Topology Qualifying exam, Fall 2008

You have three hours to answer these questions. No notes or books are allowed. All the best.

1. (20pts.) Construct a connected two dimensional CW complex X with fundamental group with the presentation:
$$\pi_1(X) = \langle a, b \,|\, a^2 = b^3.\rangle$$

2. (20pts.) Let p be a prime integer, let $M(p)$ denote the space obtained as the identification space of the 2-disc D^2 under the identification:
$$M(p) = \mathrm{D}^2/\sim, \quad x \sim y, \quad \text{if} \quad x, y \in \partial\mathrm{D}^2 = S^1, \quad \text{and} \quad x = e^{2\pi i n/p}\, y, \quad n \in \mathbb{Z}.$$
Find the values of p for which $M(p)$ is homotopy equivalent to a compact boundaryless manifold.

3. (20pts.) Can there exist a map f of degree ± 1 of the form:
$$f : \mathbb{CP}^n \longrightarrow S^{n+1} \times S^{n-1}, \quad n > 1.$$
Prove your answer.

4. (20pts.) Calculate $\pi_n(\mathbb{RP}^n \vee S^n)$ for $n > 1$.

5. (20pts.) Calculate the mod 2 cohomology ring of the space $X(m,n) = \mathbb{CP}^m \times \mathbb{RP}^n$, where m, n are positive integers. Show that $X(m,n)$ is homotopy equivalent to $X(m', n')$ if and only if $(m,n) = (m', n')$.