Planting the (Intellectual) Need From Which Learning Grows

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1. Introduction

Until recently, universities were elitist centers of knowledge that sought to impart knowledge and develop critical thinking. Today, it is recognized that such elitism no longer exists. Many more students are attending college and universities today (as a percentage) than one hundred years ago.[1]

In parallel with the increases in post-secondary enrollment has been a shift in the motivations for attending such schools. Universities had the image of being institutions that impart knowledge for the sake of knowledge. But times have changed.

Recently, Chancellor Reed of the California State University system said that attending college is no longer a luxury in our society. The financial benefits of attending college so outweigh those of not attending that one needs to go to college.[2]

Consequently, students attend universities today for a variety of reasons. Quite a few still attend for the sake of learning, but more and more are attending for the financial and social benefits of a higher education.

Students who attend college have three main reasons (needs) for being there: intellectual, economic, and social needs. It is possible that any student may have more than one such need. Also, different subjects may invoke different types of need.

For example, it is possible to have a student majoring in Mathematics and minoring in Economics because they find math to be naturally interesting and economics classes are more marketable than math courses alone.

General education courses serve to impart knowledge in a variety of areas. Some times, students find these subjects interesting, while other times they are seen as pointless. Personal experiences have shown that often students do not find themselves intellectually engaged with the material in Mathematics.[3]

Many students are required to take a year of calculus as part of some GE requirement. Those pursuing science and engineering degrees are more likely to see some utility for taking the course, while other students fail to understand why they are 'forced' to take a year of calculus. "When I am ever going to need to know how to use Lagrange Multipliers in the real world?" one student asked me.

I responded by explaining how UPS uses the dimensions of a box to charge for shipping costs. By knowing how to maximize the volume of the object you are shipping relative to the dimensions of the container, one can save a few dollars.

The student seemed intrigued by my answer, but it was clear to me that this student was not going to find an intellectual need for Mathematics, at least not for Lagrange Multipliers. At best, I managed to establish an economic need (saving a few dollars).

Students fail to realize that they are asked to take calculus not because they are going to use it later in their life, but rather because the course teaches them critical thinking skills, something that most students should know by the time that they graduate.

Furthermore, the goal of calculus, is to instill in students an intellectual need to solve problems that they are presented. Logical thinking provides students the framework with how to go about finding answers.

1.1 Three Types of Students

I claim that there are three types of students enrolled in a given calculus course. The first group is what I call attending and intellectually engaged. The second group is attending but not intellectually engaged. The last group is non-attending and not intellectually engaged.

By attending, I am referring to those students who attend lecture on a regular basis. By intellectually engaged, I mean that the student has an intellectual need to learn the material. Most often, this translates into a natural curiosity as to why a statement is true. The desire to know *why* is at the heart of intellectual need because by assessing the validity of a statement, one discovers new knowledge that is meaningful to the person who constructed it.[4]

It should be noted that one can be motivated without being intellectually engaged. As was stated earlier, many students now attend post-secondary education for social and economic reasons, not just for the sake of learning. Consequently, these students may be motivated to obtain a degree, but not necessarily for the sake of knowledge.

Harel discusses the implications of intellectual need and student's learning though his Necessity Principle. The Necessity Principle states the following:

> Students are most likely to learn when they see a need for what we intend to teach them, where by "*need*" is meant intellectual need, as opposed to social or economic need.[4]

The first type of student described above is already intellectually engaged. Such students are active participants, asking questions about the material in lecture and during office hours.

The second type of student attends lecture on a regular basis but has not been intellectually engaged by the material studied in math classes. These students pay attention to the material presented, but focus more on memorizing formulas and definitions. As will be discussed in this paper, these students do not build effective concept images and do not question the validity of statements presented in class.

The third type of student does not attend lecture regularly. These students most likely are not motivated. They fail to see any utility for mathematics, let alone possess an intellectual need to know why statements are true.

Consequently, focus should be placed on fostering the intellectual need for the second group of students. They are, for the most part, motivated to learn. Often times, the students see some usefulness in learning mathematics, but focus more on social or economic needs, rather than intellectual need.

So, the question remains: how does one foster intellectual need in students?

2. Fostering Intellectual Need

Different classes will necessitate different approaches to developing intellectual need in students. Personal experiences have shown that in upper division courses, for example, students will naturally be more curious about the material than first year calculus students. Even among first year calculus students, variations appear.

Many universities offer two calculus sequences: one aimed at science and engineering majors and the second targeted at non-science and engineering majors. More emphasis is placed on understanding (and proving) the concepts in the first sequence, while the latter examines the material less rigorously. As a result, a natural division occurs between the three types of students discussed above.

I argue that typically the first and second type of students populate the science and engineering calculus sequence. The second and the third type of students will more likely enroll in the less rigorous sequence. (Variations may be attributed to GE or major requirements.) Consequently, different teaching strategies should be used for the two sequences. Reliance on a single strategy for both classes would shortchange the intellectual needs of some students.

Harel argues that many factors contribute to a student's intellectual need. The goal of this paper is not to focus on all of them. Rather, I shall focus on three factors. Each of them is supported by other research conducted by Harel. Some may seem like common sense while others may seem counterintuitive, at least initially.

2.1 Teachers Must Understand Students' Needs

Teachers need to understand how they, as well as how their students, approach a problem and tailor their lectures accordingly. Once this is done, teachers can help students build on their intuition in modifying and refining their conceptions.[5]

For example, as was discussed above, a university may offer two calculus sequences. The students in each sequence will approach problems differently. In fact, every time a class is offered, the students will approach problems differently. Thus, the goal of the instructor is to better connect with the students by altering their lectures accordingly.

Personal Experience:

    As a tutor, I have found that no two students learn exactly the same way. Consequently, when a student asks me a question, I ask them some questions first, in order to discern their level of understanding and I target my responses according.

    For example, I had two students ask me to explain why the polynomial $f(x) = x^3 + 4x - 7$ has only one real solution. The first student was content with a graph of the function. Despite my insistence that we needed to say more and that the graph was merely an aid to help convince us of the validity of the statement, the student wanted to move on to another problem.

    The second student however, wanted to fully prove the statement. We proceeded to use the Intermediate Value Theorem (after verifying the hypotheses were satisfied) to show that a solution exists, and then used the fact that the derivative is always positive to argue that the function must be monotonically increasing, thereby showing that there can be only one solution. Clearly, both students felt that the statement was true. The second student exhibited more of an intellectual need.

