let $\Psi(x) : \mathbb{R}^n \to \mathbb{R}^{|\mathcal{C}| \times 1}$ be the vector of monomials defined by:

$$\Psi(x) = \begin{bmatrix} x^{y_1} \\ x^{y_2} \\ \vdots \\ x^{y_k} \end{bmatrix},$$

where $x = (x_1, \ldots, x_n)^T$, and we consider the differential equations,

$$(2.5) \qquad \frac{dx}{dt} = Y A_\kappa \Psi(x)$$

This system is well-defined since Y is a $|\mathcal{S}| \times |\mathcal{C}|$ matrix, A_κ is a $|\mathcal{C}| \times |\mathcal{C}|$ matrix and $\Psi(x)$ is a vector with $|\mathcal{C}|$ components. Differential equations associated to a chemical reaction network that are constructed as in (2.5) are the **Complexes Representation** of the chemical reaction network. $\blacksquare$

Remark 2.5. In the above construction, we must always pick an ordering for the complexes and species to proceed. For example, if we had the reaction,

$$2\text{H} + \text{O} \to \text{H}_2\text{O},$$

we could label the species as $S_1 = H$, $S_2 = O$, and $S_3 = H_2O$. Picking an ordering is always a matter of preference, and there is no unique way to do it. Indeed we could have ordered the species above as $S_1 = O$ and $S_2 = H$. Thus, two people may pick different ordering and then come up with different matrices Y and A_κ, but any two such constructions for the same set of chemical reactions will have the same Y, A_κ, and $\Psi(x)$ after row and column permutations, and will always yield the same system of differential equations.

We now present an example to illustrate such a construction.

Example 2.6. Suppose we had the set of reactions,
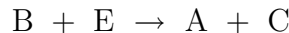
$$\text{A} + \text{C} \rightleftharpoons \text{D}$$

$$\text{D} \to \text{B} + \text{E}$$

$$\text{B} + \text{E} \to \text{A} + \text{C}$$

Then we let our set of species be,

$$\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$$

with $S_1 = A$, $S_2 = B$, $S_3 = C$, $S_4 = D$, and $S_5 = E$. Notice that we have 3 complexes, $y_1 = S_1 + S_3$, $y_2 = S_4$, $y_3 = S_2 + S_5$, so that our set of complexes is,

$$\mathcal{C} = \{y_1, y_2, y_3\}$$

We now have our set of reactions as,

$$\mathcal{R} = \{y_1 \to y_2, y_2 \to y_1, y_2 \to y_3, y_3 \to y_1\}.$$

This means we will only have rate constants $\kappa_{1\to2}$, $\kappa_{2\to1}$, $\kappa_{2\to3}$, and $\kappa_{3\to1}$. Hence our matrix A is,

$$A = \begin{bmatrix} 0 & \kappa_{1\to2} & 0 \\ \kappa_{2\to1} & 0 & \kappa_{2\to3} \\ \kappa_{3\to1} & 0 & 0 \end{bmatrix}.$$

A simple matrix computation gives:

$$A_\kappa = A^T - diag(A \cdot \mathbf{1}) = \begin{bmatrix} -\kappa_{1\to2} & \kappa_{2\to1} & \kappa_{3\to1} \\ \kappa_{1\to2} & -\kappa_{2\to1} - \kappa_{2\to3} & 0 \\ 0 & \kappa_{2\to3} & -\kappa_{3\to1} \end{bmatrix}.$$

We also have,

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Another quick computation gives,

$$(2.6) \qquad \Psi(x) = \begin{bmatrix} x_1 x_3 \\ x_4 \\ x_2 x_5 \end{bmatrix}$$

Thus, the Complexes Representation of this chemical reaction network is precisely:

$$\frac{dx}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \end{bmatrix} = YA_\kappa\Psi(x) = \begin{bmatrix} -\kappa_{1\to2}x_1x_3 + \kappa_{3\to1}x_2x_5 + \kappa_{2\to1}x_4 \\ \kappa_{2\to3}x_4 - \kappa_{3\to1}x_2x_5 \\ -\kappa_{1\to2}x_1x_3 + \kappa_{3\to1}x_2x_5 + \kappa_{2\to1}x_4 \\ \kappa_{1\to2}x_1x_3 - \kappa_{2\to1}x_4 - \kappa_{2\to3}x_4 \\ \kappa_{2\to3}x_4 - \kappa_{3\to1}x_2x_5 \end{bmatrix}$$

One may also check that the Stoichiometric Representation of this network is

$$(2.7) \qquad \frac{dx}{dt} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \kappa_{1\to2}x_1x_3 \\ \kappa_{2\to1}x_4 \\ \kappa_{2\to3}x_4 \\ \kappa_{3\to1}x_2x_5 \end{bmatrix} = Sv(x)$$

and this yields the same system of differential equations as the Complexes Representation. ∎

Notice that in the above example, $v(x)$ looks almost like $\Psi(x)$ but with rate constants attached to monomials, and the columns of matrix S in (2.7) are linear combinations of the columns of matrix Y in the Complexes Representation. This is not a coincidence; that is, for arbitrary chemical reaction networks, we will always have

$$(2.8) \qquad \frac{dx}{dt} = Sv(x) = YA_\kappa\Psi(x)$$

and the final results in this section will tell us exactly this. First, some definitions are needed.

Definition 2.7. Let $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa)$ be an arbitrary chemical reaction network. For each complex $y_j \in \mathcal{C}$, we denote the **multiplicity** of the complex as

$$\beta_j := \text{card}(\{y_i \in \mathcal{C} : y_j \to y_i \in \mathcal{R}\}).$$

Indeed, the multiplicity of a complex is the number of reactions in the network where the complex serves as an input to the reaction. Thus, complexes that are outputs to a reaction, but are not an input to any reaction have multiplicity 0. For example, if we had the set of reactions $\{A \to 2B, A + C \leftrightharpoons 2D, 2D \to B + E, B + E \to A + C\}$, then the multiplicities of the complexes A, 2B, A+C, 2D, B+E are 1,0,1,2, and 1 respectively. It follows immediately from our definition that

$$\sum_{j=1}^{|\mathcal{C}|} \beta_j = |\mathcal{R}|.$$

Also, for each $j = 1, \ldots, |\mathcal{C}|$ we will write $\{y_j \to y_{j_1}, \ldots, y_j \to y_{j_{\beta_j}}\}$ for the set of reactions corresponding to complex j. Our definition allows such a set to be empty.

With the notion of multiplicities, we can develop a more convenient form for writing the Stoichiometric Representation of a chemical reaction network.

Let $\frac{dx}{dt} = Sv(x)$ be the Stoichiometric Representation of the chemical reaction network. Recall that the columns of S correspond to the reactions of the network, and that if we order the reactions differently, we will just be permuting the columns of S and the corresponding entries of $v(x)$ without changing the actual system of differential equations. With this in mind, let $\mathcal{C} = \{y_1, \ldots, y_n\}$ be the ordered complexes for the reaction network. Denote $\beta_j :=$ multiplicity of y_j, and let B_j be the $|\mathcal{S}| \times \beta_j$ matrix given by

$$B_j := \begin{bmatrix} | & | & & | \\ y_{j_1} - y_j & y_{j_2} - y_j & \cdots & y_{j_{\beta_j}} - y_j \\ | & | & & | \end{bmatrix}$$

and let $v_{B_j} \in \mathbb{R}^{\beta_j}$ denote the vector

$$v_{B_j} := \begin{bmatrix} \kappa_{y_j \to y_{j_1}} x^{y_j} \\ \kappa_{y_j \to y_{j_2}} x^{y_j} \\ \vdots \\ \kappa_{y_j \to y_{j_{\beta_j}}} x^{y_j} \end{bmatrix}$$

for $j = 1, \ldots, n$. If we order our set of reactions as

$$\mathcal{R} = \{y_1 \to y_{1_1}, \ldots, y_1 \to y_{1_{\beta_1}}, \ldots, y_j \to y_{j_1}, \ldots, y_j \to y_{j_{\beta_j}}, \ldots, y_n \to y_{n_1}, \ldots, y_n \to y_{n_{\beta_n}}\},$$

then S and $v(x)$ are precisely

$$S = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}, \quad v(x) = \begin{bmatrix} v_{B_1} \\ \vdots \\ v_{B_n} \end{bmatrix}.$$

In this case, S and $v(x)$ we call a **Stoichiometric Canonical Form**.

