Qualifying Exam in Real Analysis, September 7, 2005

Instructions: Answer all six problems. No books or notes may be used in this exam. You may cite without proof any result in the text by Folland, in any problem.

1. (45 pts.) Prove the following. Each follows in a straightforward way by applying theorems. Be sure to name each theorem when you use it.

(a) (15 pts.) Let $\{f_j\}$ be a sequence of real-valued functions in $L^1(\mu)$ such that $f_1 \geq f_2 \geq \ldots \geq 0$. Then

$$\lim \int f_j d\mu = \int \lim f_j d\mu.$$

(b) Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous. Then for all $\epsilon > 0$ there exists $N > 0$ and continuous functions $g_j, h_j : [a, b] \to \mathbb{R}$ such that

$$|f(x, y) - \sum_j g_j(x)h_j(y)| < \epsilon \quad \forall (x, y) \in [a, b] \times [c, d].$$

(c) Let $C([0, 1], \mathbb{R})$ be the space of all real valued functions on $[0, 1]$ with the uniform norm topology. Suppose that

$$C([0, 1], \mathbb{R}) = \bigcup_j F_j,$$

where each F_j is closed. Then there exists $\epsilon > 0$, $j_0 \in \mathbb{N}$, and $f_0 \in F_{j_0}$ such that

$$\sup_{x \in [0,1]} |f(x) - f_0(x)| < \epsilon \implies f \in F_{j_0}$$

2. (15 pts.) Find Lebesgue measurable sets $A, B \subset \mathbb{R}^2$ such that $A + B$ is not Lebesgue measurable.

3. (15 pts.) Show that the function $e^x \sin(e^x)$ defines a tempered distribution on $\mathbb{R}$. (This is, show that the distribution it defines extends to a tempered distribution. OK to be brief here.)

4. (20 pts.) Prove that ℓ^∞ (the space of all bounded sequences of complex numbers with the sup norm) is not separable. (Hint: Try proof by contradiction.)

5. (25 pts.) Let $\mathcal{H}$ be a Hilbert space. Suppose that there is a sequence $\{x_j\}$ in $\mathcal{H}$ such that the finite linear combinations of the x_j are dense in $\mathcal{H}$ and

$$| < x_j, x_k > | \leq 1/2^{|j-k|} \quad \forall j, k \in \mathbb{N}.$$

Prove that $x_j \to 0$ weakly.

6. (30 pts.) Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and $0 \leq f_1 \leq f_2 \leq \ldots \leq f$ be nonnegative measurable functions on X with $\lim f_j(x) = f(x)$ for almost every $x \in X$.

(a) Prove that $\mu(f_j^{-1}(r, \infty]) \to \mu(f^{-1}(r, \infty))$ as $j \to \infty$ for every $r \geq 0$, $r \in \mathbb{R}$.

(b) Prove that $\int_X f d\mu = \int_0^\infty \mu(f^{-1}(r, \infty))dr$.

(Hint: You could use (a), taking f_j to be a sequence of simple functions that converge monotonically to f.)