MATH 240 QUALIFYING EXAM: FALL 2011

Instructions: Please answer all 5 questions. Partial credit will be given, so you should attempt all problems. You may quote any standard proposition/theorem from Folland or Lieb-Loss, but **not** the homeworks. **If you use a major result like MCT etc, you need to state the theorem you are using *explicitly.*** In this exam dx is integration with respect to Lebesgue measure, and $|E|$ denotes the Lebesgue measure of E.

I) a) (10 pts) Let $E \subseteq [0,1]$ be Lebesgue measurable. Suppose there exists a fixed $\epsilon > 0$ such that $|E \cap (a,b)| \geqslant \epsilon |a-b|$ for all intervals $(a,b) \subseteq [0,1]$. Show that one must have $|E| = 1$.

b) (10 pts) Give an example of a Lebesgue measurable set $E \subseteq [0,1]$ with $0 < |E| < 1$ and $|E \cap (a,b)| > 0$ for all nontrivial intervals $(a,b) \subseteq [0,1]$. Explain why this example does not contradict part a) above.

II) a) (10 pts) Let X be an infinite dimensional Banach space. Prove that X endowed with the the weak topology is not a complete metric space. (Hint: Begin by showing that any norm bounded set in X is nowhere dense in the weak topology.)

b) (10 pts) Let X be a Banach space and X^* its dual. Suppose that there exists countably many linear functionals $L_n \in X^*$ with the following property: Any sequence $x_j, x \in X$ converges weakly $x_j \rightharpoonup x$ iff $L_n(x_j) \to L_n(x)$ for all n. Show that X must be finite dimensional. (Hint: Consider the function $d(x,y) = \sum_n 2^{-n} \frac{|L_n(x-y)|}{1+|L_n(x-y)|}$.)

III) a) (15 pts) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose that $f_n \in L^1(d\mu)$ is a sequence of functions with the property that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $E \in \mathcal{M}$:

$$|E| < \delta \quad \Longrightarrow \quad \sup_n \int_E |f_n| d\mu < \epsilon \ .$$

Suppose in addition that there exists f with $f_n \to f$ μ-a.e.. Show that $f_n \to f$ in $L^1(d\mu)$.

b) (5 pts) Give a simple example to show that if one drops the finite measure assumption but keeps all the other hypotheses above, the conclusion can fail.

IV) (20 pts) Let $K \in L^1(\mathbb{R}^d)$ with Lebesgue measure. Suppose that $\psi_n \in L^2(\mathbb{R}^d)$ is a sequence of functions such that $\psi_n \rightharpoonup \psi$ (weak L^2 convergence), and also with the property that $\psi_n \equiv 0$ in $|x| > 1$. Show that $f_n(x) = \int_{\mathbb{R}^d} K(x-y)\psi_n(y)dy$ converges to $f(x) = \int_{\mathbb{R}^d} K(x-y)\psi(y)dy$ strongly in $L^2(\mathbb{R}^d)$.

V) (20 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π periodic function, i.e. $f(x) = f(x+2\pi)$, such that there exists $C > 0$ (possibly large) and $\epsilon > 0$ (possibly small) with $|f(x) - f(y)| \leqslant C|x-y|^{\frac{1}{2}+\epsilon}$. Show that the Fourier series of f converges uniformly. (Hint: Half credit will be given for the special case $\epsilon = \frac{1}{2}$, which can be treated more directly. For the general case try computing $\int_0^{2\pi} |f(x+h) - f(x)|^2 dx$ two different ways.)