Question 1: This question is about random sampling from the 5-parameter Bivariate normal model: $(X_i, Y_i) \sim$ iid $N_2\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right)$; $i = 1, \dots, n$, where ρ is the correlation coefficient, and σ_i^2 are the marginal variances. We also introduce the usual symbol $\sigma_{12} = \rho\sigma_1\sigma_2$ for the covariance. The density is written on the blackboard.

(i) Show, quoting relevant theorems, that $(\Sigma X_i, \Sigma Y_i, \Sigma(X_i-\bar{X})^2, \Sigma(Y_i-\bar{Y})^2, \Sigma(X_i-\bar{X})(Y_i-\bar{Y}))$ is a complete sufficient statistic for this problem.

(ii) Find, quoting relevant theorems, the UMVUE for σ_{12}. Explain with a brief explanation (no messy detail) if it is possible to find the UMVUE for σ_i. If so show that the reciprocal of the UMVUE for σ_i is biased for $\frac{1}{\sigma_i}$. How do you know the direction of the bias? Explain whether that bias is removable.

(iii) Show that in the one-parameter submodel $\mu_1 = \mu_2 = 0$; $\sigma_1 = \sigma_2 = 1$ $P[X_1 > 0 \;\&\; Y_1 > 0] = \frac{1}{4} + \frac{1}{2\pi}\arcsin\rho$ (even if you can't do this part you can still do what follows).

(iv) Show that although U-estimation of ρ is problematical or impossible, in the submodel a reasonable simple U-estimate of $\arcsin\rho$ is available. How do you know it is not UMVUE? Does a

UMVUE exist? explain briefly. Is your simple U-estimate of $\arcsin \rho$ consistent? $\sqrt{n}$-consistent? Does it have positive ARE compared to the MLE ($\arcsin \hat{\rho}$ where $\hat{\rho}$ is the "usual estimator")?

(v) Give a general proof that a UMVUE is unique.

(vi) Back in the big 5-parameter model one sometimes reads an unrigorous statement to the effect that $\hat{\rho} \mathrel{\dot\sim} \mathcal{N}(\rho, \frac{1}{n}(1-\rho^2)^2)$ where $\dot\sim$ means "is approximately distributed". How would you state this rigorously with a specified mode of convergence?

<u>Question II</u>. $X_1, \dots, X_n \sim$ iid $\mathcal{N}(\mu, \sigma_X^2)$, and independently $Y_1, \dots, Y_n \sim$ iid $\mathcal{N}(\mu, \sigma_Y^2)$. Give a careful mathematical proof that there is no complete sufficient statistic for this problem. (It is not enough to write the joint density in exponential-family form, and then remark on the apparent impossibility of reducing the exponent to fewer than four terms.)

<u>Question III</u>. Derive the UMVUE of $e^{-\lambda}$ from $X_1, \dots, X_n \sim$ iid $\text{Pois}(\lambda)$. With explanation, give its asymptotic distribution.

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Be concise and clear. Vagueness and confusing text will be penalized. Make sure to define any math symbol that you use, unless it was used over and over in lecture.

Problem 1. Suppose that we are testing P_0 vs P_1 (simple vs simple) at level $\alpha = 0.10$ based on a realization of X taking values in the discrete sample space $\mathcal{X} = \{1, \dots, 10\}$.

x	1	2	3	4	5	6	7	8	9	10
P_0	0.02	0.03	0.05	0.07	0.09	0.09	0.10	0.10	0.10	0.35
P_1	0.14	0.01	0.02	0.36	0.00	0.02	0.10	0.23	0.09	0.03
P_1/P_0	7.00	0.33	0.40	5.14	0.00	0.22	1.00	2.30	0.90	0.09

A. First, suppose we allow tests to be randomized. Derive all the UMP tests in this case. Explain what you are doing (and state any result you are using). Calculations need not be simplified.

B. Next, suppose we do *not* allow tests to be randomized and repeat. Calculations need not be simplified. (And if they are too hard to write explicitly, explain what you would do if you had access to a computer.)

Problem 2. Consider the setting where we draw n balls from an urn containing r red balls out of v total (known).

A. Show that there is a UMP test at level α (arbitrary in $(0, 1)$) for testing $r \leq r_0$ versus $r > r_0$, where $0 \leq r_0 \leq v - 1$. If not explain why. (State any result you use.)

B. Concretely, assume that $n = 5$ and $v = 20$, and that $r_0 = 10$, and that we are testing at level 0.05. Write down the UMP test, or any other reasonable test (define it first) for that particular setting. (The following are rounded to the nearest 2 digits.)

k	0	1	2	3	4	5
$\frac{\binom{10}{k}\binom{10}{5-k}}{\binom{20}{5}}$	0.02	0.14	0.35	0.35	0.14	0.02

Problem 3. Let $X_1, \ldots, X_m$ be iid Bernoulli(p) and (independently) let $Y_1, \ldots, Y_n$ be iid Bernoulli(q). We want to test $p \leq q$ versus $p > q$ at level α.

A. Write down the likelihood.

B. Is there a UMPU test? Explain and specify the result(s) that you use. If there is one, exhibit such a test (be as precise as you can).

Problem 4. Consider an experiment that happens in two stages. In Stage 1, a sample size N is drawn according to the geometric distribution with parameter λ (unknown in $(0,1)$), so that $P_\lambda(N=n) = (1-\lambda)^{n-1}\lambda$ for $n \geq 1$. Conditional on $N = n$, in Stage 2, we draw $X_1, \dots, X_n$ iid from the exponential distribution with parameter θ (unknown in $(0,\infty)$) which has density $\theta\exp(-\theta x)$ over $x > 0$.

A. Write down the likelihood. Specify the dominating measure.

B. Consider testing $\theta \leq \theta_0$ at level α, where $\theta_0 > 0$ is given. Is there a UMPU test? Explain and specify the result(s) that you use. If there is one, exhibit such a test (be as precise as you can).