Math 281 A,B **Qualifying Exam.** **March 1999**

(1) Derive the general formula for the expectation of a quadratic form in terms of a random vector where all second moments exist.

Then write down the standard estimator of σ^2 for the (equal variances) linear model and show that this estimator is unbiased.

(2) Given a random sample from the Poisson distribution, denote by M and V the sample mean and sample variance, respectively. What is E(V|M) ? (Don't make a major project out of this !)

(3) A discrete random variable X takes on exactly two values, namely, -1 and +1 where P[X=-1] = P[X=+1] = 1/2.

a. Show that the mean is 0 and variance is 1.

b. Show that the characteristic function is given by

$$\varphi(t) = \cos(t) \ .$$

c. Finally, if $X_1, X_2, \dots, X_n$ are i.i.d. from this distribution, Use the result of (3) b. to show that

$$\sqrt{n}\ \overline{X}_n \xrightarrow{L} Z \sim N(0,1) \ .$$

(4) Discuss (briefly!) the key results that justify your work in (3) c.

(5) Write down a complete statement of the Central Limit Theorem in the Euclidean Space setting. Then write out a proof (for d=1).

(6) Given below is a "Slutsky-like" theorem and its proof. As you read the proof, provide justification for steps (a) through (d).

Theorem: Let $\{X_n, Y_n\}$ be a sequence of pairs of random variables.

$$\text{Then } |X_n - Y_n| \xrightarrow{P} 0 \text{ and } Y_n \xrightarrow{L} Y \Rightarrow X_n \xrightarrow{L} Y.$$

Proof: Denote by F_{X_n} the c.d.f. of X_n and by F_Y that of Y. Let $Y_n - X_n = Z_n$, and let x be a continuity point of F_Y. Then

$$F_{X_n}(x) = P(X_n < x) = P(Y_n < x + Z_n)$$

$$= P(Y_n < x + Z_n,\ Z_n < \varepsilon) + P(Y_n < x + Z_n,\ Z_n \geq \varepsilon)$$

(a) $$\leq P(Y_n < x + \varepsilon) + P(Z_n \geq \varepsilon).$$

Taking limits, we obtain

(b) $\limsup_n F_{X_n}(x) \leq F_Y(x+\varepsilon)$.

Similarly, we have

(c) $\liminf_n F_{X_n}(x) \geq F_Y(x-\varepsilon)$. (Go through the analogue of the development that led to (a) and (b).)

(d) Finally we obtain $\lim_n F_{X_n}(x) = F_Y(x)$ and the proof is complete.

(7) Use the theorem in Exercise (6) to show that

$$X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{L} X.$$