Mathematical Statistics (281ABC)
Qualifying Exam, September 7, 2010

Problem 1 Let X_1, ..., X_n be i.i.d. with density function

$$f(x;\theta) = \theta x^{\theta-1}, \ \ 0 < x < 1, \theta > 0.$$

This is a sub-family of the Beta distribution. Let $T = -\sum_{i=1}^n \log(X_i)/n$.

(a) Verify that this is an exponential family, write down the natural parameter and the sufficient statistic; (*5 points*)

(b) Use properties of the exponential family to show that $E(T) = 1/\theta$ and $\text{Var}(T) = 1/n\theta^2$; (*10 points*)

(c) Compute the Fisher information for this problem; (*10 points*)

(d) Deduce that T achieves the information inequality lower bound, and therefore is UMVU for $1/\theta$. (*5 points*)

Problem 2 Let X_1, ..., X_n be i.i.d. with density function

$$f(x;\theta) = \theta x^{\theta-1}\exp(-x^\theta), \ \ x > 0, \theta > 0.$$

This is a sub-family of the Weibull distribution.

(a) Explain if it belongs to the location, scale, or exponential family. (*5 points*)

(b) Show that there is a unique interior maximum of the likelihood function. (*5 points*)

(c) Find the maximum likelihood estimate $\hat{\theta}$ given the data. (*5 points*)

(d) Estimate the variance of $\hat{\theta}$. (*5 points*)

Problem 3 Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from from $\mathcal{N}(0, \sigma^2)$. We are interested in testing $H : \sigma \le \sigma_0$ versus $K : \sigma > \sigma_0$.

(a) Fix $\sigma_1 > \sigma_0$. Write down the likelihood ratio for H versus $K_1 : \sigma = \sigma_1$ and deduce that the test that rejects for large values of $\sum_i (X_i - \xi_0)^2$ is most powerful. (*10 points*)

(b) Give an explicit form for a UMP test for H versus K at level $\alpha \in (0, 1)$. (*10 points*)

(c) Is this the only UMP test? (*5 points*)

Problem 4 Suppose $X_1, \ldots, X_s$ are independent with X_i having the Poisson distribution with mean λ_i. Consider testing $H : \sum_i \lambda_i \le a$ versus $K : \sum_i \lambda_i > a$, where $a > 0$ is fixed.

(a) Fix an alternative $(\lambda'_1, \ldots, \lambda'_s)$ with $\sum_i \lambda'_i > a$. Write down the likelihood ratio. (*5 points*)

(b) For a given alternative $(\lambda_1', \ldots, \lambda_s')$, consider the prior on $(\lambda_1, \ldots, \lambda_s)$ where $\lambda_i = a\lambda_i' / \sum_j \lambda_j'$. Give a most power test for this simple versus simple situation. (*5 points*)

(c) Show that the power of the test that rejects for large values of $\sum_i X_i$ is monotone in $\sum_i \lambda_i$. (*5 points*)

(d) Deduce a UMP test for H versus K. (*5 points*)

Problem 5 $X_1, \dots, X_n \sim$ iid uniform $[\alpha - 1, \alpha + 1]$ $(\alpha > 1)$.

Do an ARE comparison of $\hat{\alpha}_1 = \bar{X}$ and $\hat{\alpha}_2 = \text{med}\{X_i\}$.

Are both limit-law and risk comparisons available as a basis for this ARE determination? If so pick either one for your analysis. What size n does an $\hat{\alpha}_2$-statistician need to match the performance of an $\hat{\alpha}_1$-statistician using $n = 1$ million observations? Note with comment that neither estimator is optimal. What would happen if you compared $\hat{\alpha}_3 = \hat{\alpha}_{MLE}$ and (bonus points) $\hat{\alpha}_4 = e^{\bar{Y}}$ (where $Y_i = \ln X_i$) against $\hat{\alpha}_2$ in an asymptotic analysis.

Prove that at least one of these estimators is inadmissible.