TOPOLOGY - QUAL EXAM, SPRING 2022

The Topology - Qual Exam, Spring 2022 is on Gradescope at 1:00 - 4:00 pm, San Diego local time, Thursday, May 12, 2022. After you finish, you will have 15 more minutes to upload your solutions, as pdf files, on Gradescope. You should assign pages for each problem on Gradescope. This exam on Gradescope will close at 4:15pm, San Diego local time. It is your responsibility to make sure that you have a working internet, but in case you encounter unexpected issues to upload your solutions on Gradescope, you may email them to me (xuzhouli@ucsd.edu).

This exam won't be proctored. You are NOT allowed to use books or notes during the exam. The instructor reserves the right to conduct an oral follow-up exam if he notices a possible academic integrity violation.

1. (50 pts) Let $X = \mathbb{R}P^3$ and $Y = S^1 \vee S^1$.

- **(a)** (5 pts) Let $A = X \vee Y$ be their wedge sum (one-point union). Provide a presentation of the group $\pi_1(A)$.
- **(b)** (5 pts) Let $B = X \times Y$ be their product. Provide a presentation of the group $\pi_1(B)$.
- **(c)** (5 pts) Compute $H_*(A;\ \mathbb{Z})$.
- **(d)** (5 pts) Compute $H_*(B;\ \mathbb{Z})$.
- **(e)** (5 pts) Compute $H^*(B;\ \mathbb{Z})$ as groups.
- **(f)** (5 pts) Compute $H^*(A;\ \mathbb{Z}/2)$ as a ring.
- **(g)** (5 pts) Compute $H^*(B;\ \mathbb{Z}/2)$ as a ring.
- **(h)** (5 pts) Are all maps $f : X \to Y$ null-homotopic? Explain your reason.
- **(i)** (5 pts) Are all maps $g : Y \to X$ null-homotopic? Explain your reason.
- **(j)** (5 pts) Let C be a double cover (2-sheeted covering space) of B. Compute the Euler characteristic of C.

2. (10 pts) Determine whether $X = S^2 \vee S^3 \vee S^5$ is homotopy equivalent to

(a) a closed manifold (compact, no boundary),

(b) a manifold.

3. (10 pts) Let X be a 5-dimensional simply-connected closed manifold. If $H_2(X) = 0$, show that X is homotopy equivalent to S^5.

4. (30 pts, 3 pts each)

Write down "True" or "False" for each of the following statements. No justification needed.

(a) For any CW complex X, $H_*(X)$ is a finitely generated abelian group.

(b) There exist two connected simply connected CW complexes X and Y such that they have isomorphic homotopy groups but non-isomorphic homology groups.

(c) Denote by Σ_g the orientable surface with genus g. Then Σ_n is a covering space of Σ_2 for all $n \geq 3$.

(d) Let M be a connected closed n-dimensional manifold. Then $H_{n-1}(M;\mathbb{Z})$ is free.

(e) Let R be a PID, and M, N be R-modules. Then $\mathrm{Ext}^n_R(M, N) = 0$ for $n \geq 1$.

(f) As a ring, $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[c]/(c^{n+1})$, where w has degree 2.

(g) The Klein bottle and the torus are covering spaces of each other.

(h) For positive n and m, $\mathbb{R}P^n \times \mathbb{R}P^m$ is orientable if and only if $n \cdot m$ is odd.

(i) For any CW structure of $\mathbb{R}P^n$, there is at least one cell in each dimension i for $0 \leq i \leq n$.

(j) For any spaces X and Y and any field k, we have an isomorphism of rings:

$$H^*(X \times Y; k) \cong H^*(X; k) \otimes_k H^*(Y; k).$$