Math 281 **Qualifying Exam.** **September 11, 2000**

Note: In what follows, all p.d.f.'s are densities with respect to Lebesgue measure on the real line.

(1) Suppose that three random variables X_1, X_2, and X_3 have a continuous joint distribution for which the joint p.d.f. is as follows:

$$f(x_1, x_2, x_3) = \begin{cases} 8x_1x_2x_3 & \text{for } 0 < x_i < 1 \ (i = 1,2,3) \\ 0 & \text{otherwise.} \end{cases}$$

a. Show that f is indeed a p.d.f. .

b. Are the random variables independent ?

c. Given that $Y_1 = X_1$ and $Y_2 = X_1 X_2$, find the joint p.d.f. of Y_1 and Y_2.

(2) Assume that an n-dimensional random vector $X = (X_1, X_2, \ldots, X_n)^t$ with generic value $x = (x_1, x_2, \ldots, x_n)^t$ has p.d.f. $f(x)$ for $x \in R^n$. Now let Y = AX (nx1) in which A (nxn) is any nonsingular matrix. Show that the p.d.f. of $y = (y_1, y_2, \ldots, y_n)^t$ is given by

$$g(y) = \frac{1}{\det A} f(A^{-1}y) \quad \text{for } y \in R^n.$$

(3) Assume the setting of problem (2) along with the result. Now, if the components of X are i.i.d. N(0,1) random variables, then derive the p.d.f. of $Y \sim N_n(\theta, \Sigma)$.

(4) Write down all of results that you know that pertain to the the theory of UMVUE. (Along the way, you should write down the definitions of all pertinent terms.)

(5) Consider now an infinite sequence of Bernoulli trial for which the parameter p is unknown (0<p<1). If X is the number of "failures" that occur before the first "success" is obtained, then it is known that X has a geometric distribution with p.d.f. given by

$$f(x;\, p) = \begin{cases} pq^{x} & \text{for} \quad \text{for } x = 0,1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Show that the *only* unbiased estimator of p is given by

$$\hat{p} = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{if } X > 0 \end{cases} .$$

Comment on this estimator in view of the theory of UMVUE. Finally, is this estimator of any practical value ?

(6) Assume $X_1, X_2, \ldots, X_n$ is a random sample taken from a distribution whose p.d.f. is given by

$$f(x;\theta) = \frac{1}{2\theta^3} x^2 e^{-x/\theta}; \quad 0 < \theta < \infty, \quad 0 < x < \infty$$

$$= 0 \quad \text{elsewhere} .$$

Find the UMVUE. Is *this* estimator of practical value ? Also, compute the Cramer-Rao lower bound and thus the asymptotic variance of the maximum likelihood estimator of θ.