Name:________________________________________ I.D. #:______________________

# Math 240 (Driver) Qual Exam (9/12/2017)

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**Instructions:** Clearly explain and justify your answers. You may cite theorems from the text, notes, or class as long as they are not what the problem explicitly asks you to prove. You may also use the results of prior problems or prior parts of the same problem when solving a problem – this is allowed even if you were unable to prove the previous results. Make sure to indicate the results that you are using and be sure to verify their hypotheses. All 7 problems have equal value.

**Notation:** $m$ or $dx$ is used to denote Lebesgue measure on the Borel $\sigma$-algebra $\mathbb{R}^d$ where $d$ may be 1, 2, or an arbitrary integer in $\mathbb{N} := \{1, 2, 3, \dots\}$. For two functions, $f, g$, on $\mathbb{R}^d$, $f * g$ denotes their convolution when this makes sense. Measurable means Borel measurable on this test unless otherwise indicated. As usual $\|\cdot\|_p$ denotes the $L^p(\mu)$-norm for the measure space appearing in the given problem.

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**Exercise 1.** In each case below find $L$ (allowing for values of $\pm\infty$) and justify the calculations:

$$\text{a)} \qquad L = \lim_{n\to\infty} \int_0^{\sqrt{\pi}} e^{-n\cos(x^2)}\, dx,$$

$$\text{b)} \qquad L = \lim_{N\to\infty} \sum_{k=0}^{N} \int_0^N \frac{x^k}{k!} e^{-2x}\, dx \text{ and}$$

$$\text{c)} \qquad L = \int_0^\infty \left[\int_0^\infty e^{-y/x} e^{-x^2/2} dx\right] dy$$

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**Exercise 2.** In this problem let $(\Omega, \mathcal{B}, \mu)$ be a measure space ($\mu$ is a positive measure) and $f_n, f : \Omega \to \mathbb{C}$ be measurable functions. Determine which of the following statements are true. For the true statements give a **brief** reason and for the false statements give a counter example.

1. If $f_n \to f$ in $L^2(\mu)$, then $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for $\mu$-a.e. $\omega$.
2. Suppose that $\omega_0 \in \Omega$ is a point such that $\{\omega_0\} \in \mathcal{B}$ and $0 < \mu(\{\omega_0\}) < \infty$. If $f_n \to f$ in $L^2(\mu)$, then $f(\omega_0) = \lim_{n\to\infty} f_n(\omega_0)$.
3. If $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ for $\mu$-a.e. $\omega$, then $f_n \xrightarrow{\mu} f$, i.e. $\lim_{n\to\infty} \mu(|f - f_n| \geq \varepsilon) = 0$ for all $\varepsilon > 0$.
4. If $\mu(\Omega) < \infty$ and $f_n \to f$ in $L^3(\mu)$, then $f_n \to f$ in $L^1(\mu)$.

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**Exercise 3.** Let $H$ and $K$ be separable Hilbert spaces, $T : H \to K$ be a bounded linear operator, and $\{u_j\}_{j=1}^{\infty}$ and $\{v_k\}_{k=1}^{\infty}$ be orthonormal bases for $H$ and $K$ respectively. Show;

1. $\sum_{j=1}^{\infty} \|Tu_j\|_K^2 = \sum_{k=1}^{\infty} \|T^* v_k\|_H^2$ allowing for the possibility that one and hence both of these sums are infinite.
2. $\|T\|_{op}^2 \leq \sum_{j=1}^{\infty} \|Tu_j\|_K^2$ where $\|T\|_{op}$ denotes the operator norm of $T$.

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**Exercise 4.** Suppose that $(\Omega, \mathcal{B}, \mu)$ is a measure space, $\Omega_n \in \mathcal{B}$ with $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$. If $f : \Omega \to \mathbb{C}$ is a measurable function such that

$$\int_\Omega |f| \, |g| \, d\mu < \infty \text{ for all } g \in L^{3/2}(\mu),$$

show $f \in L^3(\mu)$, i.e. $\int_\Omega |f|^3 \, d\mu < \infty$. **Hint:** consider $f_n := 1_{\Omega_n} f 1_{|f| \le n} \in L^3(\mu)$ for $n \in \mathbb{N}$ and recall $L^3(\mu) \cong L^{3/2}(\mu)^*$.

**Exercise 5.** Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function which is essentially bounded, i.e. there exists $M < \infty$ such that $|g(x)| \le M$ for $m$-a.e. $x \in \mathbb{R}^d$. For $f \in L^1(m) := L^1(\mathbb{R}^d, m)$, let

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy. \tag{1.1}$$

1. Show $(f * g)(x)$ is well defined (i.e. the integral exists) and $|(f * g)(x)| \le M \|f\|_1$ for all $x \in \mathbb{R}^d$.
2. Show $(f * g)(x)$ may also be written as
$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy.$$
3. Verify $(f_n * g)(x)$ is continuous in $x$ for any $f_n \in C_c(\mathbb{R}^d)$.
4. Show, for $f \in L^1(m)$, that $f * g$ may be written as a uniformly convergent limit of continuous functions and hence $(f * g)(x)$ is continuous in $x$.

**Exercise 6.** Suppose $\lambda \in \mathbb{C}$ and $f \in L^1(m) = L^1(\mathbb{R}, m)$ satisfies, $f * f * f(x) = \lambda f * f(x)$ for $m-$ a.e. $x$. Show $f(x) = 0$ for $m-$ a.e. $x$.

[Recall that $\|f * g\|_{L^1(m)} \leq \|f\|_{L^1(m)} \|g\|_{L^1(m)}$ for all $f, g \in L^1(m)$ and therefore, $f * f$ and $f * f * f$ are still in $L^1(m)$.]

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**Exercise 7.** Let $AC([0,1])$ denote the absolutely continuous functions on $[0,1]$ with values in $\mathbb{R}$ and for $1 \le p < \infty$, let

$$\mathcal{C}_p := \left\{ f \in AC([0,1]) : f(0) = 0 \text{ and } \int_{[0,1]} |f'(x)|^p \, dx \le 1 \right\}$$

thought of as a subset of the Banach space, $(C([0,1]), \|\cdot\|_u)$ where

$$\|f\|_u := \max_{x \in [0,1]} |f(x)| \ \ \forall \ f \in C([0,1]).$$

1. Show $\mathcal{C}_p$ is precompact in $C([0,1])$ for all $p \in (1, \infty)$.
2. Is $\mathcal{C}_1$ precompact in $C([0,1])$? You **must** justify your conclusion here.

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