    By focusing on the intellectual needs of the students in the class, the instructor can tailor lectures to help develop ways of thinking in the students that target the class' overall strengths and weaknesses. By the end of the course, both the students and the instructor have had a more enjoyable experience.

    In looking into student's proof schemes, Harel and Sowder worried that some instructors may assume too much of their students. This shows that the instructor failed to grasp the students' needs.[6]

2.2 Creating meaningful knowledge through cognitive conflict and resolution

    When a student encounters a situation where the outcome differs from the expectation, often times the student wants to know what happened. At this point, the student may have an intellectual need to know *why* the outcome was not what was expected.

    In resolving this cognitive conflict, the student will construct new knowledge, which will be meaningful to them. It may or may not be the desired knowledge (from the viewpoint of the instructor), but it is something the student created, and thus they are more likely to remember it. The knowledge formed modifies the existing mental image that the student had about the concept at hand.

    It should be noted that such a conflict may only stimulate a social need, instead of an intellectual need. For example, if a student looks at the back of a book to check the

answer and finds out it is wrong, the student may just want to fix it to get credit without caring why it was wrong.

This process is explained in more detail in the next few sections. Examples are included to better illustrate some of the ideas. It is interesting to note that cognitive conflicts can arise in almost any situation.

In fact, the first example is from grade school. If students are exposed to this process of cognitive conflict and resolution early on, they will be better prepared to deal with similar situations in later math courses.

2.2.1 Students need to experience cognitive conflict and resolution

Students need to experience conflicts between an operation chosen and an expected outcome. These problematic situations, Harel argues, lead to a student questioning his or her way of thinking and, ultimately, lead to a way to resolve them. The conflict is an intellectual need and the resolution is learning.

For example, a common error that grade school students commit is illustrated in the following two problems.[5]

A_1: A cheese weighs 5 kg; 1 kg costs 28 kr. Find out the price of the cheese. Which operation would you have to perform?

$$28 \div 5 \qquad 5 \times 28 \qquad 5 \div 28 \qquad 28 + 28 + 28 + 28 + 28$$

A_2: A cheese weighs 0.923 kg; 1 kg costs 27.50 kr. Find out the price of the cheese. Which operation would you have to perform?

$$27.50 + 0.923 \qquad 27.50 \div 0.923 \qquad 27.50 \times 0.923 \qquad 27.50 - 0.923$$

83% of the respondents chose the correct operation for A_1, but only 29% chose the correct operation for A_2.

Students realize that if 1 kg costs 27.50 kr, then since 0.923 is *less* than 1, they should pay *less* than 27.50. If a student chose the second method for A_2, then this conflict would lead the student to question the operation chosen. They have an intellectual need to know what went wrong and how to find the correct answer. Realizing that 0.923 can be treated as a multiplicand is learning.

Harel states that as students seek to understand and make sense of mathematical concepts, they encounter difficulties. Through these difficulties (and resolution), new knowledge is formed and an intellectual need is satisfied.[7]

Sometimes a cognitive conflict will not arise, though one should. For example, if a student has not been internally convinced by proofs, then it is unlikely that he or she will be cognitively disturbed by a situation where they understand each step of a proof but are still mystified by its assertion.[8]
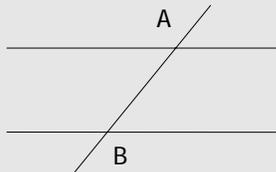
Harel and Tall discuss three types of generalizations and how they depend on one's mental constructions. The first type of generalization, *expansive generalization*, occurs when the subject expands the applicability range of an existing schema without reconstructing it. This is the goal that students should reach, but this does not seem likely to happen in only a year-long calculus sequence, for example.[9]

The second is *reconstructive generalization*. It occurs when the subject reconstructs an existing schema in order to widen its applicability range. These students appear to be experiencing some cognitive conflict and resolution as they need to reform their ideas based upon personal reflection.

The last type is *disjunctive generalization*. This occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available. This is the least desirable of the three since students fail to recognize that similar concepts are in fact, similar. Students who follow this type of generalization will learn every case of a formula and not a general rule.

> *Personal Experience*:
>
> While teaching students partial fractions, there are many different cases to consider. By giving the students the most general formula (for repeated factors) and showing them what do when the factor is not repeated, students need to learn (and remember) less. Some students will still try to learn every case.
>
> Also, when teaching partial fractions, there are two approaches to solving for the unknowns. One can equate the coefficients (which always works) or one can plug in numbers, hoping that everything cancels out.
>
> Students who do the latter encounter more difficulties since their technique becomes more difficult unless they have a special case, such as non-repeated linear factors. As they encounter a conflict, the student either a) looks for a better formula (the second type of generalization) or comes up with some rule for this exception (the third type of generalization).

Another example where disjunctive generalization appears is that students may incorrectly associate any expression that has x and y in it as a function. So, these students would feel that $x^2 + y^2 = 1$ is a function, merely by the appearance of something the students have associated as being a function.

Harel and Sowder explored student's proof schemes and categorized the results. The above example would makes students more susceptible to the ritualistic proof scheme because they are paying attention to appearance rather than strict rules of the function.[6] above

By proof scheme, Harel and Sowder are referring to what constitutes ascertaining and persuading for that person. They note that this can vary greatly among different groups of people.

The idea of proof schemes will be incorporated at other points in this paper. The similarities between intellectual need and proof schemes will be elaborated while discussing student's proof understanding, production, and appreciation (PUPA).

2.2.2 Creating Meaningful Knowledge

Another consequence of the need for cognitive conflict and resolution is that it allows students to create knowledge that is meaningful to them. Harel and Kaput argue that the extent to which a notation is elaborated is determined by the extent to which it ties to prior mathematical knowledge.[10]

Consequently, it is critical to use detailed notation for first year students whose previous experiences in mathematics are not that advanced. Over time, more details can be omitted as students see the general trends in mathematical notation.

It is preferable to have students come up with their own notation and then guide them towards mainstream notation. This way, the notation used by the students will serve a need for the student. As their notation and the mainstream notation merge, the student will adapt his or her notation and will learn what the mainstream notation represents.

