2010 Topology Qual

Three hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state clearly that you are doing so. Please try to write good clear mathematics!

1. Let M be a simply connected n-dimensional CW complex. Show that any map from M to $\mathbb{R}P^{n+1}$ is homotopic to the constant map.

2. How many distinct double covers does $\mathbb{R}P^3 \times S^1$ have? Can you identify any of them?

3. Let $n \geq 2$ be a positive integer, and let k be in the range $0 < k < n$. Let $X = \mathbb{C}P^n/\mathbb{C}P^k$ be the quotient space obtained from $\mathbb{C}P^n$ by identifying its subspace $\mathbb{C}P^k$ to a point. Calculate the integral cohomology ring of X. (You may assume the cohomology ring of $\mathbb{C}P^n$.)

4. For which n and k (as above) is $X = \mathbb{C}P^n/\mathbb{C}P^k$ homotopy equivalent to a manifold?

5. For any topological space X, whose total homology is a finitely-generated abelian group, let $\chi(X)$ denote the usual Euler characteristic

$$\chi(X) = \sum(-1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q})$$

and let $\chi_2(X)$ be the "mod-2 homology Euler characteristic"

$$\chi_2(X) = \sum(-1)^i \dim_{\mathbb{Z}_2} H_i(X;\mathbb{Z}_2).$$

Use the universal coefficient theorem to show that $\chi(X) = \chi_2(X)$.

6. Let X be a path-connected space with $\pi_{\geq 2}(X) = 0$ and whose fundamental group is a free group on a set S. Show that there is a homotopy equivalence between a bouquet of circles, indexed by S, and X.

7. Show that a closed orientable surface Σ of genus $g \geq 1$ has $\pi_{\geq 2}(\Sigma_g) = 0$, and deduce that the fundamental group of Σ_g is not a free group.

8. Let L be a solid 3-dimensional lens (a flattened ball). Identify the top and bottom surfaces via vertical translation and a twist of 120 degrees, as shown in the picture. Calculate the integral homology of the resulting space.