Math 281 Qualifying Exam. May 24, 2004

(1) Assume that $X \sim N_3(\mu, \Sigma)$ in which $\mu = (1,1,2)^T$ and

$$\Sigma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In addition, let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = (2 \ \ 1)^T, \text{ and } \ c = (-1 \ \ 1)^T.$$

a. What is the distribution of AX + b?

b. Compute Cov(AX + b, CX + d).

(2) Suppose $X_1, X_2, \dots, X_n$ are i.i.d. random variables (real) from a distribution whose first two moments are finite. The sample mean and variance are given by $\overline{X}_n$ and S_n^2, respectively. Derive the limiting distribution of

$$\left(\frac{\sqrt{n}\,(\overline{X}_n - \mu)}{S_n} \right)^2 .$$

(3) Now we have a sequence $X_1, X_2, \dots, X_n, \dots$ of (3×1) i.i.d. random vectors, $X_i = (X_i^{(1)}, X_i^{(2)}, X_i^{(3)})^T$, from some distribution with mean vector μ and covariance matrix Σ as specified in Exercise (1). Derive the limiting distribution of

$$U_n = \left(\overline{X}^{(1)}\right)^2 + \left(\overline{X}^{(2)}\right)^2 + \left(\overline{X}^{(3)}\right)^2.$$

Question 2.

a) Show that the "log-series" distributions on positive integers x,

$$f_\theta(x) = \frac{-1}{\log(1-\theta)} \frac{\theta^x}{x} \quad (0 < \theta < 1), \text{ form an exponential family.}$$

b) Find a complete sufficient reduction of iid. data $X_1, \ldots, X_n$ from f_θ above.

c) Show that UMVUEs are unique when they exist.

d) State the Lehmann-Scheffé theorem for construction of UMVUEs.

e) Find the UMVUE for $\frac{\theta}{\log(1-\theta)}$ based on one datum $X_1 \sim f_\theta$ above.

f) Find the UMVUE for θ based on one datum $X_1 \sim f_\theta$ above.

g) Find the UMVUE for $\frac{\theta}{\log(1-\theta)}$ based on two data $X_1, X_2 \sim$ iid f_θ.
(Don't sweat unduly trying to get it in fully "closed form" — just simplify as far as you can.)

Question 3.

The mean and variance of the gamma distributions

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad ; \quad x \geq 0 ; \ \alpha, \beta > 0,$$

are $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta^2}$ respectively (don't bother to show this).

a) Show that the gamma distributions form a conjugate family for the poissons, i.e. if $X_1, \ldots, X_n \sim$ iid pois(λ) and $\lambda \sim$ gamma a priori, then the posterior is again a gamma distribution.

b) Explain why with squared-error loss the minimization of Bayes risk leads to a "Bayes rule" which estimates a parameter by taking the mean of its posterior (i.e. here $\hat{\lambda}_{Bayes} = E[\lambda \mid \underset{\sim}{X}]$).

c) Write down explicitly the Bayes rule $\hat{\lambda}_{Bayes}$ for the poisson-gamma setup above. Note that it is a convex combination of the prior mean and the usual estimator $\hat{\lambda}_{UMVUE}$, with coefficients of the form $O(\frac{1}{n})$ and $1 - O(\frac{1}{n})$ respectively.

d) CHOOSE ONE OF THE FOLLOWING:

EITHER: If the data $X_1, \ldots, X_n$ arrive in a stream, and the Bayes estimate is computed for each n, give a recursive formula for $\hat{\lambda}^{(n)}_{Bayes}$ in terms of $\hat{\lambda}^{(n-1)}_{Bayes}$ and X_n.

OR: compare the posterior variance of λ with var $\hat{\lambda}_{Bayes}$; comment on why the larger of them generally has to be so.

e) Show that the ARE of $\hat{\lambda}_{Bayes}$ and $\hat{\lambda}_{UMVUE}$ is 1. Adopt whichever basis you like for defining ARE.