There is a strong tie-in with cognitive conflict and resolution and the creation of meaningful knowledge. Harel shows how many students fail to recognize functions as conceptual entities. A *conceptual entity* is defined as a cognitive object for which the mental system has procedures that can take that object as an input.

This manifests itself most prominently when first year calculus students perform differentiation and integration. Harel offers the following example:

What is the derivative of the function $f(x) = \begin{cases} \sin(x) & x \neq 0 \\ 1 & x = 0 \end{cases}$?

A common response was the following: $f'(x) = \begin{cases} \cos(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

The student who does this is attempting to use a point-wise operation, but differentiation (and integration) are conceptual entities that are actually uniform operators.

Showing the students the graph of the function of $f(x)$ would help create a cognitive conflict inside of the students. From this, ideally, the student would want to learn why their previous understanding did not work. This in turn will create meaningful knowledge for the student and thereby resolve the conflict.
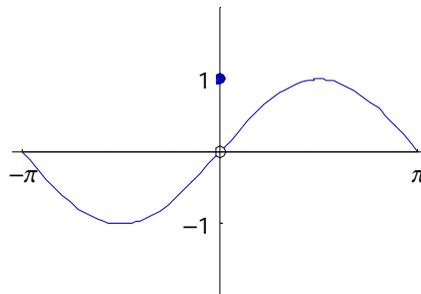
Figure 1: A graph of *f(x)*

Students should know that the derivative does not exist where the function is not continuous, since continuity is a necessary (but not sufficient) condition for the existence of a derivative.

In a series of case studies about problem solving models and local conceptual development, Lesh and Harel presented groups of students with real-life problem scenarios that needed to be solved. These non-traditional word problems instilled in the students an intellectual need.[11]

Through a process of cognitive conflict and resolution, the students were able to refine their models. Along the way, the students used their prior knowledge to form new knowledge.

Lesh and Harel conclude that students do better when they are given a clear problem and a purpose. Often times, students do not feel engaged with word problems. So, it would seem that the goal is to modify the students' ways of thinking, not create new ways of thinking. This idea ties in with the Necessity Principle and building upon existing knowledge to create new knowledge.

2.2.3 Creating Concept Images

The knowledge formed through cognitive conflict and resolution described above is not only meaningful to the student, but it also modifies the mental image that a student has regarding the concept. Harel argues it is critical for students to establish these concept images.

A *concept image* is "a mental scheme, a network, consisting of (a) what has been associated with the concept in the person's mind and (b) what the person can do in regard to this concept." This in contrast to a *concept definition*, which is "verbal definition, appearing in a textbook or written by the instructor on the blackboard, that accurately describes the concept in a non-circular way."[12]

By establishing concept images, students demonstrate that they have understood a concept. This level of understanding is more profound that of a student who just memorizes material for an exam. Those students, who only possess a concept definition, will be unable to retrieve or rebuild it on their own once it is forgotten.

The importance of concept images is revealed primarily in Linear Algebra, but as the following experience reveals, it can occur in any class. Often times, students will not learn the ways to construct the knowledge themselves, rather opting to just memorize the material.

For example, in an integral calculus class, knowing the unit circle is critical for evaluating many trigonometric integrals. One student may spend hours memorizing the unit circle for an exam, and another may learn how to construct it using existing knowledge.

Both students would do well on an exam, but I would argue that the second student possesses a better way of thinking. The student can use (existing) knowledge of 30°-60°-90° and 45°-45°-90° right triangles from high school geometry, to derive the unit circle.

The student who does this has learned how to form new knowledge from existing knowledge. This technique, ideally, will be applied to other situations as well, which should aid in the students' formation of conceptual images.

2.3 Students need to intrinsically justify a solution

One implication of the Necessity Principle is that students must intrinsically justify their solutions. Failure to do this comes from many factors. The biggest of which is a student's authoritative proof scheme.

A student who possesses this proof scheme will believe that whatever is written in a textbook or said by an instructor is true, simply due to the implicit authority of the source. This in turn, eliminates the need for justification on the part of the student, since the students already views the statement as true.

The second problem actually stems from instruction in grade school. From an early age, students are taught various strategies for dealing with multiplication and division problems. Among these strategies is the conservation-of-operation formula (defined in 2.3.2), which allows students to successfully get the correct answer without knowing what they did.

This is followed by a discussion of the key-word strategy where students are taught to solve word problems by associating various mathematical operations with certain words. This, like the conservation-of-operation formula, takes the responsibility away from students for knowing what they are doing.

Teaching for standardized tests also removes a need to justify solutions. Such tests, Harel and Manaster argue, focus on results, not on reasoning. This can also be seen in a paper written by Schoenfeld.[13]

The third problem that prevents students from justifying solutions is an immature PUPA. The term, coined by Harel and Sowder, is used to describe a student's Proof Understanding and Proof Appreciation.

A student with an immature PUPA often relies solely on examples to constitute a proof, or may memorize proofs without understanding what the proof actually states. Such students may possess an authoritarian proof scheme, as described above.

Entering college often fail to see a need to justify their solutions. Typically, their experience with proofs was in a high school geometry course. At this level, students are told they need to "justify" their solutions by citing the various properties or axioms. However, most students fail to see *why* citing the axioms justifies their solution.

Each of these problems is discussed in more detail in the following sections.

2.3.1 Removing Authoritative Proof Schemes

By far, the greatest determining factor of a student's intellectual need is the need to intrinsically justify a solution. Many students may accept what a teacher tells them without question.

Such blind faith can be problematic, especially if the instructor is prone to errors. Still, this can be a problem if the instructor never makes a mistake. By justifying a solution, one seeks to know *why* the statement is true. That is at the heart of the Necessity Principle.

> *Personal Experience*:
>
> Some of my most frustrating (but worthwhile) classes were those in which the instructor repeatedly made mistakes. Very quickly, I learned that I could not trust everything that was put on the board. I would follow each step closely, making sure that the statements on the board were correct. In those classes, I was intellectually engaged with the material.
>
> When I first taught section, I tried not to make a single mistake. However, mistakes still popped up. The students were just copying what I had written. When I would find the error, students seemed to understand what went wrong, but it would have been better if they had caught the error sooner.
>
> Since then, I have decided to *purposely* make one small error in section each week. I know where the error is and can stop to test students. Section involvement has increased, as students look for the error.

