QUALIFYING EXAMS

September 9 2020
5PM-8PM (Pacific Time)

Three-hour exam. Do as many questions as you can. **No book or notes allowed.** Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly. **Even if you can not solve the whole problem, you still need to write your partial answer to receive partial credit.**

1. Show that there does **not** exist a continuous map $f : S^1 \times S^1 \to S^1$ that satisfies **both** of the following conditions:

- $f(x, x) = x$ for any $x \in S^1$;
- $f(x, y) = f(y, x)$ for any $x, y \in S^1$.

2. Let X be a path connected CW complex whose fundamental group is finite. Show that any continuous map $f : X \to \mathbb{T}^n$ is null homotopic. (Here $\mathbb{T}^n = S^1 \times \cdots \times S^1$ is the n-dimensional torus.)

3. Compute the homology group $H_k(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z})$ for all $k \geqslant 0$. (You need to show your computation of $H_k(\mathbb{R}P^2; \mathbb{Z})$.)

4. For $n \geqslant 1$, show that one can **not** cover the complex projective space $\mathbb{C}P^n$ by n open subsets $U_1, U_2, \cdots, U_n$ such that each U_i is contractible. (You may assume the ring structure of $H^*(\mathbb{C}P^n; \mathbb{Z})$.)

5. For $n \geqslant 1$, take a point $p \in S^n$ and consider the following subspace of $S^n \times S^n$

$$A = \{(x, y) \in S^n \times S^n \mid x = p \text{ or } y = p\}.$$

Show that there does not exist a retraction of $S^n \times S^n$ to A. (Namely, show that there does not exists a continuous map $r : S^n \times S^n \to A$ that fixes A pointwisely.)

6. Let M, N be two connected, closed, oriented n-dimensional manifolds ($n \geqslant 1$). Consider a continuous map $f : M \to N$ that has nonzero mapping degree. Show that the induced map

$$f^* : H^k(N; \mathbb{Q}) \to H^k(M; \mathbb{Q})$$

is injective for any k. (Recall: the mapping degree of f equals d if $f_*[M] = d[N]$, where $[M], [N]$ denote the fundamental classes.)

7. Let M be a closed, orientable n-dimensional manifold with **nonzero Euler characteristic**. Consider the map $f : M \times M \to M \times M$ defined by $f(x, y) = (y, x)$ for any $x, y \in M$. Show that any map $g : M \times M \to M \times M$ that is **homotopic** to f has a fixed point.

8. Let X be a connected CW complex such that $\pi_1(X)$ is a nontrivial finite group and $\pi_k(X) = 0$ for any $k \geqslant 2$. Show that X can not be a finite CW complex. (Namely, X must have infinitely many cells.) **Hint**: Compute the Euler characteristic of the universal covering space.