Real Analysis Qualifying Examination

Spring, 2016

Name ______________________ ID number ______________________

Problem	1	2	3	4	5	6	7	8	Total
Score									

Instructions

- This is a three-hour, closed-book, closed-note, and no-calculator exam, with 10 pages. There are 8 problems of total 200 points. To get credit, you must show your work. Partial credit will be given to partial answers.
- You may use without proof any results proved in the textbook or covered in the lecture. If you use such a result, please cite it by its name (if it has one) or explain what it is concisely. Please also verify explicitly all the hypotheses in the statement.
- You need to re-prove any result given as a homework problem, unless it is a statement proved in the text or in the lecture.
- If the statement you are asked to prove is exactly a result in the text or covered in the class, you still need to re-construct the proof instead of just citing the result.

Problem 1 (60 points). Determine if each of the following statements is true or false. If you decide that it is true, then please give a brief proof. If you decide that it is false, then please give a counterexample or prove your assertion. For your proof, you may cite a proved result from the text or lectured in the class, with a brief explanation how the conclusion follows.

(1) Let $(X, \mathcal{M}, \mu)$ be a measure space and $E_j \in \mathcal{M}$ ($j = 1, 2, \ldots$). Denote by $E = \{x \in X : x \in E_j \text{ for infinitely many } j\}$. If $\sum_{j=1}^{\infty} \mu(E_j) < \infty$, then $\mu(E) = 0$.

(2) Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and denote the Lebesgue set of f by L_f. Let $x_0 \in \mathbb{R}^n$ and suppose f is continuous at x_0. Then $x_0 \in L_f$.

(3) Let H be a real Hilbert space. Let $x_k \in H$ $(k = 1, 2, \dots)$ and $x \in H$. If $x_k \to x$ weakly in H and $\|x_k\| \to \|x\|$, then $\|x_k - x\| \to 0$.

(4) Let $\mathcal{S}$ denote the Schwartz space on $\mathbb{R}^n$. Let $f,\, g \in \mathcal{S}$. If $f * g = 0$ in $\mathbb{R}^n$ then either $f = 0$ in $\mathbb{R}^n$ or $g = 0$ in $\mathbb{R}^n$.

(5) Let $f(x) = |x|$ $(x \in \mathbb{R})$ and identify f as a distribution on $\mathbb{R}$. Then the second-order distributional derivative f'' is the 0 distribution (as the 0 linear functional) on $\mathcal{D}(\mathbb{R})$.

Problem 2 (15 points). Let μ be the Lebesgue–Stieltjes measure associated to the following increasing and right-continuous function $F : \mathbb{R} \to \mathbb{R}$:

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x + 2 & \text{if } 0 \leq x < 1, \\ 4x^2 & \text{if } 1 \leq x < \infty. \end{cases}$$

Calculate $\mu((-\infty, 0])$, $\mu(\{1\})$, and $\mu([1, 2])$.

Problem 3 (10 points). Let X be a compact Hausdorff topological space. Let K_j ($j = 1, 2, \dots$) be a sequence of decreasing, nonempty compact subsets of X. Prove that $\cap_{j=1}^{\infty} K_j \neq \emptyset$..

Problem 4 (25 points). Let $0 = x_0^{(k)} < x_1^{(k)} < \cdots x_{k-1}^{(k)} < x_k^{(k)} = 1$ $(k = 1, 2, \dots)$ and $0 \neq A_j^{(k)} \in \mathbb{R}$ $(j = 0, \dots, k;\ k = 1, 2, \dots)$. Define for any $f \in C([0,1])$

$$I[f] = \int_0^1 f(x)\,dx \qquad \text{and} \qquad I_k[f] = \sum_{j=0}^{k} A_j^{(k)} f\left(x_j^{(k)}\right) \quad (k = 1, 2, \dots).$$

(1) Assume $\sup_{k\geq 1} \sum_{j=0}^{k} \left|A_j^{(k)}\right| < \infty$ and $\lim_{k\to\infty} I_k[p] = I[p]$ for any polynomial p. Prove $\lim_{k\to\infty} I_k[f] = I[f]$ for $f \in C([0,1])$.

(2) (a) For any $k \geq 1$, I_k is a linear functional on the Banach space $C([0,1])$ (with the uniform norm). Prove that $\|I_k\| = \sum_{j=0}^{k} \left|A_j^{(k)}\right|$.

(b) Assume $\lim_{k\to\infty} I_k[f] = I[f]$ for any $f \in C([0,1])$. Prove $\sup_{k\geq 1} \sum_{j=0}^{k} \left|A_j^{(k)}\right| < \infty$.

Problem 5 (20 points). Let X be a Banach space and denote by $\mathcal{L}(X)$ the space of all linear and bounded operators from X to X. Let $T \in \mathcal{L}(X)$ be topologically isomorphic, i.e., $T : X \to X$ is linear, bijective, and both T and T^{-1} are continuous. Let $S \in \mathcal{L}(X)$ be such that $\|(S - T)T^{-1}\| < 1$. Prove that $S : X \to X$ is also topologically isomorphic.

Problem 6 (25 points). Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $1 < p < \infty$. Suppose $f_k \in L^p(\mu)$ $(k = 1, 2, \dots)$ are such that $\sup_{k \geq 1} \|f_k\|_{L^p(\mu)} < \infty$ and $f_k \to f$ in $L^1(\mu)$ for some $f \in L^1(\mu)$. Prove that $f \in L^p(\mu)$ and $f_k \to f$ in $L^q(\mu)$ for any $q \in (1, p)$.

Problem 7 (25 points). Let X be a locally compact Hausdorff topological vector space. Let $f \in C_0(X)$ and $f_k \in C_0(X)$ ($k = 1, 2, \dots$). Prove that $f_k \to f$ weakly in $C_0(X)$ if and only if $\sup_{k \geq 1} \|f_k\|_u < \infty$ and $f_k \to f$ pointwise on X.

Problem 8 (20 points). Let $f \in L^1(\mathbb{R})$ and $xf(x) \in L^1(\mathbb{R})$. Prove that the Fourier transform $\hat{f}$ is differentiable at every point $\xi \in \mathbb{R}$, and

$$\frac{d}{d\xi}\hat{f}(\xi) = \widehat{(-ixf)}(\xi) \qquad \forall \xi \in \mathbb{R}.$$