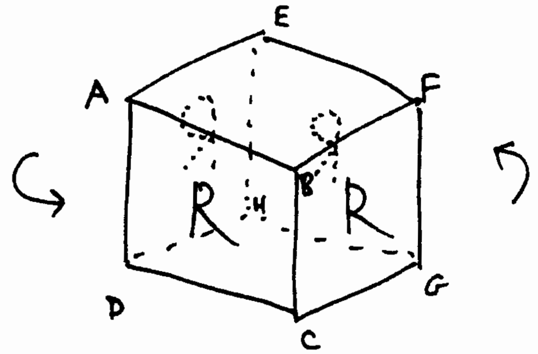
Fall 2007 Topology Qual

Three hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state clearly that you are doing so. Please try to write good clear mathematics!

1. Consider a solid cube. Four of the faces are identified together by means of rigid rotations, as pictured below. (For example, the face $ABCD$ is identified with $BFGC$ via an affine map preserving the order of vertices.) Compute the fundamental group of the resulting quotient space.



2. Let Σ_g be the closed orientable surface of genus g, that is the "g-holed torus". Describe all the possible covering spaces of the form $\Sigma_g \to \Sigma_h$, where $1 \leq g, h \leq 4$, and explain why these are the only possibilities.

3. Let X be a space whose homology groups are $\mathbb{Z}, 0, \mathbb{Z}_6$ in dimensions $0, 1, 2$ and zero otherwise. Compute the integral homology $H_*(X \times \mathbb{R}P^3; \mathbb{Z})$.

4. Let K be a (perhaps knotted) subspace of S^5 which is homeomorphic to the 3-sphere. Let N be a closed regular neighbourhood of K, so that N is homotopy equivalent to K. Let X be S^5 minus the interior of N, so that X is a compact 5-manifold with boundary. By considering the relative cohomology $H^*(S^5, N)$ and applying excision and Lefschetz duality, calculate the homology of X.

5. Let M be a closed (that is, compact and without boundary) path-connected orientable 3-manifold. Suppose that M contains a 2-dimensional orientable submanifold Σ which is *non-separating*, meaning that $M - \Sigma$ is still path-connected. Show that $H_1(M; \mathbb{Z})$ contains a subgroup isomorphic to $\mathbb{Z}$.

6. Use the Hurewicz theorem to calculate $\pi_3(\mathbb{R}P^4 \vee S^3)$.

7. Let W be a closed (i.e. compact, without boundary) 4-manifold which is 1-connected (i.e is path-connected and simply-connected). Show that its second homology group is a free abelian group (in other words, has no finite cyclic summands).

8. Show that the Euler characteristic of a closed orientable odd-dimensional manifold is zero. Is this still true if the manifold is non-orientable?