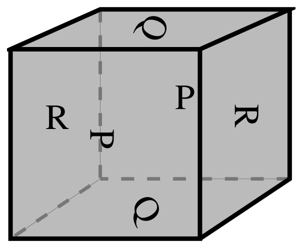
## 19. Summer 2015

**1.** Let $X$ be the space obtained by gluing opposite pairs of faces of a standard cube $I^3$ via 90 degree rotations, as shown. Compute the homology $H_*(X;\mathbb{Z})$.




**2.** Prove that there is no compact 4-manifold $M$ (with or without boundary) which is homotopy-equivalent to the suspension $\Sigma\mathbb{R}P^3$.

**3.** Prove that any map $\mathbb{R}P^2 \to T^2$ must be null-homotopic.

**4.** Let $X_n$ be a space whose homology groups are given by $H_k(X_n;\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$ for $0 \leq k \leq n$ and which vanish for $k > n$. Compute the homology $H_*(X_3 \times X_5;\mathbb{Z})$.

**5.** Let $\Sigma_3$ be the closed orientable surface of genus 3. Suppose $\mathbb{Z}_3$ acts on $\Sigma_3$; show that there must be at least two fixed points.

**6.** Let $M$ be a compact 3-manifold-with-boundary such that $H_1(M;\mathbb{Z})$ contains a torsion element (i.e. element of finite order). Prove that $M$ cannot be embedded as a submanifold of $S^3$. (Hint: if it could, we could decompose $S^3 = M \cup_\Sigma N$ as a union of two compact 3-manifolds, glued along their common boundary surface $\Sigma$.)

**7.** Let $q : S^3 \to \mathbb{R}P^3$ be the usual quotient map which identifies antipodal points. It can be used to attach a 4-ball to $\mathbb{R}P^3$, forming a space $X = \mathbb{R}P^3 \cup_q B^4$. Compute $\pi_4(X)$.

**8.** Suppose $X$ is a path-connected CW-complex with $\pi_1(X) \cong \mathbb{Z}^2$ and $\pi_{\geq 2}(X) = 0$. Show that $X$ is homotopy-equivalent to $S^1 \times S^1$. Use this to show that the fundamental group of a closed orientable surface $\Sigma_g$ of genus $g \geq 2$ cannot contain a subgroup isomorphic to $\mathbb{Z}^2$.