2011 Topology Qual

Three hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state clearly that you are doing so. Please try to write good clear mathematics!

1. Let $f, g : X \to S^2$ be continuous maps such that for all x in X, $f(x)$ is not antipodal to $g(x)$. Show that f is homotopic to g.

2. Consider the space X obtained from the cylinder $S^1 \times I$ by identifying antipodal points of the circle $S^1 \times \{0\}$, and similarly identifying antipodal points of $S^1 \times \{1\}$. Calculate the fundamental group of X.

3. Assume that X is a path-connected, locally simply-connected space with fundamental group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. How many path-connected covering spaces of X are there, up to equivalence (isomorphism)?

4. Consider the set L of 3-manifolds which can be formed by gluing together the boundaries of two solid tori $S^1 \times B^2$ using a homeomorphism. Consider the function $d : L \to \mathbb{N}$ given by the total dimension of the mod-2 homology: $d(M) = \sum \dim H_i(M; \mathbb{Z}_2)$. What is the maximal value of d on L?

5. Let M be a closed orientable 4-manifold whose second homology $H_2(M; \mathbb{Z})$ has rank 1. Show that there does not exist a free action of the group $\mathbb{Z}_2$ on M.

6. Let P be a 3-manifold whose fundamental group has order 120 and whose universal cover is S^3. Compute π_3 of the one-point union $P \vee S^3$.

7. Let $L(p)$ be a space whose integral homology groups are $\mathbb{Z}, \mathbb{Z}_p, 0, \mathbb{Z}$ in dimensions $0, 1, 2, 3$, and zero otherwise. Let Σ denote the suspension of a space. Compute the cohomology $H^*(\Sigma L(p) \times \Sigma L(q); \mathbb{Z})$, where p and q are coprime.

8. What are the integer cohomology rings of $S^2 \times S^2$ and $\mathbb{C}P^2$? Show that there is no map $S^2 \times S^2 \to \mathbb{C}P^2$ having odd degree.