Numerical Analysis Qualifying Exam
Spring 2022

May 11, 2022

Instructions:

- There are 8 problems, worth a total of 200 points.
- The 3 hour exam takes place remotely on Wednesday, May 11, 2022
  - It will be available for download at 1 PM PT and ends at 4 PM PT, with finished work due on Gradescope by 4:15 PM PT; after this time, your work will not be accepted.
- You must work by yourself on these problems, with no interaction with other people. You are allowed all other resources including books and notes, online or otherwise, and calculators or computers, however, all your work should be your own and not copied from any source.
- Have pen or pencil ready, and enough paper. Also have ready the necessary equipment to promptly upload the finished pages onto Gradescope.
- Start each question on a separate page, upload clear and legible work, and properly label the page locations of each problem on Gradescope. If you do not, you risk losing points.
- Show the details of your work, such as your scratch work, on each problem to receive credit.
- For questions, prior to or during the exam, email `lcheng@math.ucsd.edu`
  - Additionally, be sure to monitor your email during the exam, as any announcements will be sent through email

1. (25 pts) For a sparse matrix $B \in \mathbf{R}^{n \times n}$, consider a row-oriented coordinate list format given by $N \in \mathbf{Z}$, $r, c \in \mathbf{Z}^N$, $v \in \mathbf{R}^N$ satisfying:

- If $b_{ij} \neq 0$ then there exists a unique $1 \leq p \leq N$ such that $(r_p, c_p) = (i, j)$ and $v_p = b_{ij}$;
- N equals the number of nonzero entries in B;
- If $1 \leq p < q \leq N$, then $r_p \leq r_q$;
- If $1 \leq p < q \leq N$ and $r_p = r_q$, then $c_p < c_q$.

Given as input $b \in \mathbf{R}^n$, n, and the above format N, r, c, v for a sparse, upper triangular Cholesky factor U of some symmetric, positive definite $n \times n$ matrix, write a detailed, step-by-step pseudocode of an algorithm that, with good efficiency in speed and memory, first solves $U^T y = b$ using column-oriented forward substitution, then solves $Ux = y$ using row-oriented back substitution and outputs $x \in \mathbf{R}^n$.

2. (25 pts) Consider a general algorithm for constructing, for given matrix $A \in \mathbf{R}^{m \times n}$ with $m \geq n$, orthogonal matrices $Q \in \mathbf{R}^{m \times m}, V \in \mathbf{R}^{n \times n}$ and upper bidiagonal QAV^T.

Briefly describe how it works in general, then describe the purpose and result of each step of this algorithm when applied specifically to the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. (25 pts) For $Ax = b$, where $A \in \mathbf{R}^{n \times n}$ is symmetric, positive definite and $b \in \mathbf{R}^n$, consider the preconditioned linear system $\tilde{A}\tilde{x} = \tilde{b}$, where

$$\begin{aligned} \tilde{A} &= R^{-T} A R^{-1} \\ \tilde{x} &= Rx \\ \tilde{b} &= R^{-T} b, \end{aligned}$$

for some nonsingular $R \in \mathbf{R}^{n \times n}$ such that the inverse of $M = R^T R$ is considered an approximate inverse of A.

Now consider the steepest descent method, with given initial guess $\tilde{x}^{(0)}$, applied to this preconditioned linear system, giving iteration equations for computing $\tilde{x}^{(k+1)}, \tilde{r}^{(k+1)}$. Defining

$$\begin{aligned} \hat{x}^{(k)} &= R^{-1} \tilde{x}^{(k)} \\ \hat{r}^{(k)} &= b - A\hat{x}^{(k)}, \end{aligned}$$

rewrite the iteration equations so that $\hat{x}^{(k+1)}, \hat{r}^{(k+1)}$ depend just on $\hat{x}^{(k)}, \hat{r}^{(k)}, A, M^{-1}$.

4. (25 pts) Determine all polynomial fixed point functions, $\phi \in \mathbf{P}_n$ for some $n \geq 0$, that satisfy: fixed point iterations on ϕ will always converge to some fixed point of ϕ for any initial guess.

5. (25 pts) Fix nodes

$$x_0 < x_1 < \cdots < x_{100}$$

and let $f \in C^\infty([x_0, x_{100}])$. Furthermore, let p be the interpolating polynomial (of minimum degree) that satisfies $p^{(r)}(x_i) = f^{(r)}(x_i)$, for all $0 \leq i \leq 100$ and $0 \leq r \leq 2$.

Prove there exist, and find the values of, integers j,m, and k_i, for $0 \leq i \leq 100$, such that

$$f(x) = p(x) + \frac{f^{(j)}(\xi_x)}{m!} \prod_{i=0}^{100} (x - x_i)^{k_i},$$

for some $\xi_x \in (x_0, x_{100})$.

6. (25 pts) Let $x_0, \ldots, x_n \in [a, b]$ be distinct nodes and fix $c \in (a, b)$. For any $f \in C^1([a, b])$, consider a formula for approximating $f'(c)$ that satisfies:

- Has the form
$$\sum_{i=0}^{n} \alpha_i f(x_i),$$
for some $\alpha_i \in \mathbf{R}, 0 \leq i \leq n$;
- Is exact whenever $f \in \mathbf{P}_n$.

Prove

$$\alpha_i = w_i'(c)$$

for all $0 \leq i \leq n$, where

$$w_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

7. (25 pts) Consider the initial value problem with ODE

$$y' = f(t, y),$$

and initial value $y(t_0) = y_0$. Consider applying to this the collocation Runge-Kutta method with equally-spaced nodes of stepsize $h > 0$ that uses the collocation points:

- t_i for the initial value;
- t_i, $t_i + \frac{h}{3}$, t_{i+1} for the ODE;

to advance from y_i to y_{i+1}. Find the Butcher tableau of this method.

8. (25 pts) Consider the initial value problem with ODE

$$y' = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} y,$$

and initial value $y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Consider applying to this the class of Runge-Kutta methods that:

- Use equal stepsize $h > 0$;
- Are given by Butcher tableaus of the form

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \alpha & \alpha & 0 \\ \hline & c_1 & c_2 \end{array}$$

  for $\alpha > 0$, $c_1, c_2 \in \mathbf{R}$;
- Additionally satisfy

$$\begin{aligned} c_1 + c_2 &= 1 \\ \alpha c_2 &= \frac{1}{2}, \end{aligned}$$

  which are known necessary and sufficient conditions for the methods to be second order accurate.

Find the largest $H > 0$ such that at least one method in this class is linearly stable for $h \in (0, H)$, when applied to the above initial value problem.