43. Summer 2024

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students - pictures aid explanation but should not replace it! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.

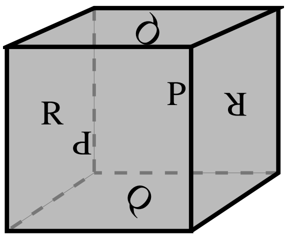
1. Let X be the union of four mutually tangent unit 2-spheres inside $\mathbb{R}^3$. Compute $H_*(X;\mathbb{Z})$.

2. The classification of closed connected surfaces says that every such surface is homeomorphic to either a connect-sum of $g \geq 0$ tori (denoted Σ_g) or a connect-sum of $h \geq 1$ projective planes (denoted N_h). *Commensurability* is the equivalence relation on spaces generated by saying that $X \sim Y$ if X is a finite cover of Y, or vice versa. What are the commensurability classes of closed connected surfaces?

3. The cone CX of a space X is $X \times I$ with $X \times \{0\}$ crushed to a point. The mapping cone of a map $f : X \to Y$ is the space C_f obtained by gluing CX to Y using the map $f : X \times \{1\} \to Y$. Show that there is a long exact sequence

$$\cdots \to H_i X \to H_i Y \to \tilde{H}_i C_f \to H_{i-1} X \to \cdots .$$

4. Let X be the space obtained by gluing opposite pairs of faces of a standard cube I^3 via 180 degree rotations, as shown. Compute the homology $H_*(X;\mathbb{Z})$.



5. Let M be the abelian group given by the following presentation with three generators and three relators: $\langle a, b, c : 2a + 3b + 5c, 3a + 5b + 2c, 5a + 2b + 3c\rangle$. Compute $\mathrm{Tor}(M, \mathbb{Z}_2)$.

6. By considering ways to attach a 6-cell to $S^3 \vee S^3$, show that $\pi_5(S^3 \vee S^3) \neq 0$.

7. Let X be the CW complex formed by attaching k two-cells $e^2_1, \ldots e^2_k$ to the circle S^1 $(= e^0 \cup e^1)$ via attaching maps with degrees $n_1, n_2, \ldots, n_k$. Compute $\pi_2(X)$ in terms of $n_1, \ldots, n_k$.

8. Show that if M is a closed 4-manifold which is homotopy-equivalent to the suspension ΣX of some path-connected topological space X, then $H_*(M;\mathbb{Z}) = H_*(S^4;\mathbb{Z})$ (that is, M must be a *homology sphere*).