∎

Theorem 2.8. *Let $(\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathcal{K})$ be a chemical reaction network. Suppose $\frac{dx}{dt} = Sv(x)$ is the Stoichiometric Representation associated with the chemical network, and let $\frac{\tilde{dx}}{dt} = YA_\kappa \Psi(x)$ be the Complexes Representation of the chemical network. Then*

$$\frac{dx}{dt} = Sv(x) = YA_\kappa \Psi(x).$$

Proof. Let $\mathcal{C} = \{y_1, \ldots, y_n\}$ be the ordered complexes for the reaction network. Denote $\beta_j :=$ multiplicity of y_j, and let B_j be the $|\mathcal{S}| \times \beta_j$ matrix given by

$$B_j := \left[\begin{array}{cccc} | & | & & | \\ y_{j_1} - y_j & y_{j_2} - y_j & \cdots & y_{j_{\beta_j}} - y_j \\ | & | & & | \end{array} \right]$$

and let $v_{B_j} \in \mathbb{R}^{\beta_j}$ denote the vector

$$v_{B_j} := \left[\begin{array}{c} \kappa_{y_j \to y_{j_1}} x^{y_j} \\ \kappa_{y_j \to y_{j_2}} x^{y_j} \\ \vdots \\ \kappa_{y_j \to y_{j_{\beta_j}}} x^{y_j} \end{array} \right]$$

for $j = 1, \ldots, n$. Thus, the Stoichiometric Canonical Form of S and v(x) is given by

$$S = \left[\begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \right], \; v(x) = \left[\begin{array}{c} v_{B_1} \\ \vdots \\ v_{B_n} \end{array} \right].$$

By direct calculation, we have

$$(2.9) \qquad Sv(x) = \sum_{j=1}^{n} B_j v_{B_j} = \sum_{j=1}^{n} \sum_{i=1}^{\beta_j} \kappa_{y_j \to y_{j_i}} x^{y_j} (y_j - y_{j_i})$$

Let $A_\kappa = (a_{ij})$ be as in the Complexes Representation of the chemical reaction network, and denote $\alpha_1, \ldots, \alpha_n$ as the columns of A_κ written symbolically as $\alpha_j = (a_{1j}, \ldots, a_{nj})^T$. Then by definition,

$$(\alpha_j)_i = a_{ij} = \begin{cases} -\sum_{k=1}^{\beta_j} \kappa_{y_j \to y_{j_k}} & \text{if } i = j \\ \kappa_{y_j \to y_{j_k}} & \text{if } i = j_k \text{ for some } k \in \{1, \ldots, \beta_j\} \\ 0 & \text{otherwise} \end{cases}$$

Direct computation of $Y\alpha_j$, which is precisely the j'th column of YA_κ, gives

$$Y\alpha_j = \left[\begin{array}{ccc} | & & | \\ y_1 & \cdots & y_n \\ | & & | \end{array} \right] \alpha_j = (-\sum_{i=1}^{\beta_j} \kappa_{y_j \to y_{j_i}}) y_j + \sum_{i=1}^{\beta_j} \kappa_{y_j \to y_{j_i}} y_{j_i}$$

$$= \sum_{i=1}^{\beta_j} \kappa_{y_j \to y_{j_i}} (y_{j_i} - y_j) =: \gamma_j$$

Thus, the computation of $YA_\kappa\Psi(x)$ gives

$$YA_\kappa\Psi(x) = \begin{bmatrix} | & & | \\ \gamma_1 & \cdots & \gamma_n \\ | & & | \end{bmatrix} \begin{bmatrix} x^{y_1} \\ \vdots \\ x^{y_n} \end{bmatrix} = \sum_{j=1}^{n} x^{y_j}\gamma_j$$

$$= \sum_{j=1}^{n} x^{y_j} \sum_{i=1}^{\beta_j} \kappa_{y_j \to y_{j_i}}(y_{j_i} - y_j) = \sum_{j=1}^{n}\sum_{i=1}^{\beta_j} \kappa_{y_j \to y_{j_i}} x^{y_j}(y_{j_i} - y_j) = Sv(x)$$

where the last equality follows directly from (2.9). ∎

In many situations, it is more useful to have an explicit algorithm to decompose the Stoichiometric Representation of a chemical network into the Complexes Representation of a chemical network, and vice versa. The proof of the following proposition tells us precisely how to do this.

Proposition 2.9. *Let $(\mathcal{S}, \mathcal{C}, \mathcal{K}, \mathcal{R})$ be an arbitrary chemical reaction network with*

$$\frac{dx}{dt} = Sv(x) = YA_\kappa\Psi(x)$$

as its Stoichiometric and Complexes Representation. Then there exists a $|\mathcal{C}| \times |\mathcal{R}|$ matrix $\mathcal{G}$ and a $|\mathcal{R}| \times |\mathcal{C}|$ matrix $\mathcal{K}_0$ such that

$$\frac{dx}{dt} = Y\mathcal{G}\mathcal{K}_0\Psi(x) \text{ with } Y\mathcal{G} = S, \ \mathcal{G}\mathcal{K}_0 = A_\kappa, \text{ and } \mathcal{K}_0\Psi(x) = v(x).$$

Moreover, we may take $\mathcal{G}$ to be a matrix consisting of entries in the set $\{-1, 0, 1\}$, and $\mathcal{K}_0$ as a block diagonal matrix (whose blocks are actually vectors) consisting of 0's and rate constants.

Proof. Let $\mathcal{C} = \{y_1, \ldots, y_n\}$ be the ordered complexes for the reaction network. Denote $\beta_j :=$ multiplicity of y_j, and let B_j be the $|\mathcal{S}| \times \beta_j$ matrix given by

$$B_j := \begin{bmatrix} | & | & & | \\ y_{j_1} - y_j & y_{j_2} - y_j & \cdots & y_{j_{\beta_j}} - y_j \\ | & | & & | \end{bmatrix}$$

and let $v_{B_j} \in \mathbb{R}^{\beta_j}$ denote the vector

$$v_{B_j} := \begin{bmatrix} \kappa_{y_j \to y_{j_1}} x^{y_j} \\ \kappa_{y_j \to y_{j_2}} x^{y_j} \\ \vdots \\ \kappa_{y_j \to y_{j_{\beta_j}}} x^{y_j} \end{bmatrix}$$

for $j = 1, \ldots, n$. Thus, the Stoichiometric Canonical Form of S and v(x) is given by

$$S = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}, v(x) = \begin{bmatrix} v_{B_1} \\ \vdots \\ v_{B_n} \end{bmatrix}.$$

Each column $y_{j_i} - y_j$ for $i = 1, \ldots, \beta_j$ of B_j is a linear combination of two columns of Y and hence, is given by $y_{j_i} - y_j = Y\gamma^{ij}$ where $\gamma^{ij} := \{\gamma_1^{ij}, \ldots, \gamma_{|\mathcal{C}|}^{ij}\}^T$ is a column vector of 0's, 1's, and -1's given by

$$\gamma_k^{ij} = \begin{cases} -1 \text{ if } k = j \\ 1 \text{ if } k = j_i \\ 0 \text{ otherwise} \end{cases}$$

