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Question I. $X_1, \ldots, X_n \sim_{iid}$ Poisson (λ).

a) Find the Fisher information $I(\lambda)$ for such a sample.

b) Derive (with explanation) the UMVU of i) $\text{var}_\lambda X$
ii) $\lambda e^{-\lambda}$

c) What is the Cramér-Rao bound for the variance of unbiased estimators of $\lambda e^{-\lambda}$? Does the UMVUE attain the bound? (Hint: avoid tedious sampling variance calculations for the last part.)

Question II. $X_1, \ldots, X_n \sim_{iid} \mathcal{N}(0, \sigma^2)$

Then $EX_1^{10} = 945\sigma^{10}$ (check at home),
but $\frac{1}{945n} \sum X_i^{10}$ is not UMVU for σ^{10}.

Why not? Is there a UMVUE for σ^{10}?

(I tried the above estimator on a million $\mathcal{N}(0,1)$'s, and got an estimate of 1.02 for σ^{10}. At home try to figure out ~~of~~ with what probability a "good" estimator would do this "badly.")

Question III. The Weibull family of distributions $\{W(\theta, \gamma)\}$ is a two-parameter family with c.d.f.'s $F(x) = F_{\theta,\gamma}(x) = 1 - e^{-(x/\theta)^\gamma}$; $x \geq 0$; $\theta > 0$, $\gamma > 0$.

a) If $X \sim W(\theta, \gamma)$ find cdf.s for i) X^γ, ii) $\log X$
(log X is said to have a "Type I extreme value distribution for minima".)

b) If $X \sim W(\theta, \gamma)$ give the density of X.

c) Define sufficiency and completeness of statistics (in the context of a "statistical problem").

Determine whether the following d)-i) are true or false (explain either way)

d) The exponential distributions $\{E(\lambda)\}$ (having cdf. $1-e^{-\lambda x}$; $x \geq 0$, some $\lambda > 0$) form a subfamily of the Weibulls. (?)

e) The Weibulls form an exponential family. (?)

If $X_1, \dots, X_n$ $(n>1)$ $\sim_{iid}$ $E(\lambda)$, then

f) $\bar{X}$ is complete (?)

g) $\bar{X}$ is sufficient (?)

(suggest an alternative statistic to make either of these facts true if it is false.)

If $X_1, \dots, X_n$ $(n>1)$ $\sim_{iid}$ $W(\theta, \gamma)$, then

h) $\bar{X}$ is complete (?)

i) $\bar{X}$ is sufficient. (?)

(suggest an alternative statistic to make either of these facts true if it is false.)

Question 4.

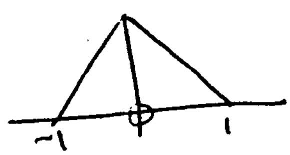
What is meant by saying an estimator is minimax?

We showed that for data $X_1, \dots, X_n \sim$ iid $\mathcal{N}(\mu, 1)$ $(\mu \in \mathbb{R})$ $\bar{X}$ was unique minimax. Show that if the parameter space $\mathbb{R}$ is replaced by $[0, \infty)$ $\bar{X}$ is still minimax but no longer uniquely. Give an example with explanation of another minimax estimator (different from $\bar{x}$ with probability > 0). Explain why $\bar{X}$ is not admissible when $\mu \in [0, \infty)$.

Now further restricting the parameter space to a bounded interval, $[0,1]$ say, show that $\bar{x}$ is neither admissible

nor minimax. Use a Bayesian idea to <u>sketch</u> the construction of an admissible estimator for this problem whose support is the whole interval $[0,1]$ (ie. avoid trivial constant estimators).

Question 5.

$f_0(x)$ is  ; $X_1, \dots, X_n \sim$ iid $f_\theta(x)$, where

$f_\theta(x) = f_0(x - \theta)$. Compute the ARE of the sample mean to the sample median for estimating θ, and also the ARE of the better of these to the MLE

Provide concise and precise answers (vagueness will be penalized). Make sure the presentation is neat (easy to read). Define any notation that you introduce. Name any result you use.