It is important to note that by creating a need to justify steps, we refer to developing an intellectual need to justify, not just because a teacher told students to justify their reasoning. Through problems of repeated reasoning, students will learn (themselves) to internalize the need to justify a solution. It is not something that can be taught explicitly, since this would be another example of blindly following what a teacher says to do.

As was discussed in the previous section, there are situations where a student may justify their work, but not fully understand what they are writing. These students do not have an intellectual need, though the casual observer might think otherwise.

Harel and Sowder believe that the reason why so many students lack the intellectual curiosity to wonder why a theorem or formula is true lies in the fact that current mathematics curricula emphasize truth rather than reasons for truth.[6]

This leads students to believe that mathematics is a subject that does not require intrinsic justification. Students appeal to the authority of a textbook or a teacher for assurance that they are performing valid mathematics.

When students look to a textbook or teacher for assurance, they are exhibiting an authoritative proof scheme. This type of proof scheme removes the need to justify any solution or thought process.[6]

One consequence of the authoritarian proof scheme is that students often ask for help on a certain problem without first making a serious effort to solve it on their own. While many students are capable of solving the problem, they need the confirmation of authority before arriving at their solution.

Personal Experience:

   For some of my own lower-division courses, I would wait for section so that the TA could do the "hard" questions for me. Part of it was laziness, but also part of it was that I was not confident in my own abilities to solve the problem.

   Now, as an instructor, I see a variety of students. There are some who attend section for the sole purpose of getting answers. There are others who want the confirmation that they did the problems correctly, and then there are others who are genuinely confused and are seeking help in understanding the fundamental concepts of the course.

   My goal as a TA has been to instill confidence in the students so that they will believe in their answers. On an exam, for example, the students will not have the luxury of asking how to do a problem; instead, they have be confident that they are capable of solving the problem on their own.

Also, research done by Harel suggests that once a relationship is labeled as a theorem, students make a reduced effort, are less willing, and less able to justify a statement. Labeling something as a "theorem" transitions the statement into a formula. Now, under the authoritarian proof scheme, it is obeyed, not reasoned and discussed.[14]

If a student doubts another student's response but the teacher accepts the response, most students will not voice their doubts either. However, once one student voices an objection, then more will chime in.

This follows from the authoritarian proof scheme. The students seek the approval (or disapproval) from the instructor. In the absence of an objection, the statement is accepted as a fact.

Also, it is known from studying human behavior that the amount of effort that a person is willing to make in seeking evidence that would support or refute a statement is proportional to the faith in the statement. When someone feels passionate about something, they are more likely to exert the effort to justify it. Often times, though, students do not feel passionately about mathematics. As a result, their need to justify their solutions is diminished.

An ideal teaching environment would be one in which students are free to ask questions. At the same time, though, the instructor needs to be careful *how* he or she responds to questions asked by students. When posed with a question, the instructor needs to answer the question in a manner that does not intimidate the student.

I have seen an inexperienced instructor make students feel inferior when they ask questions. This in turn reduces the likelihood that students will ask further questions, despite available opportunities.

A classroom size of less than thirty would be required for this to be implemented. This could prove difficult to implement in a typical lecture of two hundred students or more in many first-year courses. (See the recommendations.)

Thankfully, many universities have some type of recitation section where the concepts in the course are discussed. Often, this time is used to go over homework problems. The aim, however, should be to foster discussion on how to approach problems. (See the recommendations.)

> *Personal Experience*:
>
> As a Teaching Assistant, I go over homework, but along the way, I draw out the solution strategies from the students. I try to promote different ways to approach a problem and have the class discuss the validity of the strategy. Students feel more engaged with the material and perform better as a result.

2.3.2 Understanding the Meaning Behind Operations

Often times, students perform calculations and fail to understand the meaning behind their actions. This occurs at all levels of mathematics.

Discussing the development of multiplication in children, Harel criticizes one method for finding solutions to multiplication problems: the conservation-of-operation formula.

In the conservation formula, one replaces "nasty" numbers with "friendly" numbers. For example, consider the following problem:

"How much would I pay for 3.5 pounds of cheese if one pound costs \$2.19?"

This would be translated as follows: We replace 3.5 with 3 and replace 2.19 with 2. Now, we want to know how much 3 pounds will cost if each pound weighs \$2. To solve this, we use multiplication. That is, our solution is $3 \times \$2$. Now, we replace 3 with 3.5 and \$2 with \$2.19, to get out solution: $3.5 \times \$2.19$.

This allows one to obtain the correct answer, but not question why their solution works. As a result, children feel that mathematics is a subject that does not require intrinsic justification.

It should be noted that mathematicians often recognize a problem that they are working on as having the same structure as some already formulated mathematical knowledge. In this case, they are forming an isomorphism between the two problems.[15]

A casual observer may not see the difference between the two ways of thinking, but the second represents a deep understanding of the underlying structure between the two problems while the first shows little more than an ability to follow an algorithm.

Personal experience:

Early in my tutoring career, I encountered students who had difficulties understanding how to compute partial derivatives, namely in order to find f_x, they had to hold y constant.

At the time, I suggested the students replace y with some obscure number, say 17, and put it in parenthesis. Then, when they performed the calculations, they would get the correct answer, once they replaced 17 with y.

I dropped this strategy shortly after starting it, for I realized that I was merely using the conservation-of-operation formula and not helping the students understand the concept of partial derivatives.

Instead, I was able to get students to think of a partial derivative as the slope in a given direction. I was able to show students that when we change in the x direction, we naturally have to hold y constant. I think that once they understood why the y value had to remain constant, the problems became simpler.

In integral calculus, for example, students are taught various techniques for evaluating integrals. Students learn (or memorize, see above) these strategies, but often times they do not understand the underlying concepts that are presented in class.

Discussions with students were enlightening, if not worrisome. When I asked two different students why $\int dx = x$, both said "you just cross out the *d*." This shocked me. When I asked one student what happens for $\int x^2 dx$, they said that again the *d* goes away and that's why the power of x increases by one.

It is possible that the students recognized that *dx* represented a width (which is measured with the variable *x*). And since they had been taught that integrals measure area, then by multiplying the height, *f(x)*, by the width, they should get the area. Then we take the limit as we sum over each small area and get the area under the curve.

