Name:______________________________ I.D. #:__________________

Math 240 Real Analysis Qual Exam (5/25/2018)

Instructions: Clearly explain and justify your answers. You may cite theorems from the text, notes, or class as long as they are not what the problem explicitly asks you to prove. You may also use the results of prior problems or prior parts of the same problem when solving a problem – this is allowed even if you were unable to prove the previous results. Make sure to indicate the results that you are using and be sure to verify their hypotheses. All 8 problems have equal value.

Notation: m or dx is used to denote Lebesgue measure on the Borel σ-algebra $(\mathcal{B})$ on $\mathbb{R}$. For $f \in L^1(\mathbb{R}, m)$ and μ a finite positive measure on $(\mathbb{R}, \mathcal{B})$, let

$$\hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx \quad \text{and} \quad \hat{\mu}(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} d\mu(x).$$

Exercise 1. Compute the following two limits allowing for the values of $\pm\infty$;

$$\text{a) } \lim_{n\to\infty} \int_0^\infty \frac{x^n}{1+x^{n+2}} e^{-x/n} dx \text{ and b) } \lim_{n\to\infty} \int_0^\infty \frac{x^n}{1+x^{n+1}} e^{-x/n} dx.$$

Exercise 2. Let $\{f_n\}_{n\geq 1}$ be a sequence of functions in $L^2([0,1],m)$ such that $\|f_n\|_2 \leq 1$, for all $n \geq 1$. Assume that $\lim_{n\to\infty} f_n(x) = 0$, for almost every $x \in [0,1]$.

1. Show that $\lim_{n\to\infty} \|f_n\|_1 = 0$.
2. Given an example showing that we do not necessarily have that $\lim_{n\to\infty} \|f_n\|_2 = 0$.

Exercise 3. Let H be a Hilbert space and $\{\xi_n\}_{n\geq 1}$ be a sequence of vectors in H. Assume that $\|\xi_n\| = 1$, for all $n \geq 1$, and that $\lim\limits_{n,m\to\infty} \|\xi_n + \xi_m\| = 2$. Show that there exists $\xi \in H$ such that $\lim\limits_{n\to\infty} \|\xi_n - \xi\| = 0$.

Exercise 4. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^1-function such that $M := \sup_{x\in\mathbb{R}} [|\varphi(x)| + |\varphi'(x)|] < \infty$.

1. If $f \in C_c^1(\mathbb{R})$, show
$$\left|\int_{\mathbb{R}} f(x)\,\varphi'(\lambda x)\,dx\right| \le M \cdot \|f'\|_{L^1(\mathbb{R},m)} |\lambda|^{-1} \text{ for all } \lambda > 0.$$
2. If $f \in L^1(\mathbb{R}, m)$, show
$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} f(x)\,\varphi'(\lambda x)\,dx = 0.$$

Exercise 5. Let μ be a finite positive measure on $(\mathbb{R},\mathcal{B})$, $f:\mathbb{R}\to\mathbb{C}$ be a Borel measurable function, and for $x\in\mathbb{R}$, let

$$(\mu * f)(x) := \begin{cases} \int_{\mathbb{R}} f(x-y)\,d\mu(y) & \text{if} \quad \int_{\mathbb{R}} |f(x-y)|\,d\mu(y) < \infty \\ 0 & \text{otherwise.} \end{cases}$$

1. For $1\le p<\infty$ and $f\in L^p(m)$, show $\|\mu * f\|_{L^p(m)} \le \mu(\mathbb{R})\cdot\|f\|_{L^p(m)}$.

   [If you can not do this for general p at least do it for $p=1$.]
2. If $f\in L^1(\mathbb{R},m)$, show $\widehat{\mu * f}(k)=\sqrt{2\pi}\hat{\mu}(k)\hat{f}(k)$ for all $k\in\mathbb{R}$.
3. If f and $\hat{f}$ are in $L^1(\mathbb{R},m)$, then

$$(\mu * f)(x) = \int_{\mathbb{R}} \hat{\mu}(k)\hat{f}(k)e^{ikx}dk \text{ for } m \text{ a.e. } x.$$

Exercise 6. Let $f : \mathbb{R} \to [0,1]$ be a Borel measurable function. For $t \geq 0$, let $\varphi(t) = m(\{x \in \mathbb{R} | f(x) \geq t\})$.

1. Assume that $f \in L^1(\mathbb{R}, m)$. Prove that $\varphi(t) \leq \frac{1}{t}\|f\|_1$, for all $t > 0$.
2. Prove that $\int_{\mathbb{R}} f \,\mathrm{d}m = \int_0^\infty \varphi(t) \,\mathrm{d}t$.
3. Assume that $\varphi(t) \leq \frac{1}{\sqrt{t}}$, for all $t > 0$. Prove that $f \in L^1(\mathbb{R}, m)$.

Exercise 7. Let (X, d) be a compact metric space.

1. Prove that X is a separable metric space, i.e. there exists a countable dense subset, $\{x_n\}_{n\geq 1} \subset X$.
2. Let $\{f_n\}_{n\geq 1} \subset C\,(X, \mathbb{R})$ be the sequence of functions defined by $f_n(x) = d(x, x_n)$ for every $n \geq 1$ and $x \in X$.

   Prove that this sequence, $\{f_n\}_{n\geq 1}$, separates points in X.
3. Prove that $C(X, \mathbb{R})$ is separable when endowed with the uniform norm.

Exercise 8. Let $\ell^{\infty}(\mathbb{N})$ be the Banach space of bounded complex sequences, $x = (x_1, x_2, x_3, \dots)$, such that $\|x\|_{\infty} = \sup_n |x_n| < \infty$ and let V be the subspace of $\ell^{\infty}(\mathbb{N})$ defined as;

$$V = \{x \in \ell^{\infty}(\mathbb{N}) \mid \lim_{n\to\infty} \frac{1}{n}(x_1 + x_2 + ... + x_n) \text{ exists in } \mathbb{C}\}.$$

1. Prove that there exists $\varphi \in \ell^{\infty}(\mathbb{N})^*$ such that $\varphi(x) = \lim_{n\to\infty} \frac{1}{n}(x_1 + x_2 + ... + x_n)$, for every $x \in V$.

   The rest of this problem is independent of part 1 and for the remainder of this problem, let $\psi \in \ell^{\infty}(\mathbb{N})^*$ be any continuous linear functional such that $\psi(x) = \lim_{n\to\infty} \dfrac{1}{n}(x_1 + x_2 + ... + x_n)$ when $x \in V$.
2. Show $\psi(\tilde{x}) = \psi(x)$, for every $x = (x_1, x_2, x_3, x_4, \dots) \in \ell^{\infty}(\mathbb{N})$ where $\tilde{x} := (x_2, x_3, x_4, \dots) \in \ell^{\infty}(\mathbb{N})$.
3. Show there is no $y \in \ell^1(\mathbb{N})$ such that $\psi(x) = \sum_{n=1}^{\infty} x_n y_n$ for all $x \in \ell^{\infty}(\mathbb{N})$.
