Name:______________________________________ I.D. #:________________________

Math 240 Real Analysis Fall Qual Exam (9/4/2018)

Instructions: Clearly explain and justify your answers. You may cite standard theorems as long as they are not what the problem explicitly asks you to prove. You may also use the results of prior problems or prior parts of the same problem when solving a problem − this is allowed even if you were unable to prove the previous results. Make sure to indicate the results that you are using and be sure to verify their hypotheses. All 8 problems have equal value.

Notation: m or dx is used to denote Lebesgue measure on the Borel σ-algebra ($\mathcal{B}$) on $\mathbb{R}$ and for $f \in L^1(\mathbb{R}, m)$ let

$$\hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx.$$

Exercise 1. Let $\{f_n\}_n \subset L^1(\mathbb{R}, m)$ be a sequence of functions such that $f_n \to 0$, m-almost everywhere. Assume that there exists $M < \infty$ such that

$$\int_{\mathbb{R}} \max(|f_1|, |f_2|, ..., |f_n|) \, \mathrm{d}m \leq M, \text{ for every } n \geq 1.$$

Prove that $\lim_{n\to\infty} \|f_n\|_1 = 0$.

Exercise 2. Let H be a Hilbert space and $\{\xi_n\}_n$ be a sequence of vectors in H such that $\|\xi_n\| = 1$, for all n.

1. Assume that $\{\xi_n\}_n$ is an orthonormal set. Prove that ξ_n converges weakly to 0.
2. Assume that ξ_n converges weakly to a vector $\xi \in H$ such that $\|\xi\| = 1$. Prove that $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$.

   Recall. ξ_n converges weakly to ξ iff $\langle \xi_n, \eta \rangle \to \langle \xi, \eta \rangle$, for every $\eta \in H$.

Exercise 3. Let $f \in L^1(\mathbb{R}, m)$. Prove that $\lim_{n\to\infty} f(n^2 x) = 0$, for almost every $x \in \mathbb{R}$.

Exercise 4. Let $f \in L^1([0,1], m)$, m^2 be Lebesgue measure on $\mathbb{R}^2$, and

$$A_\varepsilon = \{(x,y) \in [0,1] \times [0,1] \mid |x-y| \leq \varepsilon\} \text{ for all } \varepsilon > 0.$$

Prove that

$$1)\ \int_{A_\varepsilon} |f(x) - f(y)| \, \mathrm{d}m^2(x,y) \leq 4\varepsilon \cdot \|f\|_{L^1([0,1],m)} \text{ and}$$

$$2)\ \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_{A_\varepsilon} |f(x) - f(y)| \, \mathrm{d}m^2(x,y) = 0.$$

Exercise 5. Let $1 < p < \infty$, and $\{f_n\}_n \subset L^p([0,1], m)$ be a sequence such that $M := \sup_n \|f_n\|_p < \infty$ and $\lim_{n\to\infty} \|f_n\|_1 = 0$.

1. Prove that $\lim_{n\to\infty} \|f_n\|_r = 0$, for every $r \in [1, p)$.
2. Let $1 < q < \infty$ such that $1/p + 1/q = 1$. Prove that $\lim_{n\to\infty} \int_0^1 f_n g \, \mathrm{d}m = 0$, for every $g \in L^q([0,1], m)$.

Exercise 6. Consider the Banach space $L^\infty([0,1],m)$ and its vector subspace

$$V = \{f \in L^\infty([0,1],m) \mid \text{the limit } \lim_{n\to\infty} n \int_{[0,1/n]} f \,\mathrm{d}m \text{ exists}\}.$$

1. Prove that there exists $\varphi \in L^\infty([0,1],m)^*$ such that $\varphi(f) = \lim_{n\to\infty} n \int_{[0,1/n]} f \,\mathrm{d}m$, for every $f \in V$.
2. Let $\varphi \in L^\infty([0,1],m)^*$ such that $\varphi(f) = \lim_{n\to\infty} n \int_{[0,1/n]} f \,\mathrm{d}m$, for every $f \in V$.

   Prove that there does not exist $g \in L^1([0,1],m)$ such that $\varphi(f) = \int fg \,\mathrm{d}m$, for every $f \in L^\infty([0,1],m)$.

Exercise 7. Let $g \in L^1([0,1], m)$ and $h : [0,1] \to \mathbb{R}$ be a continuous function.

1. If $f : [0,1] \to \mathbb{R}$ is absolutely continuous and satisfies $f'(x) = h(x)$ for a.e. x, show $f'(x) = h(x)$ for **all** $x \in (0,1)$.
2. Now suppose that
$$f(x) := \int_0^1 \min(x,y)\, g(y)\, dy.$$
Show $f'(x) = \int_x^1 g(y)\, dy$ for **all** $x \in (0,1)$.

Exercise 8. Let $-\infty < a < b < \infty$, $C([a,b],\mathbb{R})$ be the Banach space of continuous functions on $[a,b]$ equipped with the supremum norm,

$$\mathcal{F} := \left\{ f \in L^1(\mathbb{R}, m) \cap C(\mathbb{R},\mathbb{R}) : \int_{\mathbb{R}} (1+|k|) \left| \hat{f}(k) \right| dk \le 1 \right\}, \text{ and}$$
$$\mathcal{F}_{[a,b]} := \left\{ f|_{[a,b]} : f \in \mathcal{F} \right\}.$$

Show $\mathcal{F}_{[a,b]}$ is a precompact subset of $C([a,b],\mathbb{R})$.