Set

$$\Gamma_j := [\gamma^{1j}, \gamma^{2j}, \ldots, \gamma^{\beta_j j}]$$

so that we have the equality

$$B_j = Y\Gamma_j.$$

The matrix $\mathcal{G}$ is row obtained as

$$\mathcal{G} := [\Gamma_1, \ldots, \Gamma_n],$$

a $|\mathcal{C}| \times |\mathcal{R}|$ matrix with entries in $\{-1, 0, 1\}$ and we have the computation

$$Y\mathcal{G} = [Y\Gamma_1, \ldots, Y\Gamma_n] = [B_1, \ldots, B_n] = S.$$

Let $A_\kappa = (a_{ij})$ be as in the Complex Representation of the chemical reaction network, and denote $\alpha_1, \ldots, \alpha_n$ as the columns of A_κ written symbolically as $\alpha_j = (a_{1j}, \ldots, a_{nj})^T$. Then by definition,

$$(\alpha_j)_i = a_{ij} = \begin{cases} -\sum_{k=1}^{\beta_j} \kappa_{y_j \to y_{j_k}} & \text{if } i = j \\ \kappa_{y_j \to y_{j_k}} & \text{if } i = j_k \text{ for some } k \in \{1, \ldots, \beta_j\} \\ 0 & otherwise \end{cases}$$

By construction, we have the calculation

$$\Gamma_j \begin{bmatrix} \kappa_{y_j \to y_{j_1}} \\ \vdots \\ \kappa_{y_j \to y_{j_{\beta_j}}} \end{bmatrix} = \alpha_j$$

for each $j = 1, \ldots, n$.

Now, we turn to the matrix $\mathcal{K}_0$. We let $\mathcal{K}_j$ be a column vector with β_j entries given by

$$\mathcal{K}_j := \begin{bmatrix} \kappa_{y_j \to y_{j_1}} \\ \vdots \\ \kappa_{y_j \to y_{j_{\beta_j}}} \end{bmatrix}$$

for each $j = 1, \ldots, n$. Note that our definition of multiplicities allows for $\mathcal{K}_j$ to be an empty vector. Since $\sum_{j=1}^n \beta_j = |\mathcal{R}|$, the matrix

$$\mathcal{K}_0 := \begin{bmatrix} \mathcal{K}_1 & \mathbf{0}_{\beta_1} & \cdots & \mathbf{0}_{\beta_1} \\ \mathbf{0}_{\beta_2} & \mathcal{K}_2 & \cdots & \mathbf{0}_{\beta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{\beta_n} & \mathbf{0}_{\beta_n} & \cdots & \mathcal{K}_n \end{bmatrix}$$

is a $|\mathcal{R}| \times |\mathcal{C}|$ block diagonal matrix consisting of rate constants and 0's, where $\mathbf{0}_{\beta_i}$ denotes a 0-vector with β_i entries. At this point, we have an important remark: If $\beta_j = 0$ for

some $j \in \{1, \ldots, n\}$, then the rows corresponding to $\mathcal{K}_j$ will not appear in the matrix $\mathcal{K}_0$ above. In this case, column j will still appear and it will consist entirely of 0's.

A direct computation now shows

$$\mathcal{G}\mathcal{K}_0 = [\Gamma_1 \mathcal{K}_1, \ldots, \Gamma_n \mathcal{K}_n] = A_\kappa.$$

Our final computation gives

$$\mathcal{K}_0 \Psi(x) = \begin{bmatrix} \mathcal{K}_1 x^{y_1} \\ \vdots \\ \mathcal{K}_n x^{y_n} \end{bmatrix} = \begin{bmatrix} v_{B_1} \\ \vdots \\ v_{B_n} \end{bmatrix} = v(x).$$

Hence, we have

$$\frac{dx}{dt} = Sv(x) = Y\mathcal{G}\mathcal{K}_0\Psi(x) = YA_\kappa\Psi(x),$$

and this completes the proof. ■

We now provide an example illustrating the use of Proposition 2.9.

Example 2.10. *This example gives an explicit computation using Proposition 3.9 in decomposing the Stoichiometric Representation ($\frac{dx}{dt} = Sv(x)$) into the Complexes Representation ($\frac{dx}{dt} = YA_\kappa\Psi(x)$).*

Suppose we had the set of reactions,

$$A \to B, A \to C, A \to D$$
$$B \to C, B \to A + D$$
$$C \to A + D$$
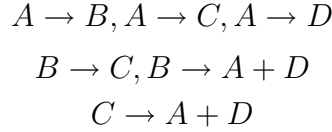
The ordered species for this network are

$$\mathcal{S} = \{A, B, C, D\}.$$

Hence, the set of complexes, reactions, and rate constants are precisely

$$\mathcal{C} = \left[y_1 = (1,0,0,0)^T, y_2 = (0,1,0,0)^T, y_3 = (0,0,1,0)^T, y_4 = (0,0,0,1)^T, y_5 = (1,0,0,1)^T \right]$$

$$\mathcal{R} = \{y_1 \to y_2, y_1 \to y_3, y_1 \to y_4, y_2 \to y_3, y_2 \to y_5, y_3 \to y_5\}$$

$$\mathcal{K} = \{\kappa_{y_1 \to y_2}, \kappa_{y_1 \to y_3}, \kappa_{y_1 \to y_4}, \kappa_{y_2 \to y_3}, \kappa_{y_2 \to y_5}, \kappa_{y_3 \to y_5}\}$$

With the notation in the proof of Proposition 3.9, we have $\Gamma_1, \Gamma_2, \Gamma_3, \mathcal{G}, \mathcal{K}_0$ given by

$$\Gamma_1 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{G} = [\Gamma_1, \Gamma_2, \Gamma_3] = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\mathcal{K}_0 = \begin{bmatrix} \kappa_{y_1 \to y_2} & 0 & 0 & 0 & 0 \\ \kappa_{y_1 \to y_3} & 0 & 0 & 0 & 0 \\ \kappa_{y_1 \to y_4} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{y_2 \to y_3} & 0 & 0 & 0 \\ 0 & \kappa_{y_2 \to y_5} & 0 & 0 & 0 \\ 0 & 0 & \kappa_{y_3 \to y_5} & 0 & 0 \end{bmatrix}.$$

We also have Y and $\Psi(x)$ as in the Complexes Representation of the reaction network given by

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Psi(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_1 x_4 \end{bmatrix}$$

A quick matrix computation for S, A_κ, $v(x)$ as in the usual Complexes and Stoichiometric Representation shows

$$S = Y\mathcal{G}, \ A_\kappa = \mathcal{G}\mathcal{K}_0, \ v(x) = \mathcal{K}_0\Psi(x).$$

∎

Now that we know the Stoichiometric Representation and the Complexes Representation are the exact same system of differential equations, perhaps there are other properties that are shared between these two decompositions. One of the most important properties is the number of equilibrium points of a differential equation associated to a chemical reaction network. This will be the topic of the next section.

3. Determinant Expansions versus Deficiency

We now turn to various methods used to analyze equilibria of differential equations arising from chemical reaction networks. Indeed, for those who are familiar with chemical network theory, **determinant expansions** (see [HKG08]) are commonly used to analyze equilibria for the Stoichiometric Representation, and **deficiency** (see [G03]) is used to investigate equilibria of the Complexes Representation. Since these two representations exhibit the same system of differential equations as shown in Proposition 2.9, one might expect a relationship between determinant expansions and deficiency since both of these tools are used to determine unique equilibria for the differential equations as shown in the Introduction.

In this section we examine determinant expansions and deficiency with concrete examples to show that they are unrelated when it comes to investigating equilibria for a chemical network.

3.1. **Determinant Expansions versus Deficiency.** In this subsection, we provide examples that suggest determinant expansions and deficiency are generally unrelated when they are used to analyze equilibria of systems of differential equations associated to chemical reaction networks. Before doing so, we provide the necessary background

definitions. Some of these concepts were introduced in §1, but we present them again here for the reader's convenience.