However, I doubted that this was the case. I proceeded to explain what *dx* was to the student. Through the course of our subsequent discussion, it became apparent that the student did not possess the high level of thinking described above. Instead, they saw *dx* as a part of the structure of an integral; to student, its appearance lacked any meaning.

Regrettably, this problem arose early in the discussion of integration. I would have liked to have asked the student what $\int \sin(x)dx$ would become.

Harel and Behr discuss the idea of the key-word strategy when it comes to solving multiplication problems. It is interesting to note one of the key words was 'of' means multiplication. This could explain some students' belief that $\dfrac{\sin(x)}{x}$ can be simplified, that is, students will just write $\dfrac{\sin(\cancel{x})}{\cancel{x}} = \sin$, since sin(x) is read as "sine of *x*" and students have been conditioned to think that 'of' means multiplication. Again, the students are performing operations with no clear understanding of their meaning.[16]

In public schools, a lot of emphasis is placed on measurable results. Often times, school funding or an instructor's job rides on the success (or failure) of their students. Harel feels that too much emphasis is placed on test results and not enough on the meaning behind the mathematical operations in K-12 education. This is supported by the work of Manaster researching eighth grade students.[17]

> *Personal Experience*:
>
> When teaching a section on vectors for a Pre-calculus class, I used the analogy that vectors are like dogs. What do dogs do? They sniff each other. From there, I went on to discuss vector addition.
>
> Students remembered the image later on, but I feel that I prevented them from seeking to ask *why* vector addition can be represented geometrically. Instead, they had a mental image in their heads which prepared them for their exam, but I failed to engage the students intellectually.

Teaching for test results can lead to students retaining knowledge, but not fully learning the material, nor being intellectually engaged. In essence, the students are being trained to memorize in novel ways.

This lack of intrinsic justification occurs at all levels of mathematics, not just with grade school children. As can be seen above, it would appear that some of this carries over to college.

2.3.3 Fostering Students' PUPAs

A great deal of work done by Harel focuses on students' proof understanding and proof appreciation (PUPA). It should be noted that a student's PUPA and intellectual need are very closely related. As proof understanding and appreciation builds, one becomes more likely to ask *why* something is true, thereby presenting an intellectual need.

However, a difference can be noted. A student may appreciate proofs, but lacks the ability to produce them. So, it would seem that by producing proofs, one satisfies an intellectual need (since knowledge is brought together to complete a proof).

In order to foster a student's PUPA, proofs need to be tangible to the students. That is, the proofs need to include mathematical objects that have meaning for students, present clear underlying ideas, and include a need to justify each step of the proof.[8]

Tangible proofs create an intellectual need in students by giving them something familiar to start with. Students are more likely to want to know why a statement that they know about is true, for example why the product of two odd numbers is an odd number, as compared to an abstract proof in Differential Geometry. A tangible proof also provides a clear understanding of what needs to be proven. This, coupled with a student's way of thinking, leads to a need to justify the steps of the proof. (The proof alone cannot instill in the students a need to prove each step.)

At the same time, though, instructors need to convey the mental labor and thought process involved in constructing proofs. Along the way, justification for each step can, and must, be shown. Students need to be active participants in the development of the proof and not just proof-observers.

Most teachers who put a proof on the board have done so many times before. It is easy to forget how difficult the proof was initially. Consequently, students see a rather complicated proof flow out of the professor's brain and onto the chalkboard. It is easy for a student to get frustrated when he or she is unable to do the same at first.

By slowing down and asking for student's input throughout the proof, the teacher can engage the class and bring the students into the construction of the proof. Incorrect thinking needs to be followed to see why it fails. Students need to realize that creating a proof is no easy task.

Sadly, not all instructors engage the students in the proof production. In fact, to save time, some instructors omit proofs to key theorems and instead provide numerical examples to assert the validity of the statement and leave it up to the students to find the proof in their textbook.

While I can understand the reasons for doing so, merely providing examples sends a bad signal to the students. It tells the student that instead of a formal proof, plugging in a few numbers is sufficient justification for a statement.[8]

To "prove" a statement using an example is referred to as an empirical proof scheme, more precisely, an inductive empirical proof scheme. For example, a student may pick a few random numbers and test a formula. If it holds, then they feel the statement is true.[18]

In some sense, the students are justifying a proof using probability. That is, they feel that the likelihood that their "random" guess would hold if it were not true is small enough for them to feel confident about the result. One's proof scheme is about what convinces oneself and persuades others. These students feel that they would be able to convince others based on their random guess.

Unfortunately, students do not choose numbers at random. Typically, students choose small numbers. There are many examples where a formula would hold for most values, but fail in only one or two instances. For example, consider the claim that $n^2 + n + 41$ is prime for all n. Numerical examples will seem to support the claim. In fact, the first counterexample is when $n = 40$. (At that point, $n^2 + n + 41 = 1681 = 41^2$.) Even if the numbers chosen by the student were truly random, the examples alone can only *suggest* the validity of the statement.

Harel and Sowder discuss a case study of one student, Ann, and how her PUPA never fully matured during her undergraduate career in mathematics. "Ann memorized proofs, she was failing to seek new knowledge. Instead, she would memorize for an exam and then forget the proof." As was discussed in section 2.2.3, this ties in with Harel's observation that once concept definitions are forgotten, students are unable to recreate them on their own.

Personal Experience:

As a student, I memorized all of the formulas for Vector Calculus, never bothering to learn why they were true. I did quite well in the course, earning a solid A. However, my understanding of the material was at a level far below that.

From the point of view of a teacher, I have tried to refrain from telling students to just memorize formulas. In most classes, students have been permitted the use of a formula sheet. To prevent pure memorization, we ask conceptual questions that use the formulas, but not in an algorithmic fashion. This forces students to apply what they have learned.

I conclude this section on proofs with a short discussion on proofs by contradiction and the need for justification. Aristotle argued that one does not understand something until one knows why it works. It is this natural curiosity as to why something works that is at the heart of intellectual need and the Necessity Principle.

Harel argues that proofs should explain, in addition to prove. If proofs were presented this way, students' intellectual needs would be addressed.

The argument against a proof by contradiction is interesting. Proponents argue that one does not learn from a proof by contradiction, rather one only determines certainty. This supports Aristotle's belief that one should know why something is true.

