June 1, 2007

Qualifying Exam in Real Analysis

Instructions. You may use without proof anything which is proved in the text by Folland or in the notes on distributions, unless otherwise stated. Either state the theorem by name, if it has one, or say what the theorem says. However, you must reprove items which were given as exercises. Unless indicated otherwise, $(X, \mathcal{M}, \mu)$ is a measure space.

1. (20 pts.) Prove Chebyshev's Inequality: If $f \in L^p(X, \mu)$, with $0 < p < \infty$, then for any positive number r,

$$\mu(\{x : |f(x)| > r\}) \leq \left[\frac{\|f\|_p}{r}\right]^p.$$

2.(25 pts.) Prove that if $f_n, g_n, f, g \in L^1(X, \mu)$ with $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \leq g_n$ and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Hint: The proof is similar to that of the Dominated Convergence Theorem. Be sure to justify all your steps.)

3. (90 pts.) True or false. For each part, determine if it is always true or sometimes false. If true give a brief proof. Be sure to identify any If false give a counterexample or disprove it. No credit if reason is missing or incorrect. It's OK to be brief here.

(a) If X is a complete metric space and $\{x_n\}$ a bounded sequence in X, then $\{x_n\}$ has a convergent subsequence.

(b)If $f \in \mathcal{S}(\mathbb{R})$ (the space of Schwartz functions) and $\dfrac{d\hat{f}}{d\xi} = 0$, then $f = 0$. (Here $\hat{f}$ is the Fourier transform of f.)

(c)If $T \in \mathcal{S}'(\mathbb{R})$ (the space of tempered distributions) and $\dfrac{d\hat{T}}{d\xi} = 0$, then $T = 0$.

(d) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space $\mathcal{X}$ such that $\mathcal{X}$ is complete with respect to both norms. If $\|\cdot\|_1 \leq \|\cdot\|_2$, then there exists $C > 0$ such that $\|\cdot\|_2 \leq C\|\cdot\|_1$

(e) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space $\mathcal{X}$ such that $\mathcal{X}$ is complete with respect to one of the two norms. If $\|\cdot\|_1 \leq \|\cdot\|_2$, then there exists $C > 0$ such that $\|\cdot\|_2 \leq C\|\cdot\|_1$

(f) If $\mu(X) < \infty$, then $L^1(X, \mu) \subset L^2(X, \mu)$.

(g) $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$

(h) If μ_1 and μ_2 are two positive measures on a measurable space $(X, \mathcal{M})$ such that μ_1 is absolutely continuous with respect to μ_2 then there exists a measurable function f on X such that $d\mu_1 = f d\mu_2$.

4. (25 pts.) Prove that if $1 < p < \infty$ and f_n converges to f weakly in $\ell^p(\mathbb{N})$, then $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ pointwise. (The converse is also true, but you are not asked to prove it here.)

5.(30 pts.) Prove that $L^2(X, \mu)$ is complete. This is stated and proved in Folland, but you are being asked to give a proof here. For this, you may use without proof the following: A normed vector space $\mathcal{X}$ is complete if and only if every absolutely convergent series in $\mathcal{X}$ converges.

5. (25 pts.) In the following, give complete justification for your answers.

(a) Find all distributions $T \in \mathcal{D}'(\mathbb{R})$ such that $xT = 0$.

(b) Find all distributions $T \in \mathcal{D}'(\mathbb{R})$ such that $x^2T = 0$.