NA Qual: Parts A and B

May 27, 2008

Name ______________________

#1	25	
#2	20	
#3	30	
#4	30	
#5	55	
Subtotal	160	
Part C	40	
Total	200	

(25) 1. State and prove the SVD Existence Theorem (for real $m \times n$ matrices).

(20) 2. Let A be the $m \times n$, rank$(A) = r$. Use the SVD of A, $U\Sigma V^T$, to show:

(a) Nullspace $(A) = \text{span}\{v_{r+1}, \dots, v_n\}$

(b) Range $(A) = \text{span}\{u_1, \dots, u_r\}$

(30) 3. (a) Let D be an $m \times n$ diagonal matrix. Prove $\|D\|_p = \max_i |d_{ii}|$ for $1 \le p \le \infty$.

(b) Prove that if A is $m \times n$, rank$(A) = n$ and $\|E\|_p \|A^+\|_p < 1$ for some p, $1 \le p \le \infty$, then rank$(A + E) = n$.

(c) Let A be $n \times n$, nonsingular, and $A = QR$, where Q is orthogonal and R is upper triangular with positive diagonal. Prove that Q and R are unique.

(30) 4. Suppose the computed $a_{kk}^{(k)} \neq 0$ for $1 \le k \le n-1$, where A is $n \times n$; then the computed L and U satisfy $A + E = LU$, where L is unit lower triangular and U is upper triangular. Derive the bound on E:

$$|E_{ij}| \le \left\{ \begin{matrix} (3+u)(i-1)gu \text{ for } i \le j \\ [(3+u)(j-1)+1]gu \text{ for } i > j \end{matrix} \right\}, \text{ where } g \equiv \max_k \max_{i,j} |a_{ij}^{(k)}|$$

a typo?

and u = unit roundoff.

(55) 5. (20) (a) Show that if the single shift QR method converges, then the convergence is: (a) quadratic for general matrices, (b) cubic for symmetric matrices.

(25) (b) Let $A_0 = A$, where A is symmetric positive definite,

for $k = 1, 2, \dots$

$A_{k-1} = G_k G_k^T$ (Cholesky)

$A_k \equiv G_k^T G_k$

Prove that if $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with $a \ge c$ then $A_k \to \text{diag}(\lambda_1, \lambda_2)$, where $\lambda_1 \ge \lambda_2 > 0$.

(10) (c) Let $S = \begin{bmatrix} 0 & -B^T \\ B & 0 \end{bmatrix}$, where B is $n \times n$. Relate the eigenvalues and eigenvectors of S to the SVD of B, $B = U\Sigma V^T$.

NA Qual. Part C: Approximation. Interpolation. and Numerical Quadrature.

Question 3.1. [20 points]

(1) Let $f \in C[-1,1]$ be an even function. Let $p_n \in \mathcal{P}_n$ be the best uniform approximation of f in $\mathcal{P}_n$. Prove that p_n is also an even function.

(2) Let $n \geq 1$ be an integer. Let $l_0(x), \ldots, l_n(x)$ be the Lagrange basic interpolation polynomials associated with $n+1$ distinct points $x_0, \ldots, x_n$, i.e.,

$$l_k(x) = \prod_{j=1, j \neq k}^{n} \frac{x - x_j}{x_k - x_j}, \qquad k = 0, \ldots, n.$$

Prove that

$$x^m = \sum_{j=0}^{n} x_j^m l_k(x), \qquad m = 1, \ldots, n.$$

Question 3.2. [20 points]

Let $n \geq 1$ be an integer and $-\infty < a < b < \infty$. Consider the numerical quadrature

$$\int_a^b f(x)\, dx \approx \sum_{k=1}^{n} A_k f(x_k),$$

where $x_1, \ldots, x_n \in [a,b]$ are distinct points and $A_1, \ldots, A_n \in \mathbb{R}$. Let m denote the degree of precision of this numerical quadrature. Prove the following:

(i) $m \leq 2n - 1$;

(ii) If this is an interpolatory quadrature, then $m \geq n - 1$;

(iii) That $m = 2n - 1$ if and only if this is a Gaussian quadrature.