## 44. Fall 2024

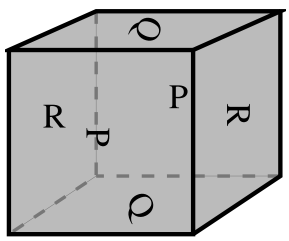
*Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students - pictures aid explanation but should not replace it! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.*

**1.** Let $\Sigma_g$ be the closed orientable surface of genus $g \geq 0$. Show that if $g < h$, there does not exist a degree 1 map $\Sigma_g \to \Sigma_h$.

**2.** Give an example (a) of a space $X$ with $\pi_2(X) = 0$ and $H_2(X) \neq 0$ and (b) of a space $Y$ with $\pi_2(Y) \neq 0$ and $H_2(Y) = 0$.

**3.** Let $K$ be a CW-complex with one 0-cell $v$, and let $f : (K, v) \to (K, v)$ be a basepoint-preserving map. Let $T_f$ be the mapping torus of $f$: that is, the space obtained from $K \times I$ by gluing $(x, 1) \sim (f(x), 0)$ for all $x$. Show that a presentation of $\pi_1(T_f)$ may be obtained from a presentation of $\pi_1(K)$ by adjoining one new generator $t$ and certain relations involving $f_*$.

**4.** Let $X$ be the space obtained by gluing opposite pairs of faces of a standard cube $I^3$ via 90 degree rotations, as shown. Compute the homology $H_*(X; \mathbb{Z})$.




**5.** Let $\Sigma$ be a one-holed torus (delete an open disc from a standard torus) whose boundary is identified with the standard $S^1$. Let $(X, x_0)$ be a based path-connected space, $\alpha$ a loop based at $x_0$, and $[\alpha]$ the corresponding class in $\pi_1(X, x_0)$. Show that $[\alpha]$ may be expressed algebraically as a commutator $yzy^{-1}z^{-1}$ of two elements $y, z \in \pi_1(X, x_0)$ if and only if "$\alpha$ extends to a map of $\Sigma$", meaning that $\alpha$ may be expressed as $f|_{\partial\Sigma}$ for some map $f : \Sigma \to X$. Is there an analogue of this result for one-holed higher-genus surfaces?

**6.** Use Mayer-Vietoris to show that the reduced homology of $S^n$ is $\mathbb{Z}$ in dimension $n$ and zero otherwise, for all $n \geq 0$.

**7.** What is the fundamental group of $X = \mathbb{R}P^3 \# \mathbb{R}P^3$? Give a description of the universal cover $\tilde{X}$ of $X$ and use this to calculate $\pi_2(X)$.

**8.** Compute $\mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}_4, \mathbb{Z}_6 \oplus \mathbb{Z}_8)$.