MATH 240: Real Analysis
Qualifying Exam. May 26, 2006

General instructions: 3 hours. No books or notes. Be sure to motivate all (nontrivial) claims and statements. You may use without proof any result proved in the text. You need to reprove any result given as an exercise.

Specific instructions: Do problems 1, 5, and 6. Choose *two* of the three problems 2, 3, and 4. Be sure to indicate which two of these you want graded.

Notation: $m = dx$ denotes the Lebesgue measure. $(X, \mathcal{M}, \mu)$ is a measure space.

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1. (50p) Determine if the statements below are **True** or **False**. If **True**, give a brief proof. If **False**, give a counterexample (or prove your assertion in another way, if you prefer). If you claim an assertion follows from a theorem in the text, name the theorem (or describe it otherwise) and explain carefully how the conclusion follows.

(a) If $f \in C([0,1])$, f' exists a.e. (m) and $f' = 0$ a.e. (m), then f is constant.

(b) If $X \supset E_1 \supset E_2 \supset \ldots$ are $\mathcal{M}$-measurable sets such that $\mu(\cap_j E_j) = 0$, then $\lim_{j \to \infty} \mu(E_j) = 0$.

(c) Suppose $\{f_j\}$ is a sequence in $L^1(X, \mu)$ with $f_1 \geq f_2 \geq \ldots \geq 0$ a.e. (μ), and let $f(x) = \lim_{j \to \infty} f_j(x)$ a.e. (μ). Then $\int_X f d\mu = \lim_{j \to \infty} \int_X f_j d\mu$.

(d) Let ν be a finite signed measure on X, and $|\nu|$ its total variation. Then, there is $f \in L^1(X, |\nu|)$ such that for every $g \in L^1(X, \nu)$, $\int_X g d\nu = \int_X g f d|\nu|$.

(e) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}^*$ its dual. Let $\{x_j^*\}$ be a sequence in $\mathcal{X}^*$ such that $\lim_{j \to \infty} x_j^*(x)$ exists (as a complex number) for every $x \in X$. Then, there is $x^* \in X^*$ such that $x^*(x) = \lim_{j \to \infty} x_j^*(x)$ for every $x \in X$.

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2. (30p) Let $\{f_j\}$ be a sequence in $AC([0,1])$ such that that $\lim_{j \to \infty} f_j(0) = c$ and, for some $g \in L^1([0,1], dx)$, $f_j' \to g$ in L^1.

(a) Show that $f(x) = \lim_{j \to \infty} f_j(x)$ exists for all $x \in [0,1]$.

(b) Show that $f \in AC([0,1])$ and $f' = g$ a.e. (m).

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3. (30p) Let $\{a_k\}$ be a sequence of complex numbers such that $\sum_{k=0}^{\infty} a_k$ is convergent. Set $S_m^n := \sum_{k=m}^{n} a_k$.

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(a) Show that, for $0 \leq x \leq 1$,

$$\sum_{k=m}^{n} a_k x^k = \sum_{j=m}^{n-1} S_m^j (x^j - x^{j+1}) + S_m^n x^n.$$

(b) Show that

$$\lim_{x \to 1^-} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k.$$

Hint: Estimate the left hand side of the formula in (b).

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4. (30p) Let $\{f_j\}$ be a sequence of real-valued functions in $L^1(X, \mu)$ such that $f_j \to f$ a.e. with $f \in L^1(X, \mu)$. Suppose $\{g_j\}$ is a sequence of functions in $L^1(X, \mu)$ such that $|f_j| \leq g_j$, $g_j \to g$ a.e. for some $g \in L^1(X, \mu)$, and also $g_j \to g$ in L^1. Prove that

$$\int_X f d\mu = \lim_{j \to \infty} \int_X f_j d\mu$$

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5. Let $\mathcal{H}$ be a Hilbert space, $P \colon \mathcal{H} \to \mathcal{H}$ a bounded linear operator, $P^* \colon \mathcal{H} \to \mathcal{H}$ its adjoint, and $\mathcal{R}(P)$, $\mathcal{N}(P)$ its range and nullspace, respectively.

(a) (10p) Show that $\mathcal{N}(P^*) = \mathcal{R}(P)^\perp$ and $\overline{\mathcal{R}(P^*)} = \mathcal{N}(P)^\perp$.

(b) (15p) Show that $\mathcal{R}(P^*)$ is closed if $\mathcal{R}(P)$ is closed.

(c) (15p) Assume that $P^2 = P$. Show that the following are equivalent (you may use parts (a) and (b) even if you did not do them):

(i) $\mathcal{R}(P)$ is closed and $\|x - Px\| = \inf_{y \in \mathcal{R}(P)} \|x - y\|$ for every $x \in \mathcal{H}$.

(ii) $P = P^*$.

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6. (30p) For each $N \in \mathbb{R}$, define in $\mathbb{R}^n$ the measures $d\mu_N := (1 + |x|)^N dx$ and, for each $1 \leq p < \infty$, the norms

$$\|f\|_{p,N} := \|f\|_{L^p(\mathbb{R}^n, d\mu_N)} = \left(\int_{\mathbb{R}^n} |f(x)|^p (1 + |x|)^N dx \right)^{1/p}.$$

(a) Show that for every $t > 0$, there is a constant C_t such that for every $1 \leq r < p < \infty$ and every

$$N_t := \frac{Np + n(p - r) + t(p - r)}{r}$$

the estimate

$$\|f\|_{r,N} \leq C_t \|f\|_{p,N_t}$$

holds.

(b) Show that the constants C_t can be chosen such that $C_t \to 0$ as $t \to \infty$.

Hint: For part (a), observe that $1 = (1 + |x|)^M / (1 + |x|)^M$.