Topology Qualifying exam, Spring 2008

You have three hours to answer these questions. No notes or books are allowed. All the best.

1. (20pts.) Calculate the fundamental group of $\mathbb{RP}^2 \# \mathbb{RP}^2$.

2. (20pts.) Given a topological space X, let I denote the directed set of compact subsets of X under inclusion. Show that the following canonical map is an isomorphism:

$$\mathrm{colim}_{K \in I} H_i(K; R) \longrightarrow H_i(X; R),$$

   for any ring R.

3. (20pts.) Show that there is no continuous map f with the following properties:

$$f : S^n \longrightarrow S^m, \quad f(-x) = -f(x), \quad n > m \geq 1,$$

   where $-x$ denotes the antipode of x.

4. (20pts.) Let M^n and N^n be compact, oriented, boundaryless manifolds of dimension n with fundamental classes μ_M and μ_N respectively. Assume that $f : M^n \to N^n$ is a continuous map, and $U \subseteq N^n$ is an open set with the property that $f^{-1}(U) = \coprod_{i=1}^k U_i$ is a finite disjoint union, such that f restricts to an orientation preserving homeomorphism $f : U_i \to U$. Show that $f_*(\mu_M) = k\,\mu_N$.

5. (10pts.) Given a compact, oriented, boundaryless n-manifold M^n, show that there always exists a continuous map $f : M^n \to S^n$, such that $f_*(\mu_M) = \mu_{S^n}$.

6. (10pts.) Calculate the integral homology of any compact, oriented, boundaryless manifold N^n, which admits a continuous map $f : S^n \to N^n$ with $f_*(\mu_{S^n}) = \mu_N$.

7. (20pts.) Let $T^2 = S^1 \times S^1$ be the 2-torus. Let $f : T^2 \to T^2$ be a self-map of degree 1. Show that the map $f^* : H^1(T^2, \mathbb{Z}) \to H^1(T^2, \mathbb{Z})$ defines an element of $\mathrm{SL}_2(\mathbb{Z})$. Here, we identify $H^1(T^2, \mathbb{Z})$ with $\mathbb{Z} \oplus \mathbb{Z}$ using the identification $T^2 = S^1 \times S^1$.