2.4 Application of Intellectual Need: Teaching Mathematical Induction

The following section discusses how Harel applies the Necessity Principle to the teaching of mathematical induction through a gradual refinement. In another paper, Harel also shows how the Necessity Principle can be applied to teaching Linear Algebra.[19]

Harel argues that there are some fundamental instructional deficiencies when it comes to the presentation of mathematical induction.[18]

First, he feels that the concept is introduced abruptly. As a result, the students do not see a need to solve problems with mathematical induction, nor how it follows from previous experiences. Consequently, the students are unable to build upon existing knowledge to form new knowledge, or to satisfy an intellectual need.

Secondly, the problems in a typical math book do not challenge students to think about the meaning behind mathematical induction, rather they teach students that induction is a formalistic (mechanical) procedure, i.e. prove the base case, assume the formula holds for n, show that $n + 1$ follows logically, and conclude the formula holds for all n.

An example of such a problem would be to show that 5 divides $7^n - 2^n$. Plugging in $n = 1$, students can see that the $5 \mid 7 - 2$, which establishes the base case of $n = 1$. Assuming the formula holds for n, they show that $n + 1$ follows. Through some algebraic manipulation, the student can show that $7^{n+1} - 2^{n+1} = 7(7^n - 2^n) + 5(2^n)$. Since $5 \mid 7^n - 2^n$, by assumption, and 5 divides the second term, the student concludes the formula holds for $n + 1$, thus completing the proof.

Here, the student can prove the statement without much awareness of the steps that are being performed. The thinking is mechanical; there is no intellectual need present. These types of problems can lead to a result pattern generalization.

Result pattern generalization is when one's way of thinking is based on regularity of the result. On the other hand, implicit recursion problems, such as the Tower of Hanoi Problem where one must find the minimal number of moves required to transfer n disks from one peg to another, lead students to focus on process pattern generalization.

Process pattern generalization is defined to be how one's way of thinking is based on regularity of the process, though it might be initiated by regularity in the result. This way of thinking operates at a higher level than result pattern generalization and, consequently, is more desirable to instill in students.

However, implicit recursion problems are lacking (or omitted) in most books. Rather, typical identity, inequality and divisibility problems which require little understanding of mathematical induction are asked. Thus, students may not understand mathematical induction as a way of thinking.

The easiest problems of mathematical induction may be too simple for students. That is, instead of using mathematical induction, students look for alternative (easier, in their eyes) ways to prove the statements. Also, some students perceive induction as circular reasoning. This can lead many to question its validity as a method of proof.

Personal Experience:

I have tutored many students who had fundamental problems with mathematical induction. Many felt that they were assuming that the formula holds for all *n* from the beginning. Others were confused how assuming the formula holds for one specific *n* would show that the formula holds for all *n*.

Others were able to show the base case, work through the algebra and show that the formula is also valid for *n* + 1, but they did not understand that this in turn showed that the formula holds for all *n*.

In some of my early Mathematical Induction proofs, I incorrectly assumed both *n* and *n* + 1. Through algebraic manipulations, I was able to show that *n* + 1 was true, and so I thought that I had completed the proof. It was quickly pointed out to me, though, that I had assumed what I was trying to prove.

Because of the inherent problems in the presentation of mathematical induction, a lot of students believe mathematical induction because of authoritative or empirical proof schemes. The students look to the authority of an instructor or the use of examples to justify their results; there is no intellectual need.

Harel offers a solution, modeled around the Necessity Principle. He argues that mathematical induction should be developed gradually through intrinsic problems, problems the students can understand and appreciate. This is accomplished through three phases:

1. Quasi-induction as an internalized process pattern generalization
2. Quasi-induction as an interiorized process pattern generalization
3. MI as an abstraction of quasi-induction

Harel uses the term *Quasi-induction* to represent the general solution approach that emerged from the students' repeatedly applying process pattern generalization in solving implicit recursion problems.

The term *internalized process pattern generalization* means that the students used quasi-induction autonomously. Beyond this is *interiorized process pattern generalization*, where students not only used quasi-induction autonomously, but it conceptually represents a method of proof to the students.

Harel stresses that implicit recursion problems should be introduced early. This leads students to focus on process pattern generalization, not on result pattern generalization.

By using process pattern generalization, these students will construct new knowledge when they encounter a situation that is incompatible with their existing knowledge. The knowledge gained may not be what is intended (from the viewpoint of the instructor), but it is meaningful to the students, for it is driven by personal intellectual need.

3. Recommendations

Throughout the previous sections of this paper, some suggestions for fostering intellectual need have been proposed, with some discussion about their significance. The aim of this section is to lay out a series of recommendations, tying together various threads from earlier.

The first two recommendations deal with institutional changes, while the last two address what can be done in a given class.

3.1 Introduce New Concepts in Small Groups

It was discussed earlier that recitation sections are often used to discuss homework solutions. Ideally, this time could be used to do more than just discuss homework, but the class size (of around 30 to 40) makes this difficult.

I propose that discussion sections should be limited to 15 or 20 students. The number of discussion sections would need to be doubled to reduce the number of students to 15-20 students. These new discussion sections could be a venue to get students to think about new topics before they are formally presented in lecture.

The goal of the new section would be to introduce students to concepts that would appear the following week. This would give them a chance to think about the material *before* it is presented in lecture.

Discussion leaders would pose problems to the students that are within their proximal knowledge. For example, before discussing optimization problems, the following problem could be presented to students:

"A farmer wishes to enclose a pasture with fence. Due to winds, the sides that run North to South cost ten dollars per foot and the other sides cost only eight dollars per foot to construct. Find the maximum amount of area that the farmer can enclose with $2,000."

The discussion leaders would break the section into smaller groups and each group would discuss the problem and offer a solution strategy.

In the above problem, the students would be encouraged to try plugging in some numbers to get an idea of what to expect for their answers. Then, through some

suggestions and observations, students would be led to drawing a picture and setting up some equations using the information given. One goal would be to tie in this "new" problem with problems discussed the previous week in lecture and in the homework.

All the while, another goal would be to place emphasis on the underlying process. By developing the solution as a class, the students would have more desire to work through the solution. And since they were part of the solution process (as opposed to just copying down answers) they would be more likely to solve similar problems.

