Math 281AB Qualifying Exam

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Let $\mathcal{G}$ be a convex and symmetric class of functions $g : \mathbb{R} \to \mathbb{R}$ equipped with norm $\|\cdot\|_{\mathcal{G}}$. Consider a class of functions over $\mathbb{R}^d$ as follows

$$\mathcal{F}_{\text{add}} = \left\{ f : \mathbb{R}^d \to \mathbb{R} | f = \sum_{j=1}^{d} g_j \text{ for some } g_j \in \mathcal{G} \text{ with } \|g_j\|_{\mathcal{G}} \leq 1 \right\}.$$

Suppose we have n, i.i.d. samples of the form

$$y_i = f^*(x_i) + \sigma\varepsilon_i,$$

where each $x_i = (x_{i1}, \cdots, x_{id}) \in \mathbb{R}^d$, ε_i are mean zero sub-Gaussian random variables with sub-Gaussian parameter σ.

Suppose $f^* = \sum_{j=1}^{d} g_j^*$ for some $g_j \in \mathcal{G}$. We estimate f^* by constrained least-squares estimate

$$\hat{f} = \arg \min_{f \in \mathcal{F}_{\text{add}}} \left\{ n^{-1} \sum_{i=1}^{n} (y_i - f(x_i))^2 \right\}.$$

Find a high-probability, tight, upper bound on the estimation error $\|\hat{f} - f^*\|_n^2$, if we suppose that there exists a constant $K \geq 1$ such that the following inequality holds

$$\sum_{j=1}^{d} \|g_j\|_n^2 \leq K \| \sum_{j=1}^{d} g_j \|_n^2.$$

NOTE: All details of the computation must be present. Examples and Exercises from the book cannot be used as statements. Theorems and Propositions/Lemmas can be used as statements when and if bounding all the needed terms. Provide ALL the details of your work. Write legibly: points will be taken off if it is impossible to read what was written. Submit your work by sending an email to **jbradic@ucsd.edu**.

QUAL EXAM: MATH 281C – Spring 2022

Problem 3. (Hypothesis Testing for Exponential Distributions)

Let $X_1, \ldots, X_n$ be a sample from a distribution with exponential density $a^{-1}e^{-(x-b)/a}$ for $x \geq b$. Denote by $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ the order statistics of $\{X_i\}_{i=1}^n$.

(a) For testing $H_0 : a = 1$, show that there exists a UMP unbiased test given by the acceptance region

$$C_1 \leq 2\sum_{i=1}^{n}(X_i - X_{(1)}) \leq C_2,$$

where the test statistics has a χ^2-distribution with $2n-2$ degrees of freedom when $\alpha = 1$, and C_1, C_2 are determined by

$$\int_{C_1}^{C_2} \chi^2_{2n-2}(y)\mathrm{d}y = \int_{C_1}^{C_2} \chi^2_{2n}(y)\mathrm{d}y = 1 - \alpha.$$

(b) For testing $H_0 : b = 0$, show that there exists a UMP unbiased test given by the acceptance region

$$0 \leq \frac{nX_{(1)}}{\sum_{i=1}^{n}(X_i - X_{(1)})} \leq C.$$

When $b = 0$, the test statistics had probability density

$$p(u) = \frac{n-1}{(1+u)^n}, \quad u \geq 0.$$

Hint: You may use the following property to solve the above question. Define random variables

$$Z_1 = n(X_{(1)} - b), \quad Z_i = (n-i+1)(X_{(i)} - X_{(i-1)}),\ i = 2, \ldots, n.$$

Then $2Z_1/a, 2Z_2/a, \ldots, 2Z_n/a$ are independently distributed as χ^2 with 2 degrees of freedom.