Ph.D./Masters Qualifying Examination in Numerical Analysis

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9am–12pm
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2402 AP&M

Name ______________________________

#1.1	25	
#1.2	25	
#1.3	25	
#2.1	25	
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#3.1	25	
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Total	200	

- Put your name in box provided and staple page to your solutions.
- Write your name clearly on every sheet submitted.

1 Numerical Linear Algebra (270A)

Question 1.1. Consider a linear system $Ax = b$, where A is an $n \times n$ matrix with real and positive eigenvalues. Denote by λ_{min} the smallest eigenvalue of A and λ_{max} the largest. Find an expression, in terms of these eigenvalues, for the optimal parameter $\alpha \in \mathbb{R}$ for fast convergence of Richardson's iterations:

$$x^{(k+1)} = (I - \alpha A)x^{(k)} + \alpha b.$$

Question 1.2. Consider an $n \times n$ matrix A. Count the number of flops it takes to compute the LU factorization, assuming it exists, using Gaussian elimination without pivoting. The following identities may be helpful in simplifying your answer: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ and $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

Question 1.3. Let $Ax = b$ be a linear system with A an $n \times n$ nonsingular matrix and $b \neq 0$. Consider the perturbed linear system $(A + \delta A)\hat{x} = b$ satisfying $||\delta A||/||A|| < 1/\kappa(A)$, where $\kappa(A)$ is the condition number of A and $|| \cdot ||$ is an induced matrix norm.

(a) Show $A + \delta A$ is nonsingular.

(b) Show

$$\frac{||\hat{x} - x||}{||x||} \leq \frac{\kappa(A)\frac{||\delta A||}{||A||}}{1 - \kappa(A)\frac{||\delta A||}{||A||}}.$$

Be sure to label all locations where you use the facts that $b \neq 0$ and $|| \cdot ||$ is an induced norm.

2 Numerical Approximation and Nonlinear Equations (270B)

Question 2.1. Let $D \subset \mathbb{R}^n$ be an open convex set, let $F\colon D \subset \mathbb{R}^n \to \mathbb{R}^n$, $F \in C^1(D;\mathbb{R}^n)$, and assume that $F(x^*) = 0$ for some $x^* \in D$, and that $F'(x)$ is nonsingular $\forall x \in D$.

(a) State and prove the basic convergence theorem for Newton's method for solving $F(x) = 0$, establishing superlinear rate of convergence.

(b) Under the additional assumption that the Jacobian $F'\colon D \subset \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is Lipschitz continuous with Lipschitz constant γ, first show that the error in the linear model

$$L_k(x) = F(x^k) + F'(x^k)(x - x^k)$$

of $F(x)$ can be bounded as follows:

$$\|F(x) - L_k(x)\|_2 \leq \frac{1}{2}\gamma\|x - x^k\|_2^2.$$

Use this result to show Newton's method converges with quadratic rate.

(c) To make Newton's method robust, we often use the following energy functional:

$$J_F(x) = \frac{1}{2}\|F(x)\|_2^2.$$

Prove that the Newton direction is a direction of decrease for $J_F(x)$.

Question 2.2. Consider the following tabulated data for a function $f : \mathbb{R} \to \mathbb{R}$:

x	f(x)
0	0
1	2
2	10

(a) Construct the (unique) quadratic interpolation polynomial $p_2(x)$ which interpolates the data.

(b) If the function being interpolated was in fact $f(x) = x^3 + x$, derive a fairly tight upper bound on the error in using $p_2(x)$ as an approximation to $f(x)$ on $[0, 2]$.

(c) Use the composite trapezoid rule with two intervals to construct an approximation to:

$$\int_0^2 f(x)\, dx,$$

and give an expression for the error.

Question 2.3. We consider now the problem of best L^p-approximation of a (continuous) function $u(x) = x^3$ over the interval $[0, 1]$ from a subspace $V \subset L^p([0, 1])$.

(a) Determine the best L^2-approximation in the subspace of linear functions; i.e., $V = \text{span}\{1, x\}$, and justify the technique you use.

(b) Why (specifically) does this problem become much more difficult if we consider the case $p \neq 2$?

(c) If X is a Hilbert space, prove that the projection of $u \in X$ onto a subspace $U \subset X$ is unique.

3 Numerical Ordinary Differential Equations (270C)

Question 3.1. (a) Derive a 3-stage collocation Runge–Kutta method with collocation points $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = 1$, and write down the resulting method in the form of a Butcher tableau.

(b) Compute the stability function $R(h\lambda)$ for the collocation Runge–Kutta method you derived in 3.1(a), which satisfies the equation

$$y_{n+1} = R(h\lambda)y_n$$

when the Runge–Kutta method is applied to the model problem,

$$\begin{cases} y'(t) = \lambda y(t), & t > 0, \\ y(0) = 1. \end{cases}$$

(c) By using Lemma 4.3, which states that $|R(z)| < 1$ for all $\mathbb{C}^-$ iff all the poles of $R(z)$ have positive real parts and $|R(it)| \leq 1$ for all $t \in \mathbb{R}$, prove that the Lobatto IIIA method you derived above is A-stable.

Question 3.2. Consider the following α-parameterized family of linear multistep methods,

$$u_{n+1} = \alpha u_n + (1-\alpha)u_{n-1} + 2hf_n + \frac{h\alpha}{2}[f_{n-1} - 3f_n].$$

Determine how the stability (root condition), order of accuracy, and convergence (Dahlquist equivalence theorem) properties of this family of methods changes as we allow $\alpha \in \mathbb{R}$ to vary.