This recommendation has its roots in the work previously done by Harel. In the paper where Harel introduces the Necessity Principle, he suggests that small group discussion would be crucial in fostering intellectual need.[4]

The idea of introducing new topics, by connecting them to existing knowledge follows the established body of literature in education. The goal of both of Harel's alternative teaching strategies (for Mathematical Induction and Linear Algebra) rested on a gradual refinement of ideas while bridging the new concepts to the existing knowledge. Thus, new concepts are constructed within the framework of pre-existing knowledge.

A thorough discussion on the treatment of Mathematical Induction is included in section 2.4. With regards to Linear Algebra, Harel begins with a discussion in $\mathbb{R}^2$ and $\mathbb{R}^3$. Here, the students' existing knowledge can be utilized to show basic concepts and relationships. This is refined through a discussion on $\mathbb{R}^n$, and lastly through a general vector space V^n.[19]

With regards to calculus, it is critical to fully explore the underlying connections between concepts. For example, in integral calculus, students may forget the definition of an integral and fail to see how the convergence of an infinite series and volume problems are related. Or in differential calculus, students may not realize the connection between slope and optimization.

3.2 Institute an Honors Calculus Sequence

The second recommendation would be the establishment of an Honors Calculus sequence. The class would be limited in enrollment (around thirty), with some set of prerequisites. Ideally, the course would be one year long, taught by the same professor.

The goal of the course would be to have students develop deeper, more profound ways of understanding calculus *and* strengthen their proof schemes through more exposure to (and construction of) more proofs. With the small class size, the ability to halt the course and break into small discussion groups would be more reasonable.

As a result, the students and the instructor would be exploring calculus together. Students would be responsible for the development of most of the proofs.

Connecting with the idea of three types of students (attending and intellectual engaged, attending and non-engaged, and non-attending and non-engaged), this sequence could be the ideal class for the first type of student.

As the natural division of a two calculus sequence was illustrated above, the honors sequence would create a third division, better matching students and classes. For example, students who are attending and intellectually engaged would be most likely to enroll in the honors sequence.

3.3 Integrate Technology into the Course

Through classroom observations, Harel feels that the computer could be a valuable tool for helping students visualize concepts and explore concepts further.[14]

> *Personal Experience*:
>
> In a Differential Equations and Linear Algebra course, I had weekly Matlab assignments that were designed to extend some of the concepts taught in class. Often times, we used the computer to perform many tedious calculations. Other assignments illustrated basic results or theorems from the class.
>
> However, as I reflect upon what I did, I cannot recall a single assignment. At the time, it was something that needed to be done. We were told what commands we needed to enter, and we did just that. We were mindlessly using the computer with no knowledge of what it was that we were doing.

The sentiment described in the personal experience has been echoed by discussions with other students who have taken those courses. Perhaps the best incorporation would be allowing students to use a computer to assist them in doing their regular homework.

The computer should not be used to do all of the student's homework. Rather, the instructor could designate which homework problems could be done by computer. Computers should be used for the more computational-oriented problems, like Newton method or Euler's method.

Or, in the context of Harel's discussion, the powerful three dimensional modeling tools could be of great benefit to students when studying multivariable calculus.

I have found that students have much difficulty relating functions and graphs in three dimensions. The classic example is that of the Hyperbolic Paraboloid, $z = x^2 - y^2$.

A method which has helped students is to begin by plotting level curves of the surface. Once these are done (and the student observes that these are parabolas), I cut out some parabolas using scratch paper. Having the student hold some of the pieces, I model the shape in three dimensions.

After the students have a method to model such problems, I try to relate the object to their existing knowledge of the world. "Can you picture what a Pringle's chip looks like?" I often ask students. "Well, that is a Hyperbolic Paraboloid."

For students, one of the greatest advantages of having someone model an object is their ability to view the shape from all sides. With the mental image of a Pringle's chip, students can visualize themselves turning the object or looking at it from different angles.

Textbooks fail to perform this action. Any time that a three-dimensional figure is printed two-dimensionally, information is naturally lost. Shading is used to represent depth, but students not familiar with this will struggle trying to "see" the figure.

A computer with three-dimensional graphing capabilities would allow the students control over the object, permitting them to turn and rotate the object at their discretion.
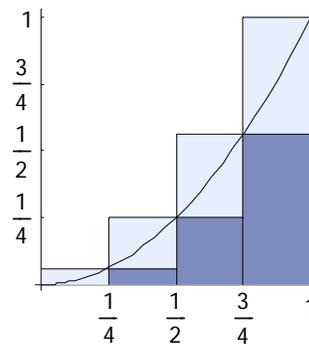
My instructor for integral calculus used Mathematica to illustrate the results of the Riemann sum. The instructor had a problem that would draw the left and right hand approximations for various functions. This can be done on the board, but it takes a lot of time to properly draw all of the boxes.

On top of that, the computer allowed us to see the refinement as n increased. We started with a simple equation: $y = x^2$ on the interval [0, 1].

Before we started, the instructor asked us to determine what a plausible area should be. After some suggestions, we noticed that the area should be less than 1/2, since the graph lies below the line $y = x$. Also, the area should be positive, since the curve lies above the x-axis. This gave us some framework for which to check our answers. We knew that if we estimated the area outside of that range, then we had done something incorrectly.
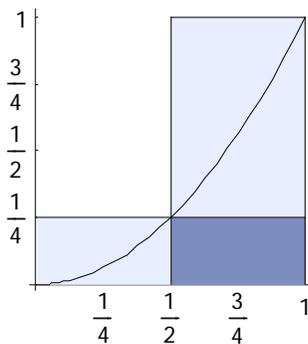
Approximations with $n = 2$ Approximations with $n = 4$

Figure 2: Left and right hand approximations of $y = x^2$

Visually, we can see how in the first figure, the left-hand approximation tells us that the area should be greater than 1/8 while the right-hand approximation tells us that the area should be less than 5/8.

In the second figure, we refined our estimates and saw the left-hand approximation was now 13/64 and the right-hand approximation was 7/16.

As we continued our refinements, we were visually able to see how the two approximations converged to what appeared to be 1/3.

We proceeded to repeat the above procedure with other functions. The clean display of the computer made it easy to visualize the concept of the Riemann sum.

The instructor used the visual examples as a spring board to develop the power rule for integration, and also to illustrate the relationship between definite integrals and areas under a curve.