Problem 1. Below $u : \mathbb{R} \mapsto \mathbb{R}$ is measurable. The underlying measure is Lebesgue's. Show that $p_\theta(x) = f(x - \theta)$, with $f(x) = C(\beta)\exp(-|x|^\beta)$, is QMD when $\beta > 1/2$.

Problem 2. Let $P_n = \mathcal{N}(0, I_n)$ and $Q_n = \mathcal{N}(\xi_n, I_n)$, multivariate normal distributions in dimension n with covariance matrix I_n (the identity matrix). Find a necessary and sufficient condition for P_n and Q_n to be contiguous as $n \to \infty$.

Problem 3. Consider a density f with respect to the Lebesgue measure on $\mathbb{R}$ which differentiable, has zero median[1], and such that the location family $\{f(\cdot - \theta) : \theta \in \mathbb{R}\}$ is QMD. We want to test $\theta = 0$ versus $\theta > 0$ based on an IID sample $X_1, \dots, X_n$ from $f(\cdot - \theta)$.

1. Define the sign test with asymptotic level α. (Be explicit so the test could be implemented using your description.)
2. Compute the asymptotic power of the test against an alternative of the form $\theta = h/\sqrt{n}$ with $h > 0$ fixed.

[1] Equivalently, $\int_{-\infty}^{0} f(x)\mathrm{d}x = 1/2$.

Theorem 11.2.4 (Multivariate Central Limit Theorem) *Let* $X_n^T = (X_{n,1}, \ldots, X_{n,k})$ *be a sequence of i.i.d. random vectors with mean vector* $\mu^T = (\mu_1, \ldots, \mu_k)$ *and covariance matrix* Σ. *Let* $\overline{X}_{n,j} = \frac{1}{n}\sum_{i=1}^n X_{i,j}$. *Then*

$$(n^{1/2}(\overline{X}_{n,1} - \mu_1), \ldots, n^{1/2}(\overline{X}_{n,k} - \mu_k))^T \xrightarrow{d} N(0, \Sigma) .$$

Theorem 11.2.5 (Lindeberg Central Limit Theorem) *Suppose, for each* n, $X_{n,1}, \ldots, X_{n,r_n}$ *are independent real-valued random variables. Assume* $E(X_{n,i}) = 0$ *and* $\sigma_{n,i}^2 = E(X_{n,i}^2) < \infty$. *Let* $s_n^2 = \sum_{i=1}^{r_n} \sigma_{n,i}^2$. *Suppose, for each* $\epsilon > 0$,

$$\sum_{i=1}^{r_n} \frac{1}{s_n^2} E[X_{n,i}^2 I\{|X_{n,i}| > \epsilon s_n\}] \to 0 \qquad \text{as } n \to \infty. \tag{11.11}$$

Then, $\sum_{i=1}^{r_n} X_{n,i}/s_n \xrightarrow{d} N(0,1)$.

Theorem 11.2.11 (Slutsky's Theorem) *Suppose* $\{X_n\}$ *is a sequence of real-valued random variables such that* $X_n \xrightarrow{d} X$. *Further, suppose* $\{A_n\}$ *and* $\{B_n\}$ *satisfy* $A_n \xrightarrow{P} a$, *and* $B_n \xrightarrow{P} b$, *where* a *and* b *are constants. Then,* $A_n X_n + B_n \xrightarrow{d} aX + b$.

