QUALIFYING EXAM: MATH 281A

PROFESSOR JELENA BRADIC

Suppose that we observe data in pairs $(x, y) \in \mathbb{R}^d \times \{\pm 1\}$ where the data come from a logistic model with $X \sim P_0$ and

$$p_{y|x}(y|x) = \frac{1}{1 + \exp\{-yx^\top \theta\}}$$

with log-loss function $l_\theta(y|x) = \log(1+\exp\{-yx^\top\theta\})$. Let $\hat{\theta}$ minimize the empirical logistic loss

$$L_n(\theta) := n^{-1} \sum_{i=1}^n l_\theta(Y_i|x_i) = n^{-1} \sum_{i=1}^n \log(1 + e^{-Y_i X_i^\top \theta})$$

for pairs (X_i, Y_i) drawn from the logistic model with parameter θ_0. Assume in addition that the data $X_i \in \mathbb{R}^d$ are i.i.d. and satisfy

$$E[X_i X_i^\top] = \Sigma > 0, \qquad \text{and} \qquad E\|X_i\|_2^4 < \infty,$$

that is the second moment of the X_i is positive definite.

(a) Let $L(\theta) = E[l_\theta(Y|X)]$ denote the population logistic loss. Show that the second order derivative evaluated at θ_0 is positive definite. You may assume that the order of differentiation and integration may be exchanged.
(b) Under these assumptions show that $\hat{\theta}$ is consistent estimator of θ_0 when $n \to \infty$. Provide details of your work (definitions, theorems used should be cited from the notes).
(c) Provide an asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta_0)$. You may assume here that $\hat{\theta}$ is consistent. Assume that $d = 1$ and even simpler setting where $x \in \{-1, 1\}$.
(d) Describe the effect of θ_0 on the efficiency of $\hat{\theta}$.

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