Similarly, a computer can easily create a table of data which can help students dealing with limits for the first time see what is going on. For example, consider the following two functions:

$$1)\ \lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} \qquad \text{and} \qquad 2)\ \lim_{x \to 2} \frac{x^2 + 3x + 2}{x - 2}$$

When first exposed to the idea of a limit, students are encouraged to graph the function as well as plug in numbers near the limit to see what is going on. This utilizes both the visual and computational centers of the students' brain.

The computer can provide both. We can see how the first limit factors, so that the limit should approach 1, but students may be surprised by this fact, since their intuition tells them that they would be dividing by 0.

Here are tables of data produced by a computer for each limit:

1)	x	y		2)	x	y
	1.9	0.9			1.9	−113.1
	1.99	0.99			1.99	−1193.01
	1.999	0.999			1.999	−11993
	2.001	1.001			2.001	12007
	2.01	1.01			2.01	1207.01
	2.1	1.1			2.1	127.1

Table 1: Radically different data for two similar limits

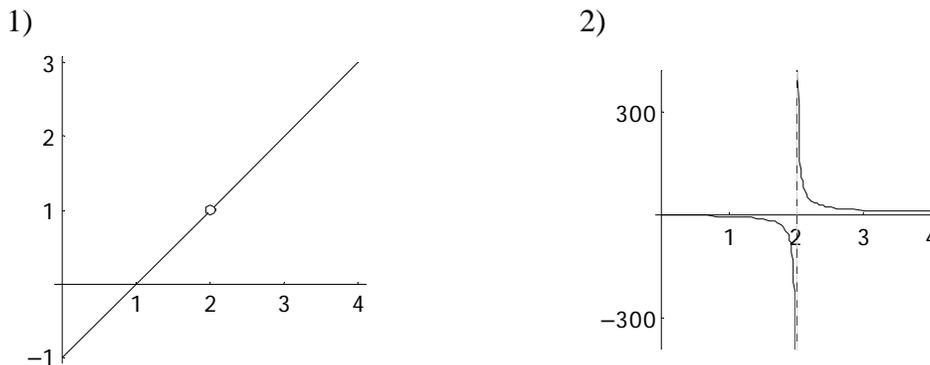The graphs of the two functions help illuminate what is going on:

1)

2)

Figure 3: The graphs of the two limits

This problem illustrates how altering a single sign can drastically change the limit, as well as the graph itself. This, in turn, can lead to students wondering *why* such a change would take place. Now, the student has an intellectual need to know why. The idea behind these types of problems leads to the next recommendation.

3.4 Emphasize Problem Solving Skills

Instructors need to emphasize problem solving skills in students. As was stated earlier, one of the goals of teaching calculus to college students is to teach problem solving.

After completing calculus, most students will never take another derivative or integral. The idea, though, is that these students have learned problem solving skills which can be applied to whatever field they choose to pursue.

That being said, exercises dealing with the mechanics of solving problem are necessary. However, such problems should be limited in amount. Instead, instructors should carefully choose problems from a textbook (or questions they wrote) that present an intellectual need for the students.

For example, an instructor could have students in a precalculus or differential calculus course examine the limits $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x - 2}$ and $\lim_{x \to 2} \dfrac{x^2 + 3x + 2}{x - 2}$. As was discussed above, this problem can create a cognitive conflict in the students.

These types of problems are particularly fruitful, since this will lead to students wanting to learn *why* they arrived at an unexpected result. It is through the resolution of this conflict that new, meaningful knowledge will be created by the student.

4. Putting It All Together

Students who pursue knowledge for the sake of knowledge no longer exclusively populate universities. As a result, instructors at universities need to find ways to develop intellectual need. Harel contends that to implement his Necessity Principle, an instructor must employ a combination of methods including combination of small group discussion, team projects, whole-class discussions, individual learning, use of technology, and lecturing.

The aim of this paper has been on fostering the intellectual need of first year calculus students. Fostering intellectual need is at the core of the Necessity Principle. In order to develop this need, though, instructors need to understand their students and their ways of thinking.

There are three types of students enrolled in a given course. The first group consists of those who regularly attend and are intellectually engaged by the material. The second group also attends, but have not been engaged yet. The last group fails to attend lecture and is not intellectually engaged.

Many universities offer two calculus sequences, which naturally distribute students. Instructional plans need to be changed for each sequence. Regardless, the target audience of an instructor should be towards the second group of students. The first group is already satisfied while the last group does not (and may never) feel an intellectual need towards the class and the material.

In addition to understanding how a student approaches a given problem, instructors need to present students with scenarios whereby the student experiences some cognitive conflict. The student will want to rectify the problem situation with their beliefs, thereby creating new knowledge.

However, care must be given to make sure that students experience this conflict and resolution. It is possible for a student to fail to realize that a conflict has taken place. Thus, instructors must focus on examples that tie in with student's existing knowledge. Again, instructors need to understand their students and their ways of thinking.

Lastly, it is crucial that instructors try to instill an intrinsic need to justify the steps of a solution. Students may accept what an instructor tells them and never think about the statement. These students are not exhibiting an intellectual need; at best, they are a sponge for knowledge.

One of the most effective strategies that I have noticed is to purposely make mistakes during a lecture. As errors will naturally occur, one should not plan too many errors. Ideally, a small error on the first day will help students to realize that they need to pay attention to catch other possible errors. As a result, students will pay closer attention to lecture and scrutinize the material presented, checking for mistakes. Of course, if no student catches the error, the instructor should draw the students' attention to the error.

Instructors need to prove theorems to the students and, ideally, have the students actively participate in the creation of the proof. Instructors need to show the mathematical thinking required to produce a proof. Students may become disheartened at their inability to produce a proof that seems to flow so easily from the professor. By showing the effort required, students may feel better if the proof does not come naturally to them.

Examples are not sufficient for a proof, especially in a first year calculus course. Presenting examples instead of proofs can lead students to believe that showing a statement is true for a few numbers is a sufficient proof. On top of that, the students do not need to question the statement, since they accept the instructor's word that the proof exists.

Teaching for test results is also a bad idea. Students are trained to pass an exam. It may appear, to the casual observer, that the student has a mastery of the subject, but in reality, all the test demonstrates is that the student knows how to do well on the exam. There is no guarantee that the student truly understands the material, let alone feels intellectually engaged.

Developing an intellectual need in students can be challenging. Many factors determine whether or not a student feels an intellectual need to learn the material in a course. Instructors should realize, though, that by planting an intellectual need in students, learning truly grows.

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