Definition 12.2.1 The family $\{P_\theta, \theta \in \Omega\}$ is *quadratic mean differentiable* (abbreviated q.m.d.) at θ_0 if there exists a vector of real-valued functions $\eta(\cdot, \theta_0) = (\eta_1(\cdot, \theta_0), \ldots, \eta_k(\cdot, \theta_0))^T$ such that

$$\int_{\mathcal{X}} \left[\sqrt{p_{\theta_0 - h}(x)} - \sqrt{p_{\theta_0}(x)} - < \eta(x, \theta_0), h >\right]^2 d\mu(x) = o(|h|^2) \tag{12.5}$$

as $|h| \to 0$.[1]

Definition 12.2.2 For a q.m.d. family with derivative $\eta(\cdot, \theta)$, define the *Fisher Information matrix* to be the matrix $I(\theta)$ with (i,j) entry

$$I_{i,j}(\theta) = 4 \int \eta_i(x, \theta)\eta_j(x, \theta)\, d\mu(x) .$$

Lemma 12.2.1 *Assume* $\{P_\theta, \theta \in \Omega\}$ *is q.m.d. at* θ_0. *Let* $h \in \mathbb{R}^k$.

(i) Under P_{θ_0}, $\langle \frac{\eta(X,\theta_0)}{p_{\theta_0}^{1/2}(X)}, h \rangle$ *is a random variable with mean 0; i.e., satisfying*

$$\int p_{\theta_0}^{1/2}(x) \langle \eta(x, \theta_0), h \rangle d\mu(x) = 0 .$$

(ii) The components of $\eta(\cdot, \theta_0)$ *are in* $L^2(\mu)$; *that is, for* $i = 1, \ldots, k$,

$$\int \eta_i^2(x, \theta_0)\, d\mu(x) < \infty .$$

Theorem 12.2.2 *Suppose Ω is an open subset of $\mathbb{R}^k$, and P_θ has density $p_\theta(\cdot)$ with respect to a measure μ. Assume $p_\theta(x)$ is continuously differentiable in θ for μ-almost all x, with gradient vector $\dot{p}_\theta(x)$ (of dimension $1 \times k$). Let*

$$\eta(x,\theta) = \frac{\dot{p}_\theta(x)}{2p_\theta^{1/2}(x)} \tag{12.9}$$

if $p_\theta(x) > 0$ and $\dot{p}_\theta(x)$ exists, and set $\eta(x,\theta) = 0$ otherwise. Assume the Fisher Information matrix $I(\theta)$ exists and is continuous in θ. Then, the family is q.m.d. with derivative $\eta(x,\theta)$.

Definition 12.3.1 Let P_n and Q_n be probability distributions on $(\mathcal{X}_n, \mathcal{F}_n)$. The sequence $\{Q_n\}$ is *contiguous* to the sequence $\{P_n\}$ if $P_n(E_n) \to 0$ implies $Q_n(E_n) \to 0$ for every sequence $\{E_n\}$ with $E_n \in \mathcal{F}_n$.

Theorem 12.3.2 *The following are equivalent characterizations of $\{Q_n\}$ being contiguous to $\{P_n\}$.*

(i) For every sequence of real-valued random variables T_n such that $T_n \to 0$ in P_n-probability, it also follows that $T_n \to 0$ in Q_n-probability.

(ii) For every sequence T_n such that $\mathcal{L}(T_n|P_n)$ is tight, it also follows that $\mathcal{L}(T_n|Q_n)$ is tight.

(iii) If G is any limit point [3] of $\mathcal{L}(L_n|P_n)$, then G has mean 1.

Corollary 12.3.2 *Assume that, under P_n, $(T_n, \log(L_n)) \xrightarrow{d} (T, Z)$, where (T, Z) is bivariate normal with $E(T) = \mu_1$, $Var(T) = \sigma_1^2$, $E(Z) = \mu_2$, $Var(Z) = \sigma_2^2$ and $Cov(T, Z) = \sigma_{1,2}$. Assume $\mu_2 = -\sigma_2^2/2$, so that Q_n is contiguous to P_n. Then, under Q_n, T_n is asymptotically normal:*

$$\mathcal{L}(T_n|Q_n) \xrightarrow{d} N(\mu_1 + \sigma_{1,2}, \sigma_1^2) \ .$$