Mathematical Statistics (281ABC) - Qualifying Exam, May 26, 2009

Problem 1 (a) Define an exponential family, including the natural parameter(s) and the associated statistic(s);

(b) Describe two useful properties of the exponential family, explain why they are useful (be as specific as you can).

Problem 2 (a) Verify that the Poisson distribution with probability function

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad \lambda > 0, \; x = 0, 1, 2, \ldots$$

belongs to the exponential family;

(b) Given data $X_1, \ldots, X_n$ i.i.d. Poisson(λ), find the UMVU estimator of λ^k, for $k = 1, 2, \ldots$;

(c) For the same data, find the UMVU estimator of $e^{-\lambda} = P(X = 0)$.

Problem 3 (a) Define the score test for a random sample from a parametric family of distributions; describe what hypotheses it tests, and give its asymptotic distribution under the null hypothesis (when regularity conditions are met). What do you know about its distribution under the alternative hypothesis?

(b) Describe Pearson's chi-squared goodness-of-fit test for a multinomial distribution with known probabilities.

(c) Show that the above Pearson's test is a score test.

Problem 4 Let (X_i, δ_i), $i = 1, \ldots, n$, be independent, possibly right-censored survival data, where $X_i = \min(T_i, C_i)$, $\delta_i = I(T_i \leq C_i)$. Assume that C_i follows the random censorship assumption, and is independent of T_i. Assume that T_i follows the exponential distribution with hazard $\lambda_i = \exp(\beta' z_i) = \exp(\beta_0 + \beta_1 z_{i1} + \ldots + z_{ip})$, where z_i is a vector of subject specific covariates, and β is a vector of unknown parameters.

(a) Write down the likelihood function $L(\beta)$ for this data;

(b) Does the likelihood function have a unique maximum? under what conditions?

(c) Derive the maximum likelihood estimate (MLE) $\hat{\beta}$ of β, and explain how you would estimate its variance.

(d) Now consider the special case of two group comparison, i.e. $z_i = 0$ for the control group, and $z_i = 1$ for the experimental group. Provide the formula for MLE of β_0 and β_1 and their estimated variances (Hint: you can also do this part directly, regardless of the previous parts). How does this relate to the one-sample exponential estimation?