Ph.D./Masters Qualifying Examination
in Numerical Analysis

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10am–1pm
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2402 AP&M

NAME ______________________

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- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Numerical Linear Algebra (270A)

Question 1.1. Consider the PDE

$$\frac{\partial^2 u}{\partial y^2}(y,z) + \frac{\partial^2 u}{\partial z^2}(y,z) = 1$$

for $(y,z) \in (0,1)\times(0,1)$ with the condition $u(y,z) = y+z$ on the square boundary. Let $h = 1/3$ and let $y_i = ih$ and $z_j = jh$ for $i,j = 1,2$. Set up a linear system $Ax = b$, explicitly writing out A, x, and b, that for calculating the approximations $U_{i,j}$ of $u(y_i,z_j)$ for $i,j = 1,2$ using

$$\frac{\partial^2 u}{\partial y^2}(y_i,z_j) \approx \frac{u(y_i+h,z_j) - 2u(y_i,z_j) + u(y_i-h,z_j)}{h^2}$$
$$\frac{\partial^2 u}{\partial z^2}(y_i,z_j) \approx \frac{u(y_i,z_j+h) - 2u(y_i,z_j) + u(y_i,z_j-h)}{h^2}.$$

Question 1.2. Let $A \in \mathbb{F}^{n\times n}$ ($\mathbb{F}$ is the set of machine numbers) admit an LU factorization where $L = (l_{ij})$ is computed through the formula

$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}}{u_{jj}},$$

for $i > j$, and $U = (u_{ij})$ through another formula. In the presence of roundoff errors, $\hat{L} = (\hat{l}_{ij})$ and $\hat{U} = (\hat{u}_{ij})$ are computed instead, in machine numbers. Determine an ordering for the floating point operations in the formula for l_{ij} such that there exists e_{ij} satisfying, for $i > j$:

$$a_{ij} = \sum_{k=1}^{j} \hat{l}_{ik}\hat{u}_{kj} + e_{ij}$$

and

$$|e_{ij}| \le nu \sum_{k=1}^{j} |\hat{l}_{ik}||\hat{u}_{kj}| + \mathcal{O}(u^2)$$

when $u > 0$, the unit roundoff error, is small enough.

Question 1.3. Let $A \in \mathbb{R}^{n\times n}$ and let D, L, U be diagonal, strictly lower triangular, and strictly upper triangular matrices, respectively, such that $A = D - L - U$. Prove if

- A is nonsingular, has nonzero diagonal elements, and is consistently ordered ($\det(\alpha D^{-1}L + \alpha^{-1}D^{-1}U - \kappa)$ is independent of $\alpha \in \mathbb{C}, \alpha \ne 0$ for all $\kappa \in \mathbb{C}$);
- the eigenvalues μ of $B_J = D^{-1}(L+U)$ satisfy $\mu \in \mathbb{R}$ and $|\mu| < 1$;
- $0 < \omega < 2$;

then SOR, with iteration matrix $B_{SOR} = (D-\omega L)^{-1}[\omega U + (1-\omega)D]$, is convergent. (You may use the fact that $x^2 - bx + c = 0$ with $b, c \in \mathbb{R}$ implies $|x| < 1$ if and only if $|c| < 1$ and $1 + c - |b| > 0$).

2. Numerical Approximation and Nonlinear Equations (270B)

Question 2.1. Let $F(x) : D \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ be continuously differentiable on an open convex set D, assume that $F(x^*) = 0$ for some $x^* \in D$, and assume that $F'(x)$ is nonsingular on D.

(a) State and prove the basic convergence theorem for Newton's method for solving $F(x) = 0$, establishing superlinear rate of convergence.

(b) Under the assumption that the Jacobian $F'(x) : D \subset \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ is Lipschitz continuous with Lipschitz constant γ, show that the error in the linear model

$$L_k(x) = F(x^k) + F'(x^k)(x - x^k)$$

of $F(x)$ can be bounded as follows:

$$\|F(x) - L_k(x)\| \leq \frac{1}{2}\gamma\|x - x^k\|^2.$$

(c) Assume now the Jacobian $F'(x)$ is Lipschitz. Use the result from part (b) to prove that Newton's method for $F(x) = 0$ converges with quadratic rate.

Question 2.2. Consider the following tabulated data for a function $f : \mathbb{R} \to \mathbb{R}$:

x	f(x)
0	1
1	3
2	13

(a) Construct the (unique) quadratic interpolation polynomial $p_2(x)$ which interpolates the data.

(b) If the function $f(x)$ that generated the above data was actually the cubic polynomial $P_3(x) = x^3 + x^2 + 1$, derive an error bound for the interval $[0, 2]$.

(c) Use the composite trapezoid rule with two intervals to construct an approximation to:

$$\int_0^2 f(x)\, dx,$$

and give an expression for the error.

Question 2.3. We consider now the problem of best L^p-approximation of a (continuous) function $u(x) = x^3$ over the interval $[0, 1]$ from a subspace $V \subset L^p([0, 1])$.

(a) Determine the best L^2-approximation in the subspace of linear functions; i.e., $V = \text{span}\{1, x\}$, and justify the technique you use.

(b) Why (specifically) does this problem become much more difficult if we consider the case $p \neq 2$?

(c) Let X be a Hilbert space, and let $U \subset X$ be a subspace. Given $u \in X$, prove that the decomposition $u = u_0 + z$, with $u_0 \in U$ and $z \in U^\perp$, as provided by the Hilbert Space Projection Theorem, is a unique decomposition.

3. Numerical Ordinary Differential Equations (270C)

Question 3.1. Consider the following Runge–Kutta method for the differential equation $y' = f(t, y)$, where f is smooth:

$$y_{n+1} = y_n + \alpha h f(t_n, y_n) + \frac{h}{2} f(t_n + \beta h, y_n + \beta h f(t_n, y_n)).$$

(a) Write down the Butcher tableau for this Runge–Kutta method.

(b) For what values of $\{\alpha, \beta\}$ is the method consistent?

(c) For what values of $\{\alpha, \beta\}$ is the method stable?

(d) For what values of $\{\alpha, \beta\}$ is the method most accurate?

Question 3.2. Consider the 3-step Adams-Bashforth method,

$$y_{n+3} = y_{n+2} + h\left[\frac{23}{12} f(t_{n+2}, y_{n+2}) - \frac{4}{3} f(t_{n+1}, y_{n+1}) + \frac{5}{12} f(t_n, y_n)\right]$$

(a) Determine the order of accuracy of this linear multistep method.

(b) Find the leading error constant for this method.

(c) Is the method convergent? Justify your answer.