For this section, we will employ the terminology used in Gunawardena's paper [G03, §3]. Let $\{\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathcal{K}\}$ be an arbitrary chemical reaction network as defined in §2. For the Stoichiometric matrix S as defined in §2, the quantity rank(S) is commonly referred to as the **dimension of the stoichiometric subspace**. Now, let $y, y' \in \mathcal{C}$ and say that y is *directly linked* to y', denoted $y \leftrightarrow y'$ if either $y \rightarrow y' \in \mathcal{R}$ or $y' \rightarrow y \in \mathcal{R}$. By definition, if $y \leftrightarrow y'$ then $y' \leftrightarrow y$. Now take the reflexive, transitive closure of $\leftrightarrow$ to obtain an equivalent relation.

Definition 3.1. If $y, y' \in \mathcal{C}$ then y is said to be linked to y', denoted $y \sim y'$, if either $y = y'$ or there are $y_1, \ldots, y_m \in \mathcal{C}$ such that $y = y_1 \leftrightarrow y_2 \leftrightarrow \cdots \leftrightarrow y_m = y'$.

The equivalence classes of complexes under $\sim$ are termed **linkage classes**. For example, the set of reactions $\{A \rightleftharpoons 2B, A + C \rightleftharpoons D, D \rightarrow B + E, B + E \rightarrow A + C\}$ has 2 linkage classes: $\{A, 2B\}$ and $\{A + C, D, B + E\}$. A vital tool used to determine unique equilibria in certain affine subspaces of $\mathbb{R}^d$ (see [G03]) is called the **deficiency** of a reaction network. In the Complexes Representation $\frac{dx}{dt} = Y A_\kappa \Psi(x)$, the deficiency of a chemical network is defined to be

$$\text{deficiency} := \dim(\ker Y \cap \mathrm{Im} A_\kappa).$$

An important result regarding the deficiency is a useful bound given by

$$0 \leq \text{deficiency} \leq n - l - s,$$

where n is the number of complexes, l is the number of linkage classes, and s is the dimension of the stoichiometric subspace [G03]. For convenience, we will refer to the quantity $n - l - s$ just described as the **topological deficiency (top. deficiency)** of the chemical network. *A useful result is that deficiency = top. deficiency whenever top. deficiency equals 0 or 1, which we state here in the following Lemma (see [G03]).*

Lemma 3.2. *Suppose the topological deficiency for a certain chemical network is 1 or 0. Then deficiency = topological deficiency for the network.*

We now present a result that will allow us to construct new chemical reaction networks from old ones, and whose deficiency and top. deficiency are equal.

Proposition 3.3. *Let $\{\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathcal{K}\}$ be a chemical reaction network with Stoichiometric and Complexes Representation given by $\frac{dx}{dt} = Sv(x) = Y A_\kappa \Psi(x)$ as described in previous sections. Suppose that the deficiency of the network is either 1 or 0. Then any chemical reaction network corresponding to the stoichiometric matrix*

$$\tilde{S} = \begin{bmatrix} S & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{0} & S & \ldots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots & S \end{bmatrix}$$

where $\boldsymbol{0}$ is a $|\mathcal{S}| \times |\mathcal{R}|$ matrix of 0's, has a topological deficiency that is equal to its deficiency.

Proof. Suppose $\{\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathcal{K}\}$ is a chemical reaction network with Stoichiometric and Complexes Representation given by $\frac{dx}{dt} = Sv(x) = YA_\kappa \Psi(x)$ as described in previous sections, and assume that the deficiency of the network is 1 or 0. Let $d = |\mathcal{S}|$, $d' = |\mathcal{R}|$, $n = |\mathcal{C}|$, l be the number of linkage classes for the network, and s as the dimension of its stoichiometric subspace. Let t_1 and d_1 by the topological deficiency and deficiency of the chemical network respectively. Fix $k \in \mathbb{N}$, and let $\{\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\mathcal{K}}\}$ be a new chemical network given by Proposition 2.4 whose $dk \times d'k$ Stoichiometric matrix $\tilde{S}$ is

$$\tilde{S} = \begin{bmatrix} S & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & S & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & S \end{bmatrix},$$

where $\mathbf{0}$ is a $d \times d'$ matrix of 0's. Let $\frac{d\tilde{x}}{dt} = \tilde{Y}\tilde{A}_\kappa \tilde{\Psi}(x)$ be the Complexes Representation for the new network. By construction, $\tilde{Y}$ is a $nk \times nk$ matrix given by

$$\tilde{Y} = \begin{bmatrix} Y & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & Y & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & Y \end{bmatrix},$$

where $\mathbf{0}$ is a $n \times n$ matrix of 0's. We also have

$$\tilde{A}_\kappa = \begin{bmatrix} A_\kappa & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & B_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & B_{k-1} \end{bmatrix},$$

where $\mathbf{0}$ is a $n \times n$ matrix of 0's and B_i is a $n \times n$ matrix of rate constants, and has the same sign pattern as A_κ for each $i = 1, \ldots, k-1$. Let t_2 and d_2 be the topological deficiency and deficiency for this new network. By construction, the new chemical network has kn complexes, kl linkage classes, and ks as the dimension of its stoichiometric subspace $\Rightarrow t_2 = kn - kl - ks = k(n - l - 1) = kt_1$. Notice, basic linear algebra on direct-sum decompositions tells us

$$(3.1) \qquad \dim(\ker\tilde{Y} \cap \text{Im}\tilde{A}_\kappa) = \dim(\ker Y \cap \text{Im}A_\kappa) + \sum_{i=1}^{k-1} \dim(\ker Y \cap \text{Im}B_i).$$

If we consider a chemical network with Complexes Representation $\frac{dx}{dt} = YB_i\Psi_i(x)$ for each $i = 1, \ldots, k-1$, then its Stoichiometric matrix is S, and its topological deficiency is $n - l - s = t_1$ by construction $\Rightarrow \dim(\ker Y \cap \text{Im}B_i) = t_1$ by Lemma 3.2. Thus, $d_2 = \dim(\ker\tilde{Y} \cap \text{Im}\tilde{A}_\kappa) = kt_1 = t_2$ using equation (4.1), and this is the desired result.

$\blacksquare$

We now provide a brief introduction to determinant expansions, which are relevant to studying the Stoichiometric Representation.

Let $A = (a_{ij})$ be an $n \times n$ matrix, and S_n be the permutation group with n elements. Recall that the **determinant** of A is given by,

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

where $sgn : S_n \to \{-1, 0, 1\}$ is the usual sign function for a permutation. In the case of RF differential equations, the determinant of matrix SU described in the Introduction is usually 0 since SU is a sparse matrix. However, there are useful variants of the determinant, see [CF05], and [HKG08], which do give meaningful results. We present them here.

Definition 3.4. The Craciun-Feinberg determinant expansion [CF05] is defined to be $\text{cfd}(S) := \det(SU - tI)$ with t fixed, e.g. $t = 1$.

Definition 3.5. Another important computation that is used to analyze equilibria for a chemical network is the **core determinant** (see [HKG08, §3.2]). For the matrices $S \in \mathbb{R}^{d \times d'}$, $U \in \mathbb{R}^{d' \times d}$ defined in §1, with $r := \text{rank}(S)$, the core determinant is defined to be

$$(3.2) \qquad\qquad cd(S) := \lim_{t \to 0} \frac{1}{t^{d-r}} \det(SU - tI)$$

Note that since S is a constant matrix, the expression $\text{cfd}(S)$ and $\text{cd}(S)$ is a polynomial in the functions $U_{ij}(x)$ which are the entries of the matrix function $U(x) = v'(x)$ as described in Introduction 1.1. Thus, we will say that $\text{cfd}(S)$ and $\text{cd}(S)$ has a **Determinant Sign(DetSign)** iff all the terms in the respective determinant expansions have the same sign. Results regarding the connection between $\text{cfd}(S)$ or $\text{cd}(S)$ having a DetSign and the type of equilibria for the chemical network can be found in [CF05] and [HKG08].

The two situations of $\text{cd}(S)$, $\text{cfd}(S)$ having a DetSign and the network having deficiency 0 have important consequences regarding the equilibria for the reaction network as shown in the Introduction. Nevertheless, the next few examples will show that $\text{cd}(S)$, $\text{cfd}(S)$ having a DetSign have no relation to the deficiency of a chemical network.

Example 3.6. *The next two examples show that cfd(S) exhibiting a DetSign has no relation to a network having deficiency 0. First, we give a network where cfd(S) has a DetSign and the deficiency of the network is 1. Then, we provide an example where cfd(S) has a sign, but the network has an arbitrarily large deficiency.*

Consider the chemical network,

$$A \to B, \ 2C \to D, \ D \to E, \ C \to E$$

$$A + B \to C, \ 3A \to 4B$$
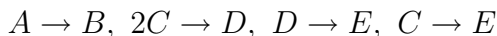
The stoichiometric matrix S and the matrix $U = v'(c)$ for this chemical network is given by,

$$S = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -3 \\ 1 & -1 & 0 & 0 & 0 & 4 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} k_{A->B} & 0 & 0 & 0 & 0 \\ c_B k_{A+B->C} & c_A k_{A+B->C} & 0 & 0 & 0 \\ 0 & 0 & 2c_C k_{2C->D} & 0 & 0 \\ 0 & 0 & 0 & k_{D->E} & 0 \\ 0 & 0 & k_{C->E} & 0 & 0 \\ 3c_A^2 k_{3A->4B} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the determinant of $SU - I$ with I being the 6×6 identity matrix is given by,

$$cfd(S) = Det(SU - I) = -1 - k_{A->B} - 9c_A^2 k_{3A->4B} - c_A k_{A+B->C} - c_B k_{A+B->C}$$

$$-2c_A k_{A->B} k_{A+B->C} - 21c_A^3 k_{3A->4B} k_{A+B->C} -$$

$$k_{C->E} - k_{A->B} k_{C->E} - 9c_A^2 k_{3A->4B} k_{C->E}$$

$$-c_A k_{A+B->C} k_{C->E} - c_B k_{A+B->C} k_{C->E} -$$

$$2c_A k_{A->B} k_{A+B->C} k_{C->E} - 21c_A^3 k_{3A->4B} k_{A+B->C} k_{C->E}$$

$$-4c_C k_{2C->D} - 4c_C k_{A->B} k_{2C->D} -$$

$$36c_A^2 c_C k_{3A->4B} k_{2C->D} - 4c_A c_C k_{A+B->C} k_{2C->D}$$

$$-4c_B c_C k_{A+B->C} k_{2C->D} - 8c_A c_C k_{A->B} k_{A+B->C} k_{2C->D} -$$

$$84c_A^3 c_C k_{3A->4B} k_{A+B->C} k_{2C->D} - k_{D->E} -$$

$$k_{A->B} k_{D->E} - 9c_A^2 k_{3A->4B} k_{D->E} - c_A k_{A+B->C} k_{D->E} -$$

$$c_B k_{A+B->C} k_{D->E} - 2c_A k_{A->B} k_{A+B->C} k_{D->E}$$

$$-21c_A^3 k_{3A->4B} k_{A+B->C} k_{D->E} - k_{C->E} k_{D->E} -$$

$$k_{A->B} k_{C->E} k_{D->E} - 9c_A^2 k_{3A->4B} k_{C->E} k_{D->E}$$

$$-c_A k_{A+B->C} k_{C->E} k_{D->E} - c_B k_{A+B->C} k_{C->E} k_{D->E} -$$

$$2c_A k_{A->B} k_{A+B->C} k_{C->E} k_{D->E}$$

$$-21c_A^3 k_{3A->4B} k_{A+B->C} k_{C->E} k_{D->E} - 4c_C k_{2C->D} k_{D->E} -$$

$$4ck_{A->B} k_{2C->D} k_{D->E} - 36c_A^2 ck_{3A->4B} k_{2C->D} k_{D->E}$$

$$-4c_A c_C k_{A+B->C} k_{2C->D} k_{D->E} -$$

$$4c_B c_C k_{A+B->C} k_{2C->D} k_{D->E} - 8c_A ck_{A->B} k_{A+B->C} k_{2C->D} k_{D->E}$$

$$-84c_A^3 ck_{3A->4B} k_{A+B->C} k_{2C->D} k_{D->E}$$

A quick observation reveals that each term in the above computation has a negative coefficient, and this means cfd(S) has a DetSign. However, this system has 9 complexes, 3 linkage classes, and the dimension of its stoichiometric subspace is 5. Thus, the deficiency of the network is $9 - 5 - 1 = 1$ by Proposition 3.3. This example may be extended to show that we may have chemical networks with an arbitrary deficiency, and still have a DetSign. ∎

Example 3.7. *Here, we show how cfd(S) may have a DetSign, but the deficiency of the network is arbitrarily large.*

So fix $k \in \mathbb{N}$. Let S be the Stoichiometric matrix in the previous examples, and let U be the corresponding 6×5 matrix written symbolically as,

$$U = \begin{bmatrix} U_{11} & 0 & 0 & 0 & 0 \\ U_{21} & U_{22} & 0 & 0 & 0 \\ 0 & 0 & U_{33} & 0 & 0 \\ 0 & 0 & 0 & U_{44} & 0 \\ 0 & 0 & U_{53} & 0 & 0 \\ U_{61} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now consider the chemical network corresponding to the $5k \times 6k$ Stoichiometric matrix $\tilde{S}$ given by,

$$\tilde{S} = \begin{bmatrix} S & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & S & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & S \end{bmatrix}$$

where $\mathbf{0}$ is a 5×6 matrix consisting of all 0's. If we consider the differential equations $\frac{dx}{dt} = \tilde{S}v(x)$, then the Jacobian $\tilde{U} = v'(x)$ will be a $6k \times 5k$ matrix written symbolically as,

$$\tilde{U} = \begin{bmatrix} U_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & U_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & U_k \end{bmatrix}$$

where $\mathbf{0}$ is a 6×5 matrix consisting of all 0's, $U_1 = U$, and U_i is a 6×5 matrix with the same sign pattern as U. Matrix multiplication gives $\tilde{S}\tilde{U}$ as a $5k \times 5k$ matrix represented in block diagonal form as,

$$\tilde{S}\tilde{U} = \begin{bmatrix} SU_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & SU_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & SU_k \end{bmatrix}$$

where $\mathbf{0}$ is a 5×5 matrix consisting of all 0's. Elementary properties of determinants gives us the direct computation,

$$cfd(\tilde{S}) = det(\tilde{S}\tilde{U} - I_{5k}) = \left| \begin{bmatrix} SU_1 - I_5 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & SU_2 - I_5 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & SU_k - I_5 \end{bmatrix} \right| =$$

$$= \prod_{i=1}^{k} det(SU_i - I_5)$$

Since $det(SU - I)$ has a DetSign as shown in the previous example, then $det(\tilde{S}\tilde{U})$ also has a DetSign by the above computation since the DetSign only depends on the sign pattern of U and not the actual entries of U. However, our construction shows that

this system has $9k$ complexes, $3k$ linkage classes, and the dimension of its stoichiometric subspace is $5k$. Hence the deficiency of this network is $9k - 3k - 5k = k$ by Proposition 3.3. ■

We now turn to examples involving the core determinant.

Example 3.8. In the next two examples, we will show that the core determinant having a DetSign has no relation to the deficiency of a chemical network. That is, we will find chemical networks whose core determinant has a DetSign, and the deficiency of the network is 1. Then we construct a network such that cd(S) has a DetSign, but the deficiency of the network is arbitrarily large.

Consider the chemical network,

$$A + F \to B, A + B \to C, C + F \to D, D \to E + F$$

$$C \to E$$
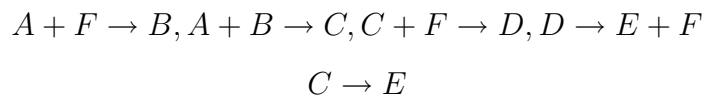
We denote $c = \{c_A, c_B, c_C, c_D, c_E, c_F\}$ as the vector of chemical concentrations. The stoichiometric matrix S and the matrix $U = v'(c)$ for this chemical network is given by,

(3.3)
$$S = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 1 & 0 \end{bmatrix}$$

(3.4)
$$U = \begin{bmatrix} c_F k_{A+F->B} & 0 & 0 & 0 & 0 & c_A k_{A+F->B} \\ c_B k_{A+B->C} & c_A k_{A+B->C} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2c_C c_F k_{D->E+F} & 0 & 0 & c_C^2 k_{D->E+F} \\ 0 & 0 & 0 & k_{D->E+F} & 0 & 0 \\ 0 & 0 & k_{C->E} & 0 & 0 & 0 \end{bmatrix}.$$

A computation in Mathematica shows $\mathrm{rank}(S) = 4$, and the core determinant for this network is given by,

$$cd(S) := \lim_{t \to 0} \frac{1}{t^{d-r}} \det(SU - tI) = \lim_{t \to 0} \frac{1}{t^{6-4}} \det(SU - tI) =$$

$$2c_A^2 c_F k_{A+F->B} k_{A+B->C} k_{D->E+F}^2 + 2c_A c_B c_C c_F k_{A+F->B} k_{A+B->C} k_{D->E+F}^2$$

$$+4c_A c_C c_F^2 k_{A+F->B} k_{A+B->C} k_{D->E+F}^2 + c_A^2 k_{A+F->B} k_{A+B->C} k_{D->E+F} k_{C \to E}$$

$$+c_A c_B k_{A+F->B} k_{A+B->C} k_{D->E+F} k_{C \to E} + 2c_A c_F k_{A+F->B} k_{A+B->C} k_{D->E+F} k_{C \to E}$$

$$+2c_A c_C^2 k_{A+F->B} k_{A+B->C} k_{D->E+F} k_{C \to E}$$

The coefficients of all the terms in $cd(S)$ are positive and thus, the core determinant for this network has a DetSign. One may even compute the Craniun-Feinberg determinant to see that it too has a DetSign. Nevertheless, this network has 8 complexes (A+F,B,A+B,C,C+F,D,E+F,E), 3 linkage classes, and the dimension of the stoichiometric subspace is 4. Thus, the deficiency for the network is 8-2-5 $= 1 \neq 0$ by Proposition 3.3. ■

We may now extend this example to find a chemical network with an arbitrary deficiency size, but whose core determinant still has a DetSign.

Example 3.9. *Here, we find a chemical network such that cd(S) has a DetSign, but the deficiency of the network is arbitrarily large.*

Fix $k \in \mathbb{N}$. Let S be the 6×5 matrix in (2.12), and U be the matrix in (2.13) written symbolically as,

$$U = \begin{bmatrix} U_{11} & 0 & 0 & 0 & 0 & U_{15} \\ U_{21} & U_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & U_{33} & 0 & 0 & U_{35} \\ 0 & 0 & 0 & U_{44} & 0 & 0 \\ 0 & 0 & U_{53} & 0 & 0 & 0 \end{bmatrix}.$$

By Proposition 2.4, there is a chemical network corresponding to the $6k \times 5k$ stoichiometric matrix $\tilde{S}$ given by,

$$\tilde{S} = \begin{bmatrix} S & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & S & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & S \end{bmatrix}$$

where $\mathbf{0}$ is a 6×5 matrix consisting of all 0's. If we consider the differential equations $\frac{dx}{dt} = \tilde{S}v(x)$, then the Jacobian $\tilde{U} = v'(x)$ will be a $5k \times 6k$ matrix written symbolically as,

$$\tilde{U} = \begin{bmatrix} U_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & U_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & U_k \end{bmatrix}$$

where $\mathbf{0}$ is a 5×6 matrix consisting of all 0's, $U_1 = U$, and U_i is a 5×6 matrix with the same sign pattern as U for each $i = 2, \ldots, k-1$. Since $r := rank(S) = 4$, and the rank of a block diagonal matrix is the sum of the ranks of the diagonal blocks, then $rank(\tilde{S}) = 4k$. Thus, using elementary properties of determinants and matrix multiplication gives us $cd(\tilde{S})$ as,

$$cd(\tilde{S}) = \lim_{t \to 0} \frac{1}{t^{6k-4k}} \det(\tilde{S}\tilde{U} - tI_{6k})$$

$$= \lim_{t \to 0} \frac{1}{t^{6k-4k}} \left| \begin{bmatrix} SU_1 - tI_6 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & SU_2 - tI_6 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & SU_k - tI_6 \end{bmatrix} \right|$$

$$= \lim_{t \to 0} \frac{1}{t^{2k}} \prod_{i=1}^{k} \det(SU_i - tI_5)$$

$$= \prod_{i=1}^{k} (\lim_{t \to 0} \frac{1}{t^2} \det(SU_i - tI_5)) = (cd(S))^k,$$

where the last equality follows if we consider $cd(S)$ as depending only on the sign pattern of U and not the particular entries of U. Hence, $cd(\tilde{S}) = (cd(S))^k$ has a sign since $cd(S)$

has a sign using the computation in the previous example. Nevertheless, this system has $8k$ complexes, $3k$ linkage classes, and the dimension of its stoichiometric subspace is $4k$ $\Rightarrow$ the deficiency of the network is $8k - 3k - 4k = k$ by Proposition 3.3. ∎

These examples suggest that even though the deficiency 0 method and certain properties of Determinant Signs yield similar conclusions regarding the uniqueness of equilibria of a chemical reaction, deficiency and Determinant Signs have no relation to each other.

Besides comparing determinant expansions and deficiency, one may wonder if the sign pattern of SU has any bearing on the deficiency of a chemical network. In the next section, we extend some matrix theoretic results regarding the sign pattern of SU, and then show that sign patterns and deficiency have no relation.

4. Sign Pattern of $f'(x) = SU$

In this section, we determine precisely when an entry of SU has a sign as described in the introduction. We then show how in certain situations when SU does not have a sign pattern, we may study an associated matrix that does have a sign pattern, and use sign definite matrix analysis on this new matrix to gain information on the determinant of SU. Finally, we provide examples to suggest that SU having a sign pattern has no relation to the deficiency of a chemical network.

4.1. **Conditions for SU having a Sign Pattern.** Here, we give concrete conditions to determine when an entry of SU has a sign.

First, in order to analyze the sign pattern of SU, we need some necessary conditions. These conditions are succinctly encompassed in [HK08, §3.1] and we state them here for completeness.

prop:JacSigns **Proposition 4.1.** *Suppose S satisfies RF.*

(1) *All the diagonal entries of SU are negative linear combinations of the U_{ij}.*

(2) *SU admits a sign pattern (that is, each of its entries is a positive or negative linear combination of monomials in U_{ij}) iff the matrix S does not contain a 2×2 submatrix with the same sign pattern as*

eq:2cycle (4.1)
$$\begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix} \quad or \quad \begin{bmatrix} -1 & +1 \\ -1 & -1 \end{bmatrix} \quad or \quad \begin{bmatrix} -1 & -1 \\ +1 & -1 \end{bmatrix} \quad or \quad \begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix}.$$

Proof. Diagonal entries of SU are of the form $\sum_j S_{ij} U_{ji}$. Since $U_{ji} = 0$ if $S_{ij} > 0$, $(SU)_{ii}$ is a negative linear combination of certain U_{ji}.

Write $S = S_+ - S_-$ for real matrices S_+, S_- with nonnegative coefficients. The (i,j)th entry of SU does not have a sign pattern iff $(S_+U)_{ij} \neq 0$ and $(S_-U)_{ij} \neq 0$. $(S_+U)_{ij} = \sum_k (S_+)_{ik} U_{kj}$, so $(S_+U)_{ij} \neq 0$ iff for some k, $(S_+)_{ik} \neq 0$ and $U_{kj} \neq 0$, i.e., $(S_+)_{ik} \neq 0$ and $S_{jk} < 0$. Similarly, $(S_-U)_{ij} \neq 0$ iff there is some ℓ with $(S_-)_{i\ell} \neq 0$ and $S_{j\ell} < 0$. Taken together this means that the 2×2 submatrix of S given by rows i, j and columns k, ℓ has the same sign pattern as one of the matrices in (4.1). ∎

In fact, the proof of this proposition gives us an even more precise result.

Corollary 4.2. *Suppose S is a $d \times d'$ matrix that satisfies RF. Let $i < j$ with $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, d'\}$. Then*

(1) $(SU)_{ij}$ *has a sign iff S does not contain a 2×2 submatrix in rows i and j and some columns u and v with the same sign pattern as*

$$\begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix} \quad or \quad \begin{bmatrix} -1 & +1 \\ -1 & -1 \end{bmatrix}$$

(2) $(SU)_{ji}$ *has a sign iff S does not contain a 2×2 submatrix in rows i and j and some columns u and v with the same sign pattern as*

$$\begin{bmatrix} -1 & -1 \\ +1 & -1 \end{bmatrix} \quad or \quad \begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix}$$

Proof. (1) "$\Rightarrow$" First, assume we are in the case where S has a 2×2 submatrix in rows i and j, $i < j$, and some columns u, v with the same sign pattern as

$$\begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix}$$

As in the proof of Proposition 4.1, write $S = S_+ - S_-$ for real matrices S_+, S_- with non-negative coefficients. Then $(SU)_{ij} = (S_+U)_{ij} - (S_-U)_{ij} = \sum_k (S_+)_{ik} U_{kj} - \sum_k (S_-)_{ik} U_{kj}$. By assumption, $S_{ju} < 0 \Rightarrow U_{uj} \neq 0$. Also, $S_{iu} > 0 \Rightarrow (S_+)_{iu} = S_{iu} > 0 \Rightarrow (S_+)_{iu} U_{uj} \neq 0 \Rightarrow \sum_k (S_+)_{ik} U_{kj} \neq 0$. Similarly, $S_{jv} < 0$ by assumption $\Rightarrow U_{vj} \neq 0$. Also, we have $S_{iv} < 0 \Rightarrow (S_-)_{iv} = S_{iv} < 0 \Rightarrow (S_-)_{iv} U_{vj} \neq 0 \Rightarrow \sum_k (S_-)_{ik} U_{kj} \neq 0 \Rightarrow (SU)_{ij}$ is not a positive or negative linear combination of the entries of U, and hence, it does not have a sign by definition. The case where S has a 2×2 submatrix with the same sign pattern as

$$\begin{bmatrix} -1 & +1 \\ -1 & -1 \end{bmatrix}$$

is proved analogously.

"$\Leftarrow$" Now suppose that $(SU)_{ij}$ for some does not have sign. More precisely, this means $(S_+U)_{ij} \neq 0$ and $(S_-U)_{ij} \neq 0 \Rightarrow 0 \neq (S_+U)_{ij} = \sum_k (S_+)_{ik} U_{kj} \Rightarrow (S_+)_{iu} \neq 0$ and $U_{uj} \neq 0$ for some $u \Rightarrow S_{iu} > 0$ and $S_{ju} < 0$ by the RF property of S. Similarly, $(S_-U)_{ij} \neq 0 \Rightarrow (S_-)_{iv} \neq 0$ and $U_{vj} \neq 0$ for some $v \Rightarrow S_{iv} < 0$ and $S_{jv} < 0$ since S is RF. Also $u \neq v$ by construction of S_+ and S_-. Thus, we have, $S_{ju}, S_{iv}, S_{jv} < 0$ and $S_{iu} > 0$. Thus S has a 2×2 submatrix in rows i, j and columns u, v with the same sign pattern as

$$\begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix} \quad or \quad \begin{bmatrix} -1 & +1 \\ -1 & -1 \end{bmatrix}$$

depending on whether $u < v$ or $u > v$ respectively.

This establishes (1). An analogous proof establishes (2) by replacing each i with j and each j with i in the above proof. ∎

Corollary 4.2 provides a way for studying the sign pattern of the more complicated matrix valued function SU by looking at the simpler constant matrix S. Such concrete

and simple conditions will allow us to look at the sign pattern of SU even if this matrix does not have a sign pattern, since the sign of each of its entries is determined by the constant matrix S. This will be the topic of the next section.

4.2. **Creating Sign Patterns for** $SU = f'(x)$**.** As shown in the previous section, SU will not be a sign definite matrix under certain conditions. In this section, we show an alternative and meaningful way to associate a sign patterned matrix to SU even if SU does not have a sign pattern. We first motivate this with a discussion on determinants.

Of most interest in the analysis of RF systems is looking at the determinant of SU. Indeed, the determinant is well-defined in this case since S is a $d \times d'$ matrix and U is a $d' \times d$ matrix. Also, the direct connection between sign patterns of matrices and their determinant can be found in [HK08, § Theorem 2.9], and we do not present their results here. For our purposes, if SU does have a sign pattern, it is often easier to analyze the determinant of SU based on its sign pattern rather than direct computation. Unfortunately, SU will sometimes not have a sign pattern according to Proposition 4.1, but this can be remedied by noticing that if we left multiply SU by an invertible matrix E, then $\det(ESU) = \det(E)\det(SU)$. Often, we may find such a matrix E, so that ESU has a sign pattern, and thus, the analysis for a sign pattern of ESU will suffice to understand the determinant of SU since the determinants of SU and ESU will merely differ by the scalar $\det(E)$. This idea is our motivation for the following definitions.

Definition 4.3. Let us say that SU **"almost" has a sign pattern** if there is an invertible matrix E such that ESU has a sign pattern.

For our next result, it will make it easier to consider matrices of the forms:

$$(1)\ i.) \begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix} \quad ii.) \begin{bmatrix} -1 & +1 \\ -1 & -1 \end{bmatrix} \quad iii.) \begin{bmatrix} -1 & -1 \\ +1 & -1 \end{bmatrix} \quad iv.) \begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix}$$

and

$$(2) \begin{bmatrix} * & -1 \\ +1 & * \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} * & +1 \\ -1 & * \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & * \\ * & +1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} +1 & * \\ * & -1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} * & -1 \\ -1 & -1 \\ +1 & * \end{bmatrix}, \begin{bmatrix} * & +1 \\ -1 & -1 \\ -1 & * \end{bmatrix}, \begin{bmatrix} -1 & * \\ -1 & -1 \\ * & +1 \end{bmatrix}, \begin{bmatrix} +1 & * \\ -1 & -1 \\ * & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ +1 & * \\ * & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & * \\ * & +1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ * & +1 \\ -1 & * \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ * & -1 \\ +1 & * \end{bmatrix}$$

Notice that the matrices in (2) are merely all possible row and column permutations of the matrix:

$$\begin{bmatrix} * & -1 \\ +1 & * \\ -1 & -1 \end{bmatrix}$$

Definition 4.4. We will refer to submatrices of S that have the same sign pattern as a matrix in (1) as **bad-submatrices**. This definition is motivated by Proposition 4.1, which tells us that SU will not have a sign pattern if S contains a submatrix with the same sign pattern as a matrix in (1).

Theorem 4.5. *Suppose S has exactly one 2×2 bad-submatrix, and S does not have a 3×2 submatrix with the same sign pattern as a matrix in (2). Then $SU = f'(x)$ almost has a sign pattern.*

Proof. [?] Without loss of generality, let us assume that rows 1 and 2, and columns 1 and 2 have the same sign pattern as (1) i.). (otherwise, we could permute the columns of matrix S and U, without affecting determinants or sign patterns). Now let $a = S_{11}$, $b = -S_{12}$, $c = -S_{21}$, and $d = -S_{22}$. First, we claim that $(SU)_{12} = aU_{12} - bU_{22} - \sum_{r=3}^{n} S_{1r}U_{r2}$, where $S_{1r} \geq 0$ whenever $U_{r2} \neq 0$ for $r = 3, 4, \ldots, n$. If not then $S_{1r} < 0$ and $U_{r2} > 0$ for some $r \in \{3, 4, \ldots, n\} \Rightarrow S_{2r} < 0$ and this means S a bad-submatrix corresponding to rows 1 and 2 and columns 1 and r, which contradicts our assumptions. Thus, $(SU)_{12} = aU_{12}+$ (negative linear combination of entries of U). By Corollary 4.2, $(SU)_{12}$ is the only such entry of SU without a sign $\Rightarrow SU_{22} = -cU_{12}+$ (negative linear combination of entries of U), since c and $d > 0$. So now, add a times row 2 of SU to row 1 to form a new matrix, $(SU)'$. Then $(SU)'_{12}$ is a negative linear combination of entries of U, and hence has a sign. We must make sure (SU)' has a sign pattern. If not, then by Corollary 4.2 (which tells us that all other entries of SU are either positive or negative linear combinations of entries of U), it must be that for some $t \neq 2$ we have $(SU)_{1t} < 0$ and $(SU)_{2t} > 0$ or vice versa. Without loss of generality, we will assume it's the former. This implies that since $(SU)_{ij} = \sum_{k=1}^{n} S_{ik}U_{kj}$, then applying this formula with 1 and t and then with 2 and t, we get that $S_{1r} < 0$ and $U_{rt} > 0$ for some r, and $S_{2s} > 0$ and $U_{st} > 0$ for some $s \neq r \Rightarrow S_{tr} < 0$ and $S_{ts} < 0$. This means S has a 3×2 submatrix, corresponding to rows 1, 2, and t, and columns r and s, with the same sign pattern as a matrix in (2), and this is a contradiction. Thus this row operation really does give a sign pattern ∎

It is indeed worth noting that we needed some very stringent assumptions on S in order to get a sign pattern after a simple row operation. Even more so, this is in the case where S has only 1 bad-submatrix. The cases where S has more bad-submatrices becomes increasingly difficult. Nevertheless, we have yet to find a counterexample where SU did not have a sign pattern after elementary row operations, and we believe that all Jacobians SU for RF systems do have a sign pattern after some elementary row operations. We leave it to the interested reader to verify if this is the case.

We now turn to comparing sign patterns and deficiency of a chemical network.

4.3. **Sign Patterns versus Deficiency.** As previously mentioned, a chemical network having 0 deficiency has significance on the type of equilibria for a chemical network, see [G03]. On the other hand, the sign pattern of SU described in the §1.3 also correlates to whether the system has equilibria in the positive orthant (see [HKG08]). A natural question is whether the sign pattern of SU has any correlation to a system having 0 deficiency and a certain number of linkage classes. The answer is no, and in this section

we give examples of chemical networks such SU that 0 deficiency does not imply a sign pattern for SU.

Example 4.6. *The next few examples illustrates how SU having a sign pattern does not correlate to a chemical network having deficiency 0. Thus, we will show that SU having a sign pattern does not imply 0 deficiency for a network.*

Consider the reaction network,

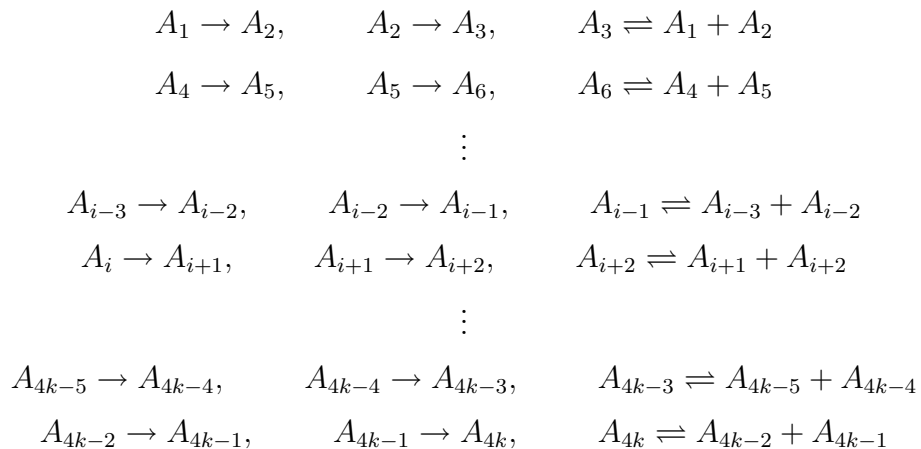$$A \to B$$
$$B \to C$$
$$C \rightleftharpoons A + B$$
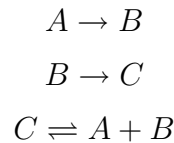
In this case we have 4 complexes, 1 linkage class, and the dimension of the stoichiometric subspace is 3. Thus, the deficiency of the system is 0 by Proposition 3.3. Nevertheless, the reactions $A \to B$ and $A + B \to C$ imply that SU will not have a sign pattern by Proposition 1.4 since in this case, we have

$$S = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

■

Example 4.7. *We may extend the construction in the previous example to find reaction networks that have $4k$ complexes, k linkage classes, $3k$ as the dimension of the stoichiometric subspace, and still do not have a sign pattern, for any $k \in \mathbb{N}$*

Fix $k \in \mathbb{N}$. Consider,

$$A_1 \to A_2, \qquad A_2 \to A_3, \qquad A_3 \rightleftharpoons A_1 + A_2$$
$$A_4 \to A_5, \qquad A_5 \to A_6, \qquad A_6 \rightleftharpoons A_4 + A_5$$
$$\vdots$$
$$A_{i-3} \to A_{i-2}, \qquad A_{i-2} \to A_{i-1}, \qquad A_{i-1} \rightleftharpoons A_{i-3} + A_{i-2}$$
$$A_i \to A_{i+1}, \qquad A_{i+1} \to A_{i+2}, \qquad A_{i+2} \rightleftharpoons A_{i+1} + A_{i+2}$$
$$\vdots$$
$$A_{4k-5} \to A_{4k-4}, \qquad A_{4k-4} \to A_{4k-3}, \qquad A_{4k-3} \rightleftharpoons A_{4k-5} + A_{4k-4}$$
$$A_{4k-2} \to A_{4k-1}, \qquad A_{4k-1} \to A_{4k}, \qquad A_{4k} \rightleftharpoons A_{4k-2} + A_{4k-1}$$

By construction, each line above represents a distinct linkage class and 3 distinct complexes $\Rightarrow$ we have k linkage classes and $4k$ complexes. We also claim that the dimension of the stoichiometric subspace is $4k$. To see this, let

(4.2) $$B = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

for $j = 1, \ldots, k$.

By construction, the Stoichiometric matrix S will be a $3k \times 4k$ block matrix,

$$S = \begin{bmatrix} B & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & B & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & B \end{bmatrix}$$

Where B is the matrix appearing in (2.5) and $\mathbf{0}$ is a 3×4 matrix consisting of all 0's. Notice that $\text{rank}(B) = 3 \Rightarrow \text{rank}(S) = 3k$. Thus, the dimension of the stoichiometric subspace is $3k \Rightarrow$ the deficiency of the network is 0. However, Proposition 4.1 implies that SU will not have a sign pattern. In fact, Corollary 4.2 implies that SU will have exactly k entries without a sign. $\blacksquare$

Remark 4.8. The examples we provide in §3 and §4 suggest that sign patterns and Determinant Signs have no relation to the deficiency of a network. There may still be other properties of deficiency and determinant expansions that do correlate, but in most chemical literature, the deficiency 0 condition and Determinant Signs have yielded the most fruitful results in studying equilibria of